Characteristic Classes of Foliated Principal GL_r-Bundles

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Introduction

Let M be a paracompact Hausdorff differentiable (C^∞) manifold of dimension n and $\mathscr{F}$ a differentiable (C^∞) condimension q foliation on M. We denote the manifold M with the foliation $\mathscr{F}$, by $(M, \mathscr{F})$. Let $GL_r = GL(r, R)$ denote the group of $r \times r$ non-singular matrices over real numbers. A *foliated* principal GL_r-bundle $E(M, p, GL_r)$ over the $(M, \mathscr{F})$ is a differentiable (C^∞) principal GL_r-bundle $p: E \to M$, such that E has a right GL_r-invariant differentiable (C^∞) foliation $\mathscr{F}_E$, where each leaf is a covering of a leaf of $\mathscr{F}$. (Cf. P. Molino [4].) $\mathscr{F}_E$ is called a *lifted foliation* of $\mathscr{F}$.

We generalize the Bott's construction of characteristic classes of a foliation (cf. R. Bott [1] and P. Molino [6]) to the foliated principal GL_r-bundles and we obtain several vanishing theorems of the characteristic classes. In particular, these theorems are remarkable in the case where $E(M, p, GL_r)$ admits a transverse projectable connection. P. Molino [6] obtains these theorems for the frame bundle of the normal bundle of the foliation $(M, \mathscr{F})$. However, if M has two foliations $\mathscr{F}$ and $\mathscr{F}'$ of condimensions q and r respectively $(q \geqq r)$ such that the tangent subbundle F of $\mathscr{F}$ is a subbundle of the tangent subbundle F' of $\mathscr{F}'$, then we can construct a foliated principal GL_r-bundle $E(M, p, GL_r)$ over $(M, \mathscr{F})$ and our generalized arguments of characteristic classes are applied to such foliated principal bundles.

Some applications of our theorems will be given in a subsequent note.

§ 1. A transverse connection

A connection on the foliated principal GL_r-bundle $E(M, p, GL_r)$ over the foliated manifold $(M, \mathscr{F})$ is said to be a *transverse* connection, if leaves of the lifted foliation $\mathscr{F}_E$ of $\mathscr{F}$ are horizontal for the connnection. (Cf. P. Molino [4].) In this section, we shall introduce characteristic classes of the foliated principal GL_r-bundle by the notion of the transverse connection.

Let gl_r denote the Lie algebra of GL_r and $I(gl_r)$ denote the algebra of invariant polynomials of gl_r. Let ∇^1 be a transverse connection on the $E(M, p, GL_r)$. It is easy to see that the $E(M, p, GL_r)$ admits a transverse connection.

LEMMA 1.1. *Let k^1 be the curvature form on M, of the transverse connection ∇^1. If the invariant homogeneous polynomial $\varphi_i \in I(gl_r)$ is of degree i, then we have*

$$\varphi_i(k^1)=0 \qquad \textit{for} \quad i>q\,.$$

PROOF. Let $k^{1\alpha}=(k^{1\alpha}_{ij})$ be a presentation of the curvature k^1 on a coordinate neighborhood U_α of the bundle $E(M, p, GL_r)$. Since the curvature k^1 vanishes along leaves of the foliation $\mathscr{F}$, $k^{1\alpha}_{ij}$ is contained in the ideal $I_\alpha(F)$ of differential forms defining the tangent subbundle F of $\mathscr{F}$ on U_α. It is obvious that

$$\varphi_i(k^{1\alpha}) \in I_\alpha(F)^{q+1}=\{0\} \qquad \text{for} \quad i>q\,.$$

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We denote a leaf of the foliation $\mathscr{F}$ by L. Let $\mathscr{F}\times R$ be the condimension q foliation on the product manifold $M\times R$ defined by the set of leaves

$$\{L\times R \mid L\in\mathscr{F}\}\,.$$

In a natual way, the map

$$p\times id\colon \quad E\times R \longrightarrow M\times R$$

defines a foliated principal GL_r-bundle $E(M\times R, p\times id, GL_r)$ over the foliated manifold $(M\times R, \mathscr{F}\times R)$, where the lifted foliation $(\mathscr{F}\times R)_E$ of $\mathscr{F}\times R$ is the foliation $\mathscr{F}_E\times R$ defined by $\{L_E\times R \mid L_E\in\mathscr{F}_E\}$. Let ∇ and $\bar{\nabla}$ be two connections on the $E(M, p, GL_r)$ with connection forms θ and $\bar{\theta}$ respectively. We denote by $T_p\,(.)$ the tangent space at a point p of a differentiable manifold. Define a gl_r-valued 1-form $\tilde{\theta}$ on the $E(M\times R, p\times id, GL_r)$ as follows. We denote the coordinate in R by t. Let $\tilde{\theta}(\partial/\partial t)=0$. If $X\in T_{(e,t)}(E\times\{t\}) = T_e(E)\times\{t\}$, define

$$\tilde{\theta}(X)=(1-t)\,\theta(X)+t\bar{\vartheta}(X)\,.$$

The 1-form $\tilde{\theta}$ determines a horizontal subspace field on the $E(M\times R, p\times id, GL_r)$ and defines a connection on it, which denoted by $\tilde{\nabla}$ or by $(1-t)\nabla+t\bar{\nabla}$, $(t\in R)$.

It is well known that a curvature k of a connection on a differentiable principal bundle is a closed form, (see, e.g., S. Kobayashi and K. Nomizu [3, Chap. XII] or R. Bott [1, Section 5].) and hence, for any invariant homogenious polynomial $\varphi_i\in I(gl_r)$, we have

$$d\varphi_i(k)=0\,.$$

Let $\bar{k}$ denote the curvature form on M, of $\bar{\nabla}$. By the differentiable homotopy invariance of de Rham cohomology, we have

$$\left[\varphi_i(k)\right] = \left[\varphi_i(\bar{k})\right] \in H^{2i}_{DR}(M)$$

for the connections ∇ and $\bar{\nabla}$ on the $E(M, p, GL_r)$. These cohomology classes are real Pontrjagin characteristic classes corresponding to φ_i. From Lemma 1.1, we obtain easily a generalized Bott's vanishing of characteristic classes as follows:

COROLLARY 1.2. (P. Molino [5]) *The real Pontrjagin characteristic classes of the foliated principal bundle $E(M, p, GL_r)$ over the $(M, \mathscr{F})$ vanishes in dimension $> 2 \cdot codim\ \mathscr{F}$.*

For the next section, we prepare the following Lemmas on the transversality of the connection $\tilde{\nabla}^1 = (1-t)\nabla^1 + t\bar{\nabla}^1$ and on the algebra $I(gl_r)$.

LEMMA 1.3. *If ∇^1 and $\bar{\nabla}^1$ are transverse connections on the $E(M, p, GL_r)$, then the $\tilde{\nabla}^1$ is a transverse connection on the $E(M \times R, p \times id, GL_r)$.*

A proof of the lemma follows immediately from the definition of the $\tilde{\nabla}^1$.

LEMMA 1.4. *If we define invariant homogeneous polynomials c_i of an $r \times r$ matrix A over the real number field R by*

$$\det(I + tA) = 1 + \sum_{i=1}^{r} t^i c_i(A),$$

then we have

$$I(gl_r) = R[c_1, \cdots, c_r].$$

For a proof of the lemma, see, e.g., R. Bott [1 Appendix A].

By using a transverse connection, one can obtain a generalized Bott's construction of characteristic classes as follows. Let ∇^0 be a Riemannian connection on the $E(M, p, GL_r)$ (that is, the connection defined by a Riemannian connection of the associated differentiable r-vector bundle) and let k^0 be the curvature form on M, of the ∇^0. We denote by $A^*_C(M)$ the algebra of differential forms on M with complex coefficients. Homomorphisms

$$\lambda(\nabla^0),\ \lambda(\nabla^0, \nabla^1):\quad I(gl_r) \longrightarrow A^*_C(M)$$

are defined by the formulas,

$$\lambda(\nabla^i)(\varphi_j) = \left(\frac{\sqrt{-1}}{2\pi}\right)^j \varphi_j(k^i) \qquad i = 0, 1,$$

$$\lambda(\nabla^0, \nabla^1)(\varphi_j) = \hat{p}_*\left(\lambda(\nabla^{0,1})(\varphi_j)|M \times I\right),$$

where $\hat{p}_*: A^*_C(M \times I) \to A^*_C(M)$ is the integration along the fibre of the projection $\hat{p}: M \times I \to M$, and $\nabla^{0,1} = (1-t)\nabla^0 + t\nabla^1$, $(t \in R)$. By the same manner as R. Bott [1, Section 10], we have, for the Riemannian connection ∇^0,

$$\lambda(\nabla^0)(c_{2i-1})=0$$

and

$$d\left\{\lambda(\nabla^0, \nabla^1)(c_{2i-1})\right\}=\lambda(\nabla^1)(c_{2i-1}), \qquad 1 \leqq 2i-1 \leqq r.$$

By Lemma 1.1, the homomorphism

$$\lambda(\nabla^1): \quad R[c, \cdots, c_r] \longrightarrow A_C^*(M)$$

annhilates all elements of degree$>q$. Let $WO_{q,r}$ be the differential algebra;

$$R[c_1, \cdots, c_s]/(\deg>q) \otimes \wedge(h_1, h_3, \cdots, h_l).$$

$$s=\min(q, r), \qquad l=\max\{2m+1 \leqq r\},$$

$$d(c_i)=0, \qquad d(h_i)=c_i \qquad 1 \leqq i \leqq s,$$

$$dh_i=0 \qquad i>s(\text{if } q>l).$$

We define an R-algebra homomorphism

$$\lambda_E: \quad WO_{q,r} \longrightarrow A_C^*(M)$$

by the formulas,

$$\lambda_E(c_i)=\lambda(\nabla^1)(c_i) \qquad 1 \leqq i \leqq s,$$

$$\lambda_E(h_i)=\lambda(\nabla^0, \nabla^1)(c_i) \qquad i=1, 3, \cdots, l.$$

λ_E is a cochain map and induces homomorphism of R-algebras,

$$\lambda_E^*: \quad H^*(WO_{q,r}) \longrightarrow H_{DR}^*(M, C).$$

PROPOSITION 1.5. *The homomorphism λ_E^* depend only on the foliated principal GL_r-bundle $E(M, p, GL_r)$, not on the choices of the transverse connection ∇^1 and the Riemanian connection ∇^0.*

PROOF. If ∇^1 and $\bar{\nabla}^1$ are transverse connections on the $E(M, p, GL_r)$, then $\tilde{\nabla}^1=(1-t)\nabla^1+t\bar{\nabla}^1(t \in R)$ is also a transverse connection on the foliated principal GL_r-bundle $E(M \times R, p \times id, GL_r)$ over the $(M \times R, \mathscr{F} \times R)$, by Lemma 1.3. Riemannian connections are differentiably homotopic through Riemannian connections. The assertion follows by the differentiable homotopy invariance of $H_{DR}^*(M; C)$.

Q. E. D.

The cohomology classes of image of the λ_E^* are called *Bott's characteristic classes* of the foliated principal GL_r-bundle $E(M, p, GL_r)$ over the $(M, \mathscr{F})$.

2. A transverse projectable connection

We consider transverse connections which have fine properties.

DEFINITION 2.1. (P. Molino [4].) If a transverse connection of the foliated principal GL_r-bundle $E(M, p, GL_r)$ over the foliated manifold $(M, \mathscr{F})$ is locally an inverse image of a connection of the restriction of the $E(M, p, GL_r)$ on a transverse q dimensional submanifold to the $\mathscr{F}$, by the local projection of the M along leaves of the $\mathscr{F}$, then the connection is called *projectable.*

Let ∇^ω be a transverse projectable connection on the $E(M, p, GL_r)$. Just as the preceding section, we have the following Bott's construction for the transverse projectable connection ∇^ω and a Riemannian connection ∇^0 on the $E(M, p, GL_r)$. Since the connection ∇^ω is induced from a connection on the manifold of dimension $q=\operatorname{codim} \mathscr{F}$, it follows that the homomorphism,

$$\lambda(\nabla^\omega): \quad R[c_1, \cdots, c_r] \longrightarrow A_C^*(M)$$

annihilates all elements of degree$>[q/2]$. Taking account with this fact, we consider the differential algebra $WO'_{q/2,r}$;

$$R[c_1, \cdots, c_s]/(\deg > q/2) \otimes \wedge (h_1, h_3, \cdots, h_l),$$

$$s' = \min\left([q/2], r\right), \qquad l = \max\{2m+1 \leqq r\},$$

$$d(c_i) = 0, \qquad d(h_i) = c_i \qquad 1 \leqq i \leqq s',$$

$$d(h_i) = 0 \qquad i > s' (\text{if } [q/2] < l).$$

We define an R-algebra homomorphism

$$\lambda'_E: \quad WO'_{q/2,r} \longrightarrow A_C^*(M)$$

by the formulas,

$$\lambda'_E(c_i) = \lambda(\nabla^\omega)(c_i) \qquad 1 \leqq i \leqq s',$$

$$\lambda'_E(h_i) = \lambda(\nabla^0, \nabla^\omega)(c_i) \qquad i = 1, 3, \cdots, l,$$

λ'_E is a cochain map and induces homomorphism of R-algebras,

$$(\lambda'_E)^*: \quad H^*(WO'_{q/2,r}) \longrightarrow H^*_{DR}(M; C).$$

In the $WO'_{q/2,r}$ linear combinations of the elements,

$$c_{i_1} \cdots c_{i_\lambda} \otimes 1,$$

and

$$c_{i_1} \cdots c_{i_\lambda} \otimes h_{j_1} \wedge \cdots \wedge h_{j_\mu}$$

$$2\left(\sum_{\alpha=1}^{\lambda} i_\alpha + \min\{j_\beta | 1 \leqq \beta \leqq \mu\}\right) > q,$$

are cocycles. On the independence of the $(\lambda'_E)^*$, from the involved connections one obtains the following.

PROPOSITION 2.2. *The cohomology class*

$$(\lambda'_E)^*[c_{i_1} \cdots c_{i_\lambda} \otimes h_{j_1} \wedge \cdots \wedge h_{j_\mu}]$$

does not depend on the choices of the transverse projectabe connection ∇^ω *and the Riemannian connection* ∇^0, *unless*

$$2\left(\sum_{\alpha=1}^{\lambda} i_\alpha + \min\{j_\beta | 1 \leqq \beta \leqq \mu\}\right) = q+1.$$

In particular, If q *is even, then the homomorphism* $(\lambda'_E)^*$ *does not depend on the choices of* ∇^ω *and* ∇^0.

PROOF. If ∇^ω and $\bar{\nabla}^\omega$ are transverse projectable connections of the $E(M, p, GL_r)$, which are locally inverse images of connections on the restrictions of the $E(M, p, GL_r)$ on transverse submanifolds N and $\bar{N}$ to the $\mathscr{F}$ by local projections of the base space M along leaves. It is obvious that there is naturally a local diffeomorphism of N and $\bar{N}$, and one can assume that ∇^ω and $\bar{\nabla}^\omega$ are locally inverse images of connections on the restriction of $E(M, p, GL_r)$ on the same transverse submanifold N to the $\mathscr{F}$ by the local projection of the M along leaves of the $\mathscr{F}$. Then, by Lemma 1.3, the connection

$$\tilde{\nabla}^\omega = (1-t)\nabla^\omega + t\bar{\nabla}^\omega \qquad t \in R$$

is a transverse connection on the principal bundle $E(M \times R, p \times id, GL_r)$ over the foliated manifold $(M \times R, \mathscr{F} \times R)$. It is obvious also that for the foliated principal GL_r-bundle structure of $E(M \times R, p \times id, GL_r)$ over the codimension $q+1$ foliation $\{\mathscr{F} \times \{t\}\}$ on $M \times R$, the connection $\tilde{\nabla}^\omega$ is transverse projectable. We have, therefore, for λ'_E: $WO'_{q/2,r} \rightarrow A^*_C(M \times R)$,

$$d\left(\lambda'_E(c_{i_1} \cdots c_{i_\lambda} \otimes h_{j_1} \wedge \cdots \wedge h_{j_\mu})\right) = 0$$

if $2\left(\sum_{\alpha=1}^{\lambda} i_\alpha + \min\{j_\beta | 1 \leqq \beta \leqq \mu\}\right) > q+1$. On the other hand, Riemannian connections are differentiably homotopic through Riemannian connections. The assertion follows by the differentiable homotopy invariance of $H^*_{DR}(M; C)$.

Q. E. D.

Proposition 2.2 is analogous to a result of J. Heitsch [2, Theorem 3]. We have a compatibility of the homomorphisms λ^*_E and $(\lambda'_E)^*$ as follows.

THEOREM 2.3. *Let $\mathscr{F}$ be a codimension q foliation on a paracompact Hausdorff differentiable (C^∞) manifold M and $E(M,p,GL_r)$, a foliated principal GL_r-bundle over the foliated manifold $(M, \mathscr{F})$, which admits a transverse projectable connection ∇^ω. Let τ: $WO_{q,r} \to WO'_{q/2,r}$ denote the cochain map defined by the natural projection homomorphism and denote by τ^* the homomorphism of cohomology rings induced by τ. Then we have the commutative diagram,*

$$\begin{array}{ccc} H^*(WO_{q,r}) & \searrow^{\lambda_E^*} & \\ \downarrow \tau^* & & \\ H^*(WO'_{q/2,r}) & \xrightarrow{(\lambda'_E)^*} & H^*_{DR}(M;\, C). \end{array}$$

PROOF. If z be any cocycle of the $WO_{q,r}$, then τz is also a cocycle of the $WO'_{q/2,r}$. By the definition of the homomorphism λ'_E, we have

$$\lambda'_E(\tau z) = \lambda^\omega_E(z),$$

where λ^ω_E denote the cochain map $WO_{q,r} \to A^*_C(M)$ defined by the transverse projectable connection ∇^ω and a Riemannian connection ∇^0. Since ∇^ω is a transverse connection, $\lambda^\omega_E(z)$ is a representative of $\lambda^*_E[z]$ by Proposition 1.5.

Q. E. D.

The cohomology classes of the image of the $(\lambda'_E)^*$ are called *Molino's characteristic classes* of the foliated principal GL_r-bundle $E(M,p,GL_r)$ over the $(M, \mathscr{F})$, *with respect to a transverse projectable connection.* These characteristic classes are "more fine" than those of Bott's characteristic classes by the above theorem. The theorem also shows vanishing of certain Bott's characteristic classes.

COROLLARY 2.4. *For the cohomology classes of*

$$\begin{aligned} \gamma &= c_{i_1} \cdots c_{i_\lambda} \otimes h_{j_1} \wedge \cdots \wedge h_{j_\mu} \\ & 1 \leqq i_1, \cdots, i_\lambda \leqq \min(q, r) = s, \\ & 1 \leqq j_1, \cdots, j_\mu \leqq l, \\ & \sum_{\alpha=1}^{\lambda} i_\alpha + \min\{j_\beta | 1 \leqq \beta \leqq \mu\} > q, \end{aligned}$$

in $H^*(WO_{q,r})$, the characteristic classes $\lambda^*_E[\gamma]$ is zero if $2\left(\sum_{\alpha=1}^{\lambda} i_\alpha\right) > q$.

3. A construction of a foliated principal GL_r-bundle

We construct a foliated principal GL_r-bundle for a "relative" foliation on a paracompact Hausdorff differentiable manifold M. We denote the tangent bundle of M by $T(M)$.

THEOREM 3.1. *Let $\mathcal{F}$ and $\mathcal{F}'$ be differentiable (C^∞) foliations of codimensions q and r respectively, on a paracompact Hausdorff differentiable (C^∞) manifold M. Let F and F' be tangent subbundles corresponding to the $\mathcal{F}$ and $\mathcal{F}'$ respectively. Suppose that the F is a subbundle of the F'. Then the frame bundle $E_T(M, p_T, GL_r)$ of the quotient vector bundle $Q=T(M)/F'$ is a foliated principal GL_r-bundle over the foliated manifold $(M, \mathcal{F})$.*

PROOF. Let λ: $T(M)\to Q$ be the projection homomorphism of vector bundles in the definition of the quotient bundle Q. Let $\Gamma(\cdot)$ denote a module of differentiable (C^∞) sections of a differentiable (C^∞) vector bundle. We take a basic connection ∇ on Q, defined by R. Bott [1, Section 6]. Then we have

$$\nabla_X(Z)=\lambda[X, \tilde{Z}] \qquad \forall X\in\Gamma(F'), \quad \forall Z\in\Gamma(Q),$$

where $\tilde{Z}\in\Gamma(T(M))$ is such that $\lambda\tilde{Z}=Z$. Let k denote the curvature form on M, of ∇. From the Jacobi identity, it follows that

$$k(X, Y)=0 \qquad \forall X, Y\in\Gamma(F').$$

We denote the frame bundle of Q by $E_T=E_T(M, p_T, GL_r)$. Let θ be the connection form (on E_T) of ∇ and H, the horizontal projection of ∇. Let U_α be a coordinate neighborhood of the bundle E_T and let $X, Y\in\Gamma(T(U_\alpha))$. If X', $Y'\in\Gamma(T(E_T|U_\alpha))$ are vector fields determined by X, Y and by the product structure $E|U_\alpha\cong U_\alpha\times GL_q$, then we have

$$k(X, Y)=d\theta(HX', HY').$$

In particular, for $X, Y\in\Gamma(F|U_\alpha)\subset\Gamma(F'|U_\alpha)$, one obtains

$$\begin{aligned} 0=k(X, Y)&=d\theta(HX', HY') \\ &=HX'\theta(HY')-HY'\theta(HX')-\theta[HX', HY'] \\ &=\theta[HX', HY']. \end{aligned}$$

That is, for horizontal lifts $\bar{X}=HX'$ and $\bar{Y}=HY'$ of X and $Y\in\Gamma(F)$ respectively, we have

$$\theta[\bar{X}, \bar{Y}]=0.$$

Now, since we have

$$p_{T*}\bar{X}=X, \qquad p_{T*}\bar{Y}=Y,$$

it follows that

$$
\begin{aligned}
p_{T*}[\bar{X}, \bar{Y}] &= [p_{T*}\bar{X}, p_{T*}\bar{Y}] \\
&= [X, Y] \in \Gamma(F|U_\alpha).
\end{aligned}
$$

Therefore, the horizontal lift $\tilde{F}$ of F is involutive, that is, integrable, and hence, a parallel displacement of a frame of Q along a differentiable (C^∞) curve on each leaf L of $\mathscr{F}$ is determined upto homotopy of the curve leaving fixed its end points. Then one obtains a differentiable (C^∞) submanifold in $E_T|L$, covering L and it is itself a differentiablly (C^∞) immersed submanifold[1)] of E_T. By the right action of element of GL_r, one obtains differentiablly (C^∞) immersed submanifold[2)] which covers L and passes through every points of fibre. These constructions of covering immersed submanifold[3)] on each leaf of $\mathscr{F}$ gives us the differentiable (C^∞) foliation $\mathscr{F}_{E_T}$, which makes E_T a foliated principal bundle on the $(M, \mathscr{F})$.

Q. E. D.

One can apply results on vanishing of Bott's characteristic classes of a foliated principal GL_r-bundle over a foliated manifold to the $E_T(M, p_T, GL_r)$. In particular, by Corollary 2. 4, we obtain,

COROLLARY 3. 2. *Let $\mathscr{F}$ and $\mathscr{F}'$ be differentiable (C^∞) foliations of codimensions q and r respectively, on a paracompact Hausdorff differentiable (C^∞) manifold M such that the tangent subbundle to $\mathscr{F}$ is a subbundle of the tangent subbundle to $\mathscr{F}'$, and $E_T(M, p_T, GL_r)$, the frame bundle of the quotient vector bundle of $T(M)$ by the tangent subbundle to $\mathscr{F}'$, which is provided with the foliated princepal GL_r-bundle structure over the $(M, \mathscr{F})$ in Theorem* 3. 1. *Let $\lambda^*_{E_T}$: $H^*(WO_{q,r}) \rightarrow H^*_{DR}(M; C)$ denote the homomorphism of Bott's characteristic classes of the $E_T(M, p_T, GL_r)$ over the $(M, \mathscr{F})$. Suppose that the $E_T(M, p_T, GL_r)$ over $(M, \mathscr{F})$ admits a transverse projectable connection. Then the characteristic classes*

$$
\begin{aligned}
&\lambda^*_{E_T}[c_{i_1} \cdots c_{i_\lambda} \otimes h_{j_1} \wedge \cdots \wedge h_{j_\mu}] \\
&1 \leqq i_1, \cdots, i_\lambda \leqq \min(q, r) = r, \qquad 1 \leqq j_1, \cdots, j_\mu \leqq l, \\
&\sum_{\alpha=1}^{\lambda} i_\alpha + \min\{j_\beta | 1 \leqq \beta \leqq \mu\} > q,
\end{aligned}
$$

is zero if $2\left(\sum_{\alpha=1}^{\lambda} i_\alpha\right) > q$.

REMARK. In particular, one applies Corollary 3.2. to the case, $\mathscr{F} = \mathscr{F}'$, and sees that if the $E_T(M, p_T, GL_q)$ over $(M, \mathscr{F})$ admits a transverse pro-

1), 2), 3) These immersions are one to one.

jectable connection, then the Godbillon-Vey class $\lambda_{E_T}^{*}((c_1)^q \otimes h_1)$ of the $(M, \mathscr{F})$ is zero.

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