On projective H-separable extensions

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Introduction

All notations and terminologies in this paper are same as those in the author's previous papers [7], [8], [9], [10] and [11]. All rings shall have identities, and all subrings of them shall have the same identities as them. Whenever we denote a ring and its subring by Λ and Γ, respectively, we shall always denote the center of Λ by C and the centralizers of Γ in Λ, i. e., $V_\Lambda(\Gamma)$, by Δ. A ring Λ is an H-separable extension of a subring Γ if $\Lambda\otimes_\Gamma\Lambda$ is Λ-Λ-isomorphic to a Λ-Λ-direct summand of a finite direct sum of copies of Λ. Some equivalent conditions and fundamental properties have been researched in [3], [4] and [7]. In case Γ is the center of Λ, this definition is same as that of Azumaya algebra, and we have found in H-separable extension many similar properties to Azumaya algebra. In §1 we shall study in what case an H-sparable extension Λ of Γ become Γ-projective. If B is an intermediate subring of Λ and Γ such that ${}_BB_\Gamma<\oplus$ ${}_B\Lambda_\Gamma$ and B is left relatively separable over Γ in Λ, Λ is left B-projective. And if furthermore B is right relatively separable over Γ in Λ, Λ is a left QF-extension of B (Theorem 1.1). In §2 we shall study some relations between H-separable extensions of simple rings and classical fundamental theorem on simple rings. The latter states that if Λ is a simple ring with its center C, and if D is a simple C-algeba ($[D:C]<\infty$) contained in Λ, then $\Gamma=V_\Lambda(D)$ is simple, $D=V_\Lambda(\Gamma)$, and some interesting commutor theorems hold in this case (see [2]). Now we shall prove that Λ is an H-separable extension of Γ in this case (Theorem 2.1). We have already found that similar commutor theorems hold in general H-separable extensions (see Theorem 1 [6]). In §3 we shall study some properties of ideals in H-separable extensions. Especially, we will see in Theorem 3.2 that if Λ is an H-separable extension of Γ such that Λ is right Γ-projective and a right Γ-generator, there exists a 1–1 correspondence between the class of left ideals of Γ and the class of left ideals of Λ which ars also right Δ-submodules.

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1. On projectivity of *H*-separable extensions

For any ring Λ and a subring Γ of Λ, we have a well known canonical Λ-Λ-homomorphism θ

$$\theta:\ \Lambda\otimes_\Gamma\Lambda \longrightarrow \mathrm{Hom}\left(\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma, \Lambda_\Gamma\right)$$

such that $\theta(x\otimes y)(f)=xf(y)$ for $x, y\in\Lambda$ and $f\in\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)$. It is obvious that if $\sum x_i\otimes y_i\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda$, that is, if $\sum x_i\otimes y_i$ is a casimir element, $\theta(\sum x_i\otimes y_i)$ is a Λ-Γ-map. Also it is well known that if Λ is left Γ-*f.g.* projective, θ is an isomorphism. On the other hand, Λ is an H-separable extension of Γ if and only if $1\otimes 1\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda\Delta$ ($\Delta=V_\Lambda(\Gamma)$), that is, if and only if there exist $\sum x_{ij}\otimes y_{ij}\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ and $d_i\in\Delta(i=1, 2, \cdots, n)$ such that $1\otimes 1=\sum_i(\sum_j x_{ij}\otimes y_{ij})d_i$. By putting $\alpha_i=\theta(\sum x_{ij}\otimes y_{ij})$, we have

LEMMA 1.1 *Let Λ be a ring and Γ a subring of Λ. Then we have;*

(1) *If Λ is an H-separable extension of Γ, there exist Λ-Γ-maps α_i of $Hom({}_\Gamma\Lambda, {}_\Gamma\Gamma)$ to Λ and $d_i\in\Delta$ such that $\sum\alpha_i(d_i\cdot f)=f(1)$ for any $f\in Hom({}_\Gamma\Lambda, {}_\Gamma\Gamma)$.*

(2) *In case Λ is left Γ-f.g. projective, Λ is an H-separable extension of Γ if and only if there exist α_i and d_i which satisfy the condition of* (1).

PROPOSITION 1.1 *Let Λ be an H-separable extension of Γ such that ${}_\Gamma\Gamma_A<\oplus{}_\Gamma\Lambda_A$ for some subring A of Γ. Then,*

(1) *Λ is isomorphic to a direct summand of a finite direct sum of copies of $Hom({}_\Gamma\Lambda, {}_\Gamma\Gamma)$ as Λ-A-module, that is, ${}_\Lambda\Lambda_A<\oplus_\Lambda(\sum^n\oplus Hom({}_\Gamma\Lambda, {}_\Gamma\Gamma))_A$.*

(2) *If furthermore, Λ is left Γ-f.g. projective and the map $\Lambda\otimes_A\Gamma\to\Lambda$ defined by $x\otimes r\to xr$ for $x\in\Lambda$, $r\in\Gamma$, splits as Λ-Γ-map, Λ is a left QF-extension of Γ.*

PROOF. (1). Let α_i and d_i be as in Lemma 2.1, and let p be the Γ-A-projection of Λ to Γ. Then clearly $d_i\circ p$ are also Γ-A-maps, and $\sum\alpha_i(d_i\circ p)=p(1)=1$. Then we have Γ-A-maps

$$G:\ \Lambda\longrightarrow\sum^n\oplus\,\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)$$
$$F:\ \sum^n\oplus\,\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)\longrightarrow\Lambda$$

such that $G(x)=(xd_1\circ p, xd_2\circ p, \cdots, xd_n\circ p)$ and $F(f_1, f_2, \cdots, f_n)=\sum\alpha_i(f_i)$, for $x\in\Lambda$ and $f_i\in\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)$. Clearly $FG=1_\Lambda$, hence we have ${}_\Lambda\Lambda_A<\oplus_\Lambda(\sum^n\oplus\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma))_A$. (2). Let G and F be as above. Since the map $\Lambda\otimes_A\Gamma\to\Lambda$ splits, there exists $\sum x_i\otimes r_i$ in $(\Lambda\otimes_A\Gamma)^\Gamma$ with $\sum x_ir_i=1$. Then the map defined by $G'(x)=\sum G(xx_i)r_i$ is a Λ-Γ-map with $GF'=1$. Since F is also a Λ-Γ-map, we see that Λ is a left QF-extension of Γ.

Now we had better give a new definition concerning as rings $A\subset\Gamma\subset\Lambda$ which satisfy the condition of Proposition 1.1 (2).

DEFINITION *Let Λ be a ring and A and Γ subrings of Λ with $A\subset\Gamma$. Then we shall call that Γ is a left relatively separable subextension of A in Λ, if map π of $\Gamma\otimes_A\Lambda$ to Λ such that $\pi(x\otimes y)=xy$ for $x\in\Gamma$, $y\in\Lambda$ splits as Γ-Λ-map. A right relatively separable subextension can be defined similarly.*

Now assume again that a ring Λ is an H-separable extension of a subring Γ. In [11], H. Tominaga proved that if Λ is left (resp. right) Γ-projective, Λ is left (resp. right) Γ-*f. g.* projective. Now we shall investigate in what case Λ is Γ-projective. First we shall note that the following isomorphisms exist for every left Λ-module M.

$$\begin{aligned}\Delta\otimes_C M &\cong \Delta\otimes_C\mathrm{Hom}({}_\Lambda\Lambda, {}_\Lambda M)\cong \mathrm{Hom}\left({}_\Lambda\mathrm{Hom}({}_C\Delta, {}_C\Lambda), {}_\Lambda M\right)\\ &\cong \mathrm{Hom}({}_\Lambda\Lambda\otimes_\Gamma\Lambda, {}_\Lambda M)\cong \mathrm{Hom}\left({}_\Gamma\Lambda, {}_\Gamma\mathrm{Hom}({}_\Lambda\Lambda, {}_\Lambda M)\right)\\ &\cong \mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma M)\end{aligned}$$

the composition η_M of the above isomorphisms is such that $\eta_M(d\otimes m)(x)=dxm$, for $d\in\Delta$, $x\in\Lambda$ and $m\in M$. Therefore, for any left Λ-modules M, N and for any left Λ-map f of N to M, we have a commutative diagram

$$\begin{array}{ccc}\Delta\otimes_C N & \xrightarrow{\;1_\Delta\otimes f\;} & \Delta\otimes_C M\\ \downarrow{\scriptstyle\eta_N} & & \downarrow{\scriptstyle\eta_M}\\ \mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma N) & \xrightarrow[\mathrm{Hom}(\Lambda, f)]{} & \mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma M)\end{array}$$

By this fact we have,

PROPOSITION 1.2 *If Λ is an H-separable extension of Γ, then for any Λ-epimorphism f of N to M and for any Γ-homomorphism g of Λ to M, there exists a Γ-homomorphism h of Λ to N such that $f\circ h=g$.*

PROPOSITION 1.3 *Let Λ be an H-separable extension of Γ. Then if there exists a subring A of Γ such that Γ is left relatively separable over A in Λ and ${}_\Gamma\Gamma_A<\oplus{}_\Gamma\Lambda_A$, we have*

(1) *Λ is left Γ-f. g. projective*

(2) *Λ is left (resp. right) A-projective if and only if Γ is left (resp. right) A-projective.*

PROOF. (1). Since $\Gamma\otimes_A\Lambda\to\Lambda$ splits, there exists $\sum r_i\otimes x_i\in(\Gamma\otimes_A\Lambda)^\Gamma$ with $\sum r_ix_i=1$. Now let f be any left Γ-epimorphism of N to M and g any left Γ-homomorphism of Λ to M, where M and N are arbitrary left Γ-modules. We can define a new Γ-map of Λ to $\Lambda\otimes_A M$ by $G(x)=\sum r_i\otimes g(x_ix)$. Since $\sum rr_i\otimes x_ix=\sum r_i\otimes x_irx$ for any $r\in\Gamma$ and $x\in\Lambda$, we see that

G is a Γ-map. Then by Proposition 1.2, there exists a left Γ-map H of Λ to $\Lambda\otimes_A N$ such that $(1_\Lambda\otimes f)\circ H=G$. Let p be the Γ-A-projection of Λ to Γ. Then we have a commutative diagram of Γ-maps

$$\begin{array}{ccccccc}
\Lambda & \xrightarrow{\ H\ } & \Lambda\otimes_A N & \xrightarrow{p\otimes 1_N} & \Gamma\otimes_A N & \xrightarrow{\ \pi_N\ } & N \\
 & \searrow^{G} & \downarrow{\scriptstyle 1_\Lambda\otimes f} & & \downarrow{\scriptstyle 1_\Gamma\otimes f} & & \downarrow{\scriptstyle f} \\
 & & \Lambda\otimes_A M & \xrightarrow{p\otimes 1_M} & \Gamma\otimes_A M & \xrightarrow{\ \pi_M\ } & M
\end{array}$$

where π_M and π_N are the contraction maps. Then $\pi_M\circ(p\otimes 1_M)\circ G=g$, since $\pi_M\circ(p\otimes 1_M)\circ G(x)=\pi_M(p\otimes 1_M)(\sum r_i\otimes g_i(x_i x))=\pi_M(\sum r_i\otimes g(x_i x))=\sum r_i g(x_i x)=g(\sum r_i x_i x)=g(x)$. Thus there exists a left Γ-homomorphism h $(=\pi_M\circ(p\otimes 1_N)\circ H)$ of Λ to N such that $f\circ h=g$. Therefore, Λ is left Γ-pojective.
(2). Suppose that Γ is right A-projective. Since ${}_\Gamma\Lambda$ is projective by (1), $\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)$ is right Γ-projective. Hence $\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)$ is right A-projective. Then, since ${}_\Lambda\Lambda_A<\oplus_\Lambda(\sum^n\oplus\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma))_A$ by Proposition 1.1 and the assumption that ${}_\Gamma\Gamma_A<\oplus_\Gamma\Lambda_A$, Λ is right A-projective. Next suppose that Γ is left A-projective. Then, since Λ is left Γ-projective, Λ is left A-projective. The converse is clear, since ${}_\Gamma\Gamma_A<\oplus_\Gamma\Lambda_A$.

In [6] and [11], we considered the class $\mathfrak{B}_l$ of subrings B of Λ such that $B\supset\Gamma$, ${}_B B_\Gamma<\oplus_B\Lambda_\Gamma$ and B is left relatively separable over Γ in Λ. Class $\mathfrak{B}_r$ is defined similarly. In case Λ is H-separable over Γ, these classes have interesting properties, because there exists a 1–1 correspondence of $\mathfrak{B}_l$ to the class of C-subalgebras D of Δ such that ${}_D D<\oplus_D\Delta$ and D is left relatively C-separable in Δ. It is easy to prove that if $B\in\mathfrak{B}_l$ (or $\mathfrak{B}_r$), Λ is H-separable over $\mathfrak{B}$. (see (0.8) [11]). Therefore by Proposition 1.3, we have;

THEOREM 1.1 *Let Λ be an H-separable extension of Γ, and let $\mathfrak{B}_l$ and $\mathfrak{B}_r$ be as above, Then, we have*

(1) *Λ is left (resp. right) B-f.g. projective for every $B\in\mathfrak{B}_l$ (resp. $\mathfrak{B}_r$).*

(2) *Λ is a QF-extension of B for every $B\in\mathfrak{B}_l\cap\mathfrak{B}_r$.*

(3) *For any B in $\mathfrak{B}_l$, B is left (resp. right) Γ-f.g. projective if and only if Λ is left (resp. right) Γ-f.g. projective.*

THEOREM 1.2 *If Λ is an H-separable extension of Γ such that ${}_\Gamma\Gamma_\Gamma<\oplus_\Gamma\Lambda_\Gamma$, Λ is left and right Γ-f.g. projective. In this case Λ is a Frobenius extension of Γ.*

PROOF. The first part is clear by Proposition 1.3. Then, since Δ is C-*f.g.* projective and separable (see Proposition 4.7 [4]), Δ is a Frobenius C-algebra by Endo-Watanabe's Theorem. Then Λ is a Frobenius extension of Γ (see Corollary 2 [8]).

2. Some remarke on separable extensions over simple rings

First we shall give an example of H-separable extension over a simple ring, which has a closed relation to the well known classical "fundamental theorem on simple rings".

THEOREM 2.1 *Let Λ be a simple ring with the center C and Δ a simple C-algebra contained in Λ, and denote $\Gamma = V_\Lambda(\Delta)$. Then Γ is a simple ring and Λ is an H-separable extension of Γ.*

PROOF. $\Lambda\otimes_C\Delta^\circ$ is a simple ring, and Λ is a left $\Lambda\otimes_C\Delta^\circ$- and right Γ-bimodule. Furthermore, we have an isomorphism $\mathrm{Hom}({}_{\Lambda\otimes_C\Delta^\circ}\Lambda, {}_{\Lambda\otimes_C\Delta^\circ}\Lambda)\cong V_\Lambda(\Delta)$, by corresponding f in $\mathrm{Hom}({}_{\Lambda\otimes_C\Delta^\circ}\Lambda, {}_{\Lambda\otimes_C\Delta^\circ}\Lambda)$ to $f(1)$. Now consider the map

$$\eta:\ \Lambda\otimes_C\Delta^\circ \longrightarrow \mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)\left(\eta(x\otimes d^\circ)(y)=xyd, \text{ for } x, y\in\Lambda,\ d\in\Delta\right)$$

Since $\Lambda\otimes_C\Delta^\circ$ is a simple ring, Λ is a $\Lambda\otimes_C\Delta^\circ$-generator. Hence Λ is right finitely generated projective over $\Gamma\cong\mathrm{End}({}_{\Lambda\otimes_C\Delta^\circ}\Lambda)$, and we have also

$$\Lambda\otimes_C\Delta^\circ\cong\mathrm{Bicom}({}_{\Lambda\otimes_C\Delta^\circ}\Lambda)\cong\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)$$

The composition of the above isomorphisms is exactly η, which is a Λ-Λ-map. Hence $\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)$ is Λ-centrally projective, i.e., $\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)$ is isomorphic to a direct summand of a finite direct sum of copies of Λ as Λ-Λ-module. Therefore Λ is an H-separable extension of Γ by Corollary 3 [10]. That Γ is a simple ring is well known. But this is clear by the fact that $\Gamma\cong\mathrm{End}({}_{\Lambda\otimes_C\Delta^\circ}\Lambda)$ and Λ is finitely generated by a simple ring $\Lambda\otimes_C\Delta^\circ$.

In case Γ is a simple ring and Λ is an H-separable extension of Γ, Λ is a simple ring, Δ is a simple C-algebra with $[\Lambda:\Gamma]_l=[\Lambda:\Gamma]_r=[\Delta:C]$, and $\Gamma=V_\Lambda(V_\Lambda(\Gamma))$ (see Theorem 1.5 [11]). Now we shall study some properties of intermediate simple ring between Λ and Γ. Before then, we shall consider a general case. Let Λ be an arbitrary ring and Γ a subring of Λ, and let M be a left Γ-module. Then $\Lambda\otimes_\Gamma M$ is a left Λ- and right Δ-bimodule by $x(y\otimes m)d=xyd\otimes m$ for $x,\ y\in\Lambda,\ d\in\Delta$ and $m\in M$. By this module structure we have,

LEMMA 2.1 *Let Γ and Λ be arbitrary rings with Γ a subring of Λ, and let M be an arbitrary porjective left Γ-module. Then for any subring D of Δ, we have $(\Lambda\otimes_\Gamma M)^D=V_\Lambda(D)\otimes_\Gamma M$.*

PROOF. Denote $B=V_\Lambda(D)$. Clearly $B\supset\Gamma$, and $B\otimes_\Gamma M$ is a submodule of $\Lambda\otimes_\Gamma M$, since M is Γ-projective. Let $\{f_i, m_i\}$ be a dual basis of ${}_\Gamma M$. i.e., $f_i\in\mathrm{Hom}({}_\Gamma M, {}_\Gamma\Gamma)$ and $m_i\in M$ such that for every $m\in M$, $f_i(m)=0$ for all but a finite number of i, and $m=\sum f_i(m)m_i$. Now for each i, define a Λ-map F_i of $\Lambda\otimes_\Gamma M$ to Λ by $F_i(x\otimes m)=xf_i(m)$ for $x\in\Lambda$ and $m\in M$.

Then clearly $\{F_i, 1\otimes m_i\}$ is a dual basis of ${}_\Lambda\Lambda\otimes_\Gamma M$. Hence we have $\alpha=\sum F_i(\alpha)\otimes m_i$ for any $\alpha\in\Lambda\otimes_\Gamma M$. Let $\beta=\sum x_j\otimes n_j$ be an arbitrary element of $(\Lambda\otimes_\Gamma M)^D$. Then $d\beta=\beta d=\sum x_j d\otimes n_j$ for every $d\in D$, and we have $dF_i(\beta)=F_i(d\beta)=F_i(\beta d)=F_i(\sum x_j d\otimes n_j)=\sum x_j df_i(n_j)=\sum x_j f_i(n_j)d=\sum F_i(\beta)d$, since $d\in D\subset\Delta$ and $f_i(n_j)\in\Gamma$. Hence $F_i(\beta)\in V_\Lambda(D)=B$ for each i. Since $\beta=\sum F_i(\beta)\otimes m_i$, we have $\beta\in B\otimes_\Gamma M$. $B\otimes_\Gamma M\subset(\Lambda\otimes_\Gamma M)^D$ is clear. Therefore we have $(\Lambda\otimes_\Gamma M)^D=B\otimes_\Gamma M$.

COROLLARY 2.1 *Let Λ be an arbitrary algebra over a commutative ring R and M a projective R-module. Then we have $(\Lambda\otimes_R M)^\Lambda=C\otimes_R M$, where C is the center of Λ.*

PROPOSITION 2.1 *Let Λ be an H-separable extension of Γ such that Λ is left Γ-projective. Then for any C-subalgebra D of Δ and for $B=V_\Lambda(D)$, we have;*

(1) *The map η_B of $B\otimes_\Gamma\Lambda$ to $Hom({}_D\Delta, {}_D\Lambda)$ defined by $\eta_B(b\otimes x)(d)=bdx$ is a B-Λ-isomorphiem.*

(2) *If Δ is a left D-generator, then B is left relatively separable over Γ in Λ.*

(3) *If Δ is left D-f. g. projective, then ${}_B B\otimes_\Gamma\Lambda_\Lambda<\oplus_B(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$.*

(4) *If furthermore B is right Γ-projective, then the map ρ_B of $B\otimes_\Gamma B$ to $Hom({}_D\Delta_D, {}_D\Lambda_D)$ defined in the same way as (1) is a B-B-isomorphism.*

PROOF. (1). Since Λ is H-separable over Γ, we have a $(\Lambda-\Delta)-(\Delta-\Lambda)$-isomorphism η of $\Lambda\otimes_\Gamma\Lambda$ to $\mathrm{Hom}({}_C\Delta, {}_C\Lambda)$ defined in the same way as η_B. Hence we have the following commutative diagram;

$$\begin{array}{ccc} B\otimes_\Gamma\Lambda & \xrightarrow{\eta_B} & \mathrm{Hom}({}_D\Delta, {}_D\Lambda) \\ \downarrow & & \downarrow \\ \Lambda\otimes_\Gamma\Lambda & \xrightarrow{\eta} & \mathrm{Hom}({}_C\Delta, {}_C\Lambda) \end{array}$$

where all vertical maps are inclusion maps, since $\otimes_\Gamma\Lambda$ is exact. Hence η_B is a monomorphism. Then since $\mathrm{Hom}({}_D\Delta, {}_D\Lambda)=[\mathrm{Hom}({}_C\Delta, {}_C\Lambda)]^D$, and $B\otimes_\Gamma\Lambda=V_\Lambda(D)\otimes_\Gamma\Lambda=(\Lambda\otimes_\Gamma\Lambda)^D$ by Lemma 2.1, we see that η_B is an epimorphism. Tnus η_B is an isomorphism. (2). Now consider the following commutative diagram of B-Λ-maps

$$\begin{array}{ccc} B\otimes_\Gamma\Lambda & \xrightarrow{\eta_B} & \mathrm{Hom}({}_D\Delta, {}_D\Lambda) \\ & {}_{\pi_B}\searrow \quad \swarrow_{\varphi} & \\ & \Lambda & \end{array}$$

where $\varphi(f)=f(1)$ for $f\in\mathrm{Hom}({}_D\Delta, {}_D\Lambda)$, and π_B is the contraction map.

Denote the left D-projection of Δ to D by p and the canonical B-Λ-isomorphism of Λ to $\mathrm{Hom}({}_D D, {}_D\Lambda)$ by ν. Then, we see $\varphi\circ\mathrm{Hom}(p, \Lambda)\circ\nu = 1_\Lambda$. Thus π_B splits as B-Λ-map. (3). Since Δ is left D-$f.g.$ projective, we have ${}_B B\otimes_\Gamma \Lambda_\Lambda \cong {}_B\mathrm{Hom}({}_D\Delta, {}_D\Lambda)_\Lambda < \oplus_B[\sum^n \oplus\mathrm{Hom}({}_D D, {}_D\Lambda)]_\Lambda \cong {}_B(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$. (4). Since B is right Γ-$f.g.$ projective and η_B in (1) is an isomorphism, we can prove (4) in the same way as (1).

Applying this to H-separable extensions over simple rings, we obtain;

PROPOSITION 2.2 *Let Γ be a simple ring and Λ an H-separable extension of Γ. Then for any simple subring B of Λ which contains Γ and for $D = V_\Lambda(B)$, we have;*

(1) *The following two maps are isomorphisms*

$$\eta_B:\ B\otimes_\Gamma\Lambda \longrightarrow Hom({}_D\Delta, {}_D\Lambda)$$
$$\left(\eta_B(b\otimes x)(d) = bdx \text{ for } x\in\Lambda,\ b\in B,\ d\in\Delta\right)$$
$$\rho_B:\ B\otimes_\Gamma B \longrightarrow Hom({}_D\Delta_D, {}_D\Lambda_D)$$
$$\left(\rho_B(a\otimes b)(d) = adb, \text{ for } a, b\in B,\ d\in\Delta\right)$$

(2) *B is left as well as right relatively separable over Γ in Λ.*

PROOF. It is well known that D is a simple C-subalgebra of Δ and $B = V_\Lambda(D)$, by classical fundamental theorem on simple ring. Therefore, the proof is immediate by Proposition 2.2.

COROLLARY 2.2 *Let Λ, Γ, B and D be as in Prop. 2.2. Then, we have;*

(1) *B is a separable (resp. an H-separable) extension of Γ, if and only if ${}_D D_D < \oplus {}_D\Delta_D$ (resp. ${}_D\Delta_D < \oplus_D(D\oplus\cdots\oplus D)_D$).*

(2) *If ${}_B B_B < \oplus_B\Lambda_B$, B is a separable extension of Γ, and D is a separable C-algebra.*

PROOF. (1). Since ρ_B defined in Prop. 2.2 is a B-B-isomorphism, the 'if' part is clear. The 'only if' part is due to (0.7) [11]. (2.) Since $B\otimes_\Gamma\Lambda \to \Lambda$ splits and ${}_B B_B < \oplus_B\Lambda_B$, B is separable over Γ by (1.4) [11]. Then D is C-separable by Theorem (1.3) [11].

3. On ideals in *H*-separable extension

It is well known that in Azumaya algebra there exists a 1–1 correspondence between the class of two sided ideals and that of ideals of its center. Therefore, it may be natural to consider this probrem for H-separable extension. The following theorems are easy to prove but are interesting. Before proving them, we need some remarks. In case Λ is an H-separable extension of Γ, we have the following three ring isomorphisms

$$\eta_l:\ \Delta\otimes_C\Lambda^\circ \longrightarrow \mathrm{Hom}({}_\Gamma\Lambda,{}_\Gamma\Lambda)$$
$$\eta_r:\ \Lambda\otimes_C\Delta^\circ \longrightarrow \mathrm{Hom}(\Lambda_\Gamma,\Lambda_\Gamma)$$
$$\eta_t:\ \Delta\otimes_C\Delta^\circ \longrightarrow \mathrm{Hom}({}_\Gamma\Lambda_\Gamma,{}_\Gamma\Lambda_\Gamma)$$

defined by $\eta_l\ (d\otimes x^\circ)(y)=dxy$, for $x, y\in\Lambda$ and $d\in\Delta$, etc., (Prop. 3.1 & 4.7 [4]).

LEMMA 3.1 *Let Λ be an H-separable extension of Γ. Then for left Δ- and right Λ-bisubmodule $\mathfrak{A}$ of Λ, and for any $f\in Hom({}_\Gamma\Lambda,{}_\Gamma\Lambda)$, we have $f(\mathfrak{A})\subset\mathfrak{A}$.*

PROOF. By the isomorphism η_l, we see $f(\mathfrak{A})=\sum d_i\mathfrak{A}x_i\subset\mathfrak{A}$ for some $d_i\in\Delta$ and $x_i\in\Lambda$.

THEOREM 3.1 *Let Λ be an H-separable extension of Γ such that Λ is right Γ-projective. Then for any left ideal $\mathfrak{A}$ of Λ which is also a rignt Δ-submodule, we have $\mathfrak{A}=\Lambda(\mathfrak{A}\cap\Gamma)$. In particular, for any two sided ideal $\mathfrak{A}$ of Λ, we have $\mathfrak{A}=\Lambda(\mathfrak{A}\cap\Gamma)\Lambda$.*

PROOF. Let $\{f_j, x_j\}$ be a dual basis of Λ_Γ. Then, since $f_j\in\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)\subset\mathrm{Hom}(\Lambda_\Gamma,\Lambda_\Gamma)$, $f_j(\mathfrak{A})\subset\mathfrak{A}\cap\Gamma$ for each j. Then for each $a\in\mathfrak{A}$, $a=\sum x_j f_j(a)\in\Lambda(\mathfrak{A}\cap\Gamma)$. Thus $\mathfrak{A}=\Lambda(\mathfrak{A}\cap\Gamma)$.

COROLLARY 3.1 *If Γ is a two sided simple ring, and if Λ is an H-separable extension of Γ such that Λ is left or right Γ-projective, then Λ is also a two sided simple ring.*

Now consider the following correspondences of ideals;

$$I:\ \mathfrak{A}\longrightarrow\mathfrak{A}\cap\Gamma \qquad M:\ \mathfrak{a}\longrightarrow\Lambda\mathfrak{a}$$

where $\mathfrak{A}$ is a left ideal of Λ which is also a right Δ-module, and $\mathfrak{a}$ is a left ideal of Γ. Then we have;

THEOREM 3.2 *Let Λ be an H-separable extension of Γ such that $\Gamma_\Gamma<\oplus\Lambda_\Gamma$ and Λ is right Γ-projective. Then we have;*

(1) *I and M are mutually converse* 1–1 *correspondences between the class of left ideals of Λ which are also right Δ-submodules and the class of left ideals of Γ.*

(2) *I and M induce* 1–1 *crrespondences between the class of left Λ- and right $\Gamma\Delta$-bisubmodules of Λ and the class of two sided ideals of Γ.*

(3) *If furthermore, $\Lambda=\Gamma\Delta$ (e.g., Λ is Γ-centrally projective), then $M(\mathfrak{a})=\Delta\mathfrak{a}$, and I and M induce* 1–1 *correspondences between the class of two sided ideals of Λ and that of Γ.*

PROOF. For any left ideal $\mathfrak{a}$ of Γ, $\mathfrak{a}\Delta=\Delta\mathfrak{a}$ and $\Lambda\mathfrak{a}$ is a Λ-Δ-submodule. Also it is obvious that $\Lambda\mathfrak{a}\cap\Gamma=\mathfrak{a}$, since $\Gamma_\Gamma<\oplus\Lambda_\Gamma$. $MI=$identity is due to

Theorem 3.1. Thus we have proved (1). (2) and (3) are easy consequences of (1).

As for two sided ideal in general case, we see

PROPOSITION 3.1 *Let Λ be an H-separable extension of Γ. Then for any two sided ideal $\mathfrak{A}$ of Λ, we have $(\mathfrak{A}\cap C)\Delta=\mathfrak{A}\cap\Delta$.*

PROOF. By (0.1) [11], Λ is H-separable over Γ if and only if $M^{\Lambda}\otimes_C\Delta \cong M^{\Gamma}$ by $m\otimes d\to md (m\in M, d\in\Delta)$, for every two sided Λ-module M. Hence $\mathfrak{A}\cap\Delta=\mathfrak{A}^{\Gamma}\cong\mathfrak{A}^{\Lambda}\otimes_C\Delta=(\mathfrak{A}\cap C)\otimes_C\Delta\cong(\mathfrak{A}\cap C)\Delta$. Thus we have $(\mathfrak{A}\cap C)\Delta=\mathfrak{A}\cap\Delta$.

Next we shall study some properties of ring homomorphisms of H-separable extensions. The author has proved the next proposition in [7].

PROPOSITION 3.2 *Let Λ be an H-separable extension of Γ, φ a ring homomorphism of Λ onto another ring $\bar\Lambda$, and denote $\bar\Gamma=\varphi(\Gamma)$, $\bar\Delta=V_{\bar\Lambda}(\bar\Gamma)$ and $\bar C=$ the center of $\bar\Lambda$. Then $\bar\Lambda$ is an H-separable extension of $\bar\Gamma$, and the map g of $\bar C\otimes_C\Delta$ to $\bar\Delta$ defined by $g(\bar c\otimes d)=\bar c\varphi(d)(\bar c\in\bar C, d\in\Delta)$ is an isomorphism. Consequently, $\bar\Delta=\bar C\varphi(\Delta)$. (Prop. 1.5 [7]).*

PROPOSITION 3.3 *Let Λ, Γ, φ, $\bar\Lambda$ and $\bar\Gamma$ be as above. Then φ induces ring homomorphism $\bar\varphi_l$ and $\bar\varphi_r$, as follows;*

$$\bar\varphi_l:\ Hom({}_\Gamma\Lambda, {}_\Gamma\Lambda)\longrightarrow Hom({}_{\bar\Gamma}\bar\Lambda, {}_{\bar\Gamma}\bar\Lambda)\quad \bar\varphi_l(f)\big(\varphi(x)\big)=\varphi\big(f(x)\big)$$
$$\bar\varphi_r:\ Hom(\Lambda_\Gamma, \Lambda_\Gamma)\longrightarrow Hom(\bar\Lambda_{\bar\Gamma}, \bar\Lambda_{\bar\Gamma})\quad \bar\varphi_r(g)\big(\varphi(x)\big)=\varphi\big(g(x)\big)$$

where $f\in Hom({}_\Gamma\Lambda, {}_\Gamma\Lambda)$, and $x\in\Lambda$. Both $\bar\varphi_l$ and $\bar\varphi_r$ are surjections.

PROOF. We need only to prove on $\bar\varphi_l$. Since $f(\ker\varphi)\subset\ker\varphi$ for every $f\in \mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Lambda)$ by Lemma 3.1, $\bar\varphi_l$ is a well defined ring homomorphism. By Prop. 3.2, $\bar\Lambda$ is H-separable over $\bar\Gamma$ and $\Delta\otimes_C\bar C\cong\bar\Delta(d\otimes\bar c\to\varphi(d)c$, for $d\in\Delta$, $c\in\bar C$). Hence $\bar\Delta\otimes_{\bar C}\bar\Lambda^\circ\cong\Delta\otimes_C\bar C\otimes_{\bar C}\bar\Lambda^\circ\cong\Delta\otimes_C\bar\Lambda^\circ$. This isomorphism induces a commutative diagram of ring homomorphisms;

$$\begin{array}{ccc}
\Delta\otimes_C\Lambda^\circ & \xrightarrow{\ \eta_l\ } & \mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Lambda)\\
\Big\downarrow 1_\Delta\otimes\varphi & & \Big\downarrow \bar\varphi_l\\
\Delta\otimes_C\bar\Lambda^\circ & \xrightarrow[\ \xi_l\ ]{} & \mathrm{Hom}({}_{\bar\Gamma}\bar\Lambda, {}_{\bar\Gamma}\bar\Lambda)
\end{array}$$

where $\xi_l(d\otimes\bar x^\circ)(\bar y)=\varphi(d)\bar y\bar x$, for $\bar x, \bar y\in\bar\Lambda$ and $d\in\Delta$. Clearly ξ_l is an isomorphism. Then since η_l and ξ_l are isomorphisms and $1_\Delta\otimes\varphi$ is a surjection, $\bar\varphi_l$ is a surjection.

PROPOSITION 3.4 *Let Λ, Γ, $\bar\Lambda$, $\bar\Gamma$ and φ be as in Prop. 3.2. If $\Lambda=\Gamma\oplus A$ as left (resp. right or two sided) Γ-module, then we have;*

(1) *$\bar\Lambda=\bar\Gamma\oplus\varphi(A)$ as left (resp. right or two sided) $\bar\Gamma$-module.*

(2) *For any two sided ideal $\mathfrak{A}$ of Λ, we have $\mathfrak{A}=(\mathfrak{A}\cap\Gamma)\oplus(\mathfrak{A}\cap A)$.*

PROOF. Suppose ${}_\Gamma\Lambda = {}_\Gamma(\Gamma \oplus A)$, and let π be the left Γ-projection of Λ onto Γ. Then since $\pi \in \mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Lambda) \cong \Delta \otimes_C \Lambda^\circ$, there exists $\sum d_i \otimes x_i^\circ \in \Delta \otimes_C \Lambda^\circ$ such that $\sum d_i x_i = 1$ and $\sum d_i A x_i = 0$. Then clearly $\sum \varphi(d_i)\varphi(x_i) = \varphi(\sum d_i x_i) = \bar{1}$ in $\bar{\Lambda}$, and $\varphi(d_i) \in V_{\bar{\Lambda}}(\bar{\Gamma})$ for each i. We also have $\sum \varphi(d_i)\varphi(A)\varphi(x_i) = \varphi(\sum d_i A x_i) = 0$. Therefore, the map $\bar{\pi}$ of $\bar{\Lambda}$ to $\bar{\Gamma}$ such that $\bar{\pi}(\bar{x}) = \sum \varphi(d_i)\bar{x}\varphi(x_i)$ for $\bar{x} \in \bar{\Lambda}$, is the left $\bar{\Gamma}$-projection of $\bar{\Lambda}$ to $\bar{\Gamma}$. Thus we have ${}_{\bar{\Gamma}}\bar{\Lambda} = {}_{\bar{\Gamma}}(\bar{\Gamma} \oplus \varphi(A))$. Similarly we can prove in case $\Lambda_\Gamma = (\Gamma \oplus A)_\Gamma$. Furthermore, since $\Delta \otimes_C \Delta^\circ = \mathrm{Hom}({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Lambda_\Gamma)$ by η_t, we can prove in case ${}_\Gamma\Lambda_\Gamma = {}_\Gamma(\Gamma \oplus A)_\Gamma$, in the same way. (2). Let $\mathfrak{A}$ be an arbtrary two sided ideal of Λ, and suppose $\Lambda = {}_\Gamma(\Gamma \oplus A)$. Let φ be the canonical map of Λ to Λ/A, and put $\bar{\Lambda} = \Lambda/A$ and $\bar{\Gamma} = \varphi(\Gamma)$. Then by (1), we have ${}_{\bar{\Gamma}}\bar{\Gamma} = {}_{\bar{\Gamma}}(\bar{\Gamma} \oplus \varphi(A))$. For any $x \in \mathfrak{A}$, we have $x = r + a$ with $r \in \Gamma$ and $a \in A$. Then $0 = \varphi(x) = \varphi(r) + \varphi(a)$, and $\varphi(r) = \varphi(a) = 0$. Therefore, $r \in \Gamma \cap \mathfrak{A}$ and $a \in A \cap \mathfrak{A}$. Thus we have $\mathfrak{A} = (\Gamma \cap \mathfrak{A}) \oplus (A \cap \mathfrak{A})$. We can prove in other cases in the same way.

4. On *H*-separable extensions over self injective rings

To begin with, we note the following interesting properties of general H-separable extensions. Let $\sum_j x_{ij} \otimes y_{ij}$, $d_i (i = 1, \cdots, n)$ be an H-system of an H-separable extension Λ of Γ, i.e., $1 \otimes 1 = \sum x_{ij} \otimes y_{ij} d_i$, $\sum x_{ij} \otimes y_{ij} \in (\Lambda \otimes_\Gamma \Lambda)^\Lambda$ and $d_i \in \Delta$. Now suppose that Γ is a left Γ-direct summand of Λ, and let p be the Γ-projection of Λ to Γ. Then for any z in Λ, we have; $z \otimes 1 = \sum z x_{ij} \otimes y_{ij} d_i = \sum x_{ij} \otimes y_{ij} z d_i$, and $z \otimes 1 = z \otimes p(1) = \sum x_{ij} \otimes p(y_{ij} z d_i)$. Thus we have an equation $z = \sum x_{ij} p(y_{ij} z d_i)$, for any z in Λ. By this formula, we have;

THEOREM 4.1 *Let Λ be an H-separable extension of Γ such that Γ is a left Γ-direct summand of Λ. Then we have*

(1) *Λ is right Γ-finitely generated.*

(2) *For any two sided ideal $\mathfrak{A}$ of Λ, we have $\mathfrak{A} = \Lambda(\Gamma \cap \mathfrak{A}) = \Lambda(\Gamma \cap \mathfrak{A})\Lambda$.*

PROOF. (1). Let $\sum_j x_{ij} \otimes y_{ij}$, $d_i (i = 1, 2, \cdots, n)$ and p be as above. Then, since $p(y_{ij} z d_i) \in \Gamma$, we see $\Lambda = \sum x_{ij}\Gamma$. (2). For any a in $\mathfrak{A}$, we have $a = \sum x_{ij} p(y_{ij} a d_i)$. But since $y_{ij} a d_i \in \mathfrak{A}$, $p(y_{ij} a d_i) \in \Gamma \cap \mathfrak{A}$ by Lemma 3.1. Hence $\mathfrak{A} \subset \Lambda(\Gamma \cap \mathfrak{A})$.

By Prop. 1.1, we see that if Λ is an H-separable extension of Γ with Γ a left direct summand of Λ, then ${}_\Lambda\Lambda < \oplus_\Lambda[\sum^n \oplus \mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)]$, i.e., $\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)$ is a left Λ-generator. On the other hand, for ${}_\Lambda M_\Omega$ and ${}_\Lambda N$, if N is Λ-injective and M is Ω-flat, $\mathrm{Hom}({}_\Lambda M, {}_\Lambda N)$ is Ω-injective. Therefore, we have

PROPOSITION 4.1 *Let Λ be an H-separable extension of Γ. Then we have*

(1) *If Γ is left self injective, then Λ is also left self injective.*

(2) *If Γ is left self injective and Λ is right Γ-flat, Λ is left Γ-injective and* $[\mathrm{Hom}(_\Gamma\Lambda, {}_\Gamma\Lambda)]^\circ$ *is a left self injective ring.*

PROOF. (1). Since Γ is left Γ-injective, $\mathrm{Hom}(_\Gamma\Lambda, {}_\Gamma\Gamma)$ is left Λ-injective, and also we have $_\Gamma\Gamma < \oplus {}_\Gamma\Lambda$. Then by Prop. 1.1, we see $_\Lambda\Lambda < \oplus_\Lambda[\sum^n \oplus \mathrm{Hom}(_\Gamma\Lambda, {}_\Gamma\Gamma)]$. Hence Λ is left Λ-injective. (2). Since Λ is left Λ-injective and right Γ-flat, $\Lambda(\cong_\Lambda\mathrm{Hom}(_\Lambda\Lambda, {}_\Lambda\Lambda))$ is left Γ-injective. Next, put $\Omega = [\mathrm{Hom}(_\Gamma\Lambda, {}_\Gamma\Lambda)]^\circ$. Since $_\Gamma\Gamma < \oplus {}_\Gamma\Lambda$, Λ is right Ω-*f.g.* projective. Then $\mathrm{Hom}(_\Gamma\Lambda, {}_\Gamma\Lambda)$ is left Ω-injective, as Λ is left Γ-injective.

By Theorem 4.1 and Proposition 4.1, we obtain

THEOREM 4.2 *If Γ is a QF-ring and if Λ is an H-separable extension of Γ, then Λ is also a QF-ring.*

PROOF. Since Γ is left as well as right self injective, Λ is left as well as right self injective. Moreover, Λ is right Γ-finitely generated, since $_\Gamma\Gamma < \oplus {}_\Gamma\Lambda$. Then Λ is right artinean, since Γ is so. Hence Λ is a *QF*-ring.

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