On a PL embedded 2-sphere in 4-manifold

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§ 0.

Let M^{2n} be a simply connected differentiable manifold, and let $\xi \in \pi_n(M^{2n})$ be a given homotopy class of maps $S^n \to M^{2n}$. It is known that if $n>2$, the class ξ can be represented by a differentiable imbedding $f\colon S^n \to M^{2n}$. This follows from a reasoning similar to the one used by H. Whitney to prove that every differentiable n-manifold can be differentiably imbedded in Euclidean $2n$ space. For $n=1$, let $F_{p,q}$ be a compact connected orientable surface of genus p with q boundary components, where q may be equal to 0. Let $a_1, \cdots, a_p, b_1, \cdots, b_p, C_1, \cdots, C_{q-1}$ be standard generators for the $H(F_{p,q}: Z)$. Here the C_i correspond to consistently oriented boundary circles (one is omitted because it is homologous to the sum of the others), and the a_i and b_i are standard curves on $F_{p,q}$, chosen so that $a_i \cap a_j = b_i \cap b_j = a_i \cap b_j = \phi$ if $i \neq j$ and a_i, b_i intersect nicely at one point. Then S. Suzuki [5] proved the following:

SUZUKI'S THEOREM. *A non zero homology class* $\sum_{i=1}^{p} \alpha_i a_i + \sum_{i=1}^{p} \beta_i b_i + \sum_{i=1}^{q-1} \gamma_i C_i$ *of* $H_1(F_{p,q}: Z)$ *is representable by a simple closed curve on* $F_{p,q}$ *if and only if one of the following two conditions is satisfied*:

(1) *Not all the* α_i *and* β_i *are zero and the g.c.d*$(\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_p) = 1$,

(2) $\alpha_i = \beta_i = 0$ *for* $1 \leqq i \leqq p$ *and* $|\gamma_i| \leqq 1$ *for* $1 \leqq i \leqq q-1$ *and all non zero* γ_i *have the same sign.*

For $n=2$, whether or not $\xi \in H_2(M^4)$ is representable by a differentiable imbedding $f\colon S^2 \to M^4$ depends on the class ξ ([2], [6]). On the other hands there exists no example whose class $\xi \in H_2(M^4)$ is not representable by a PL imbedding $f\colon S^2 \to M^4$. So we attack a problem representing a class $\xi \in H_2(M^4)$ by a PL embedding $f\colon S^2 \to M^4$. And we obtain a following.

THEOREM. *Let* M^4 *be a 1-connected closed PL 4-manifold. Then any 2-dim. homology class* $\xi \in H_2(M \# k(S^2 \times S^2))$ *is representable by a PL embedding* $f\colon S^2 \to M \# k(S^2 \times S^2)$ *for some* $k \geqq 0$ *where* $M \# k(S^2 \times S^2)$ *is a connected sum of M with k-copies of* $S^2 \times S^2$.

§ 1.

PROPOSITION 1. (*Kervaire and Milnor* [2]) *Let M be a closed PL 4-manifold containing a 2-dimensional subcomplex K such that*

(1) *$M-K$ is acyclic, and*

(2) *The boundary of a regular neighborhood of K is a 3-sphere.*

Then, every homology class $\xi \in H_2(M:Z)$ can be represented by a PL imbedded 2-sphere.

COROLLARY 1. *Let M^4 be a closed connected PL 4-manifold containing a 2-dimensional subcomplex K such that M-Int D^4 collapses to K (i.e. M-Int $D^4 \searrow K$).*

Then every homology class $\xi \in H_2(M:Z)$ can be represented by a PL imbedded 2-sphere.

PROOF. Since M-Int $D \searrow K$, $U = M$-Int D is a regular neighborhood of K in M. So $\partial U = \partial D^4 \cong S^3$ and $M-K \simeq M-(M\text{-Int } D^4) \simeq D^4$. Hence $M-K$ is acyclic. So by Proposition 1, every class $\xi \in H_2(M:Z)$ can be represented by a PL embedded 2-sphere.

COROLLARY 2. *Let M^4 be a closed connected PL 4-manifold with the following handle decomposition,*

$$M^4 = D_0^4 \cup \bigcup_i (D^1 \times D^3)_i \cup \bigcup_j (D^2 \times D^2)_j \cup D_1^4 \qquad or$$

$$M^4 = D_0^4 \cup \bigcup_j (D^2 \times D^2)_j \cup \bigcup_k (D^3 \times D^1)_k \cup D_1^4$$

(i.e. M^4 has a handle decomposition without 3-handles or without 1-handles). Then every homology class $\xi \in H_2(M^4)$ is representable by a PL imbedded 2-sphere.

PROOF. If $M^4 = D_0^4 \cup \bigcup_i (D_1 \times D^3)_i \cup \bigcup_j (D^2 \times D^2)_j \cup D_1^4$,

$$\begin{aligned}\overline{M-D_1^4} &\searrow D_0^4 \cup \bigcup_i (D^1 \times \{0\})_i \cup \bigcup_j (D^2 \times \{0\})_j \\ &\searrow \Big\{c * \big(\big(\bigcup_i (D^1 \times \{0\})_i\big) \cup \bigcup_j (D^2 \times \{0\})_j \cap \partial D_0^4\big)\Big\} \\ &\quad \cup \bigcup_i (D^1 \times \{0\})_i \cup \bigcup_j (D^2 \times \{0\})_j \equiv K\end{aligned}$$

where c is the center of D_0^4 (see also [4. P. 83]). Then K is a 2-dim subcomplex of K which satisfies the conditions (1), (2) of Proposition 1 because $\overline{M-D^4}$ is a regular neighborhood of K in M. Hence every homology class $\xi \in H_2(M:Z)$ can be represented by a PL imbedded 2-sphere. If $M^4 = D_0^4 \cup \bigcup_j (D^2 \times D^2)_j \cup \bigcup_k (D^3 \times D^1)_k \cup D_1^k$, we may take the dual handle decomposition of the above ([4, p. 82]).

Then M has a handle decomposition without 3-handles and so the conclusion follows from the above proof.

REMARK. $PC(2)$, $S^2 \times S^2$ and non-trivial S^2 boundle over S^2 satisfy the hypothesis of Proposition 1.

COROLLARY 3. *Let $P(T, m_i)$ be a 4-manifold obtained by plumbing according to the weighted tree (T, m_i) such that $\partial P(T, m_i) \cong S^3$ (see [1, p. 56]). And let $M = P(T, m_i) \underset{\partial}{\cup} D^4$.*
Then every homology class $\xi \in H_2(M : Z)$ is representable by a PL imbedded 2-sphere in M.

PROOF. $\overline{M - D^4} = P(T, m_i) \searrow S^2 \vee \cdots \vee S^2 \equiv K$

where each S^2 is a zero section of D^2-bundle over S^2. And since $\overline{M - D^4}$ is a regular nighborhood of K in M, the conclusion follows from Proposition 1.

THEOREM. *Let M^4 be a 1-connected closed PL 4-manifold. Then any 2-dim. homology class $\xi \in H_2(M \# k(S^2 \times S^2))$ is representable by a PL embedding $f: S^2 \to M \# k(S^2 \times S^2)$ for some $k \geqq 0$ where $M \# k(S^2 \times S^2)$ is a connected sum of M with k-copies of $S^2 \times S^2$.*

PROOF. Let $M^4 = D_0^4 \cup \bigcup_i (D^1 \times D^3)_i \cup \bigcup_j (D^2 \times D^2)_j \cup \bigcup_k (D^3 \times D_1)_k \cup D_1^4$.
Since every 2-dim. homology class of closed 4-manifold without 1-handles is representable by a PL imbedded 2-sphere by Corollary 2 to Proposition 1, we will show that all 1-handles of M can be eliminated by attaching $S^2 \times S^2$. Let $\phi_1 : (\partial D^1 \times D^3)_1 \to \partial D_0^4$ be an attaching map. Then

$$D_0^4 \underset{\phi_1}{\cup} (D_1 \times D^3)_1 \cong S^1 \times D^3 . \qquad \text{And let}$$

$$S^2 \times S^2 = D_2^4 \cup \bigcup_{l=1}^{2} (D^2 \times D^2)_l \underset{\phi}{\cup} D_3^4 \qquad \text{and}$$

$$W_1 = (M\text{–Int}(D_0^4 \underset{\phi_1}{\cup} (D^1 \times D^3)_1)) \underset{\alpha}{\cup} (S^2 \times S^2\text{–Int}(D_2^4 \cup (D^2 \times D^2)_1))$$

where α : $S^1 \times S^2 \to S^1 \times S^2$ is a PL homeomorphism given by $\alpha(x, y) = (x, y)$ since

$$\partial(M\text{–Int}(D_0^4 \underset{\phi_1}{\cup} (D^1 \times D^3)_1)) \cong S^1 \times S^2 \qquad \text{and}$$

$$\partial(S^2 \times S^2\text{–Int}(D_2^4 \cup (D^2 \times D^2)_1)) \cong S^1 \times S^2 .$$

Then $W_1 \cong M \# (S^2 \times S^2)$. For $S^1 \times \{0\} \subset S^1 \times D^3 \cong D_0^4 \underset{\phi_1}{\cup} (D^1 \times D^3)_1$ is homotopic to a point in M since $\pi_1(M) = \{1\}$. And there is a non-singular 2-ball B^2 with $\partial B^2 = S^1 \times \{0\}$ because $\dim M = 4$. And a regular neighborhood $U(B^2, M)$ is a 4-ball containing $S^1 \times \{0\}$ in its interior. So we may assume that there is a 4-ball D^4 in M such that $\operatorname{Int} D^4 \supset D_0^4 \underset{\phi_1}{\cup} (D^1 \times D^3)_1$.

Then $$W_1 = (M\text{-Int}(D_0^4 \underset{\phi_1}{\cup} (D^1 \times D^3)_1)) \underset{\alpha}{\cup} (S^2 \times S^2\text{-Int}(D_2^4 \cup (D^2 \times D^2)_1))$$
$$\cong (M\text{-Int}(S^1 \times D^3)) \underset{\alpha}{\cup} (D^2 \times S^2)$$
$$= (M\text{-Int}\, D^4) \underset{\partial D^4}{\cup} (D^4\text{-Int}(S^1 \times D^3)) \underset{\alpha}{\cup} (D^2 \times S^2).$$

Here $$(D^4\text{-Int}(S^1 \times D^3)) \underset{\alpha}{\cup} (D^2 \times S^2)$$
$$\cong (D^2 \times S^2\text{-Int}\, \tilde{D}^4) \underset{\alpha}{\cup} (D^2 \times S^2) \cong S^2 \times S^2\text{-Int}\, \tilde{D}^4$$

because $(D^4\text{-Int}(S^1 \times D^3)) \cong S^4\text{-Int}(S^1 \times D^3))\text{-Int}\, \tilde{D}^4 \cong (D^2 \times S^2\text{-Int}\, \tilde{D}^4)$ where $S^4 = D^4 \underset{\partial}{\cup} \tilde{D}^4$ and $\tilde{D}^4 \subset \operatorname{Int}(D^2 \times S^2)$. Hence

$$W_1 \cong (M\text{-Int}\, D^4) \underset{\partial D^4}{\cup} (S^2 \times S^2\text{-Int}\, \tilde{D}^4) = M \# (S^2 \times S^2).$$

And W_1 has a handle decomposition

$$W_1 = D_3^4 \underset{\tilde{\varphi}}{\cup} (D^2 \times D^2)_2 \cup (\underset{\tilde{\alpha}}{\cup} \underset{i \geq 1}{\cup} (D^1 \times D^3)_i \cup \underset{j}{\cup} (D^2 \times D^2)_j \cup \underset{k}{\cup} (D^3 \times D^1)_k \cup D_1^4)$$

(i.e. one 1-handle was eliminated and one 2-handle was added.) where $\tilde{\varphi}$ is given by following;

since $S^2 \times S^2$ is a closed manifold, an attaching map

$$\varphi:\ \partial D_3^4 \to \partial (D_2^4 \cup \bigcup_{l=1}^{2} (D^2 \times D^2)_l) \quad \text{is a homeomorphism}$$

onto and $$(D^2 \times \partial D^2)_2 \subset \partial (D_2^4 \cup \bigcup_{l=1}^{2} (D^2 \times D^2)_l).$$

So we define $\tilde{\varphi} = \varphi^{-1} | (D^2 \times \partial D^2)_2$. And for $\tilde{\alpha}$ if ϕ is an attaching map in M of a handle in

$$\bigcup_{i \geq 2} (D^1 \times D^3)_i \cup \bigcup_j (D^2 \times D^2)_j \cup \bigcup_k (D^3 \times D^1)_k \cup D_2^4,$$

$\alpha^{-1}\phi$ is an attaching map in W_1 of a handle in

$$\bigcup_{i \geq 2} (D^1 \times D^3)_i \cup \bigcup_j (D^2 \times D^2)_j \cup \bigcup_k (D^3 \times D^1)_k \cup D_2^4.$$

So we denote $\tilde{\alpha} = \alpha^{-1}\phi$.

By reordering lemma ([4. p. 76]) we may assume

$$W_1 = D_3^4 \underset{\tilde{\alpha}}{\cup} \bigcup_{i \geq 2} (D^1 \times D^3)_i \underset{\tilde{\varphi}}{\cup} (D^2 \times D^2)_2 \cup (\underset{\tilde{\alpha}}{\cup} \bigcup_j (D^2 \times D^2)_j \cup \bigcup_k (D^3 \times D^1)_k \cup D_1^4).$$

Next for $D_3^4 \underset{\alpha^{-1}\phi_2}{\cup} (D^1 \times D^3)_2$ we do same procedure as above and replace it by $D_5^4 \underset{\tilde{\varphi}_1}{\cup} (D^2 \times D^2)_2$ where $S^2 \times S^2 = D_4^4 \cup (D^2 \times D^2)_1 \cup (D^2 \times D^2)_2 \underset{\phi_1}{\cup} D_5^4$. We continue this steps until all 1-handles can be eliminated. Finally $W_k=M\# k(S^2\times S^2)$ has a handle decomposition without 1-handles where k is the number of 1-handles of the original handle decomposition of M.

REMARK. For a differentiable case, there is an example such that some class $\xi\oplus 0\in H_2(M)\oplus H_2(\# k(S^2\times S^2))$ can not be represented by a differentiable imbedding $f: S^2\to M\# k(S^2\times S^2)$ for any $k\geqq 0$. Let $M=S^2\times S^2$ and $\xi=2(\alpha+\beta)\in H_2(S^2\times S^2)$ where α and β are the classes representing $S^2\times p$, $q\times S^2$ respectively. Then by ([2. Cor. 1]) ξ can not be represented by a differentiable imbedding $f: S^2\to S^2\times S^2$. But since the signature $\sigma(S^2\times S^2)$ is zero, $\xi\oplus 0\in H_2(S^2\times S^2)\oplus H_2(\# k(S^2\times S^2))$ can not be represented by a differentiable imbedding $f: S^2\to\#(k+1)(S^2\times S^2)$ for any $k\geqq 0$ (see [2. Th. 1]).

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References

[1] F. HIRZEBRUCH, W. D. NEUMANN and S. S. KOH: differentiable manifold and quadratic forms, Marcel Dekker, Inc. New York 1971.

[2] M. A. KERVAIRE and J. W. MILNOR: On 2-spheres in 4-manifolds, Proc. Nat. Acad. Sci. 47 (1961) 1651–1657.

[3] J. W. MILNOR: On simply connected 4-manifolds, Symposium International de Topologia Algebrica, Mexico (1958), 122–128.

[4] C. P. ROURKE and B. J. SANDERSON: Introduction to piecewise linear topology, Ergeb. Math. und ihrer Grenzgebiete, 69, Springer Verlag.

[5] S. SUZUKI: Representations of 1-homology classes of bounded surfaces, Proc. Japan Acad. 46 (1970) 1096–1098.

[6] C. T. C. WALL: Diffeomorphism of 4-manifolds, J. London Math. Soc., 39 (1964) 131–140.

[7] C. T. C. WALL: On simply connected 4-manifold, J. London Math. Soc. 39 (1964) 141–149.

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