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On a theorem of S. Chowla

By Tadashige OKADA

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Let p be an odd prime. Then S. Chowla [3] proved the following theorem.

THEOREM. *The* $\frac{p-1}{2}$ *real numbers* $\cot(2\pi a/p)$, $a=1,2,\cdots,\frac{p-1}{2}$ *are linearly independent over the field* $\boldsymbol{Q}$ *of rational numbers.*

Other proofs were given by Hasse [4], Iwasawa [5] and by Ayoub [1], [2].

In this note, we shall show the following theorem, which is a generalization of the above theorem, by means of improving the method of Chowla's proof.

THEOREM. *Let n be an integer with $n>2$ and let T be a set of representatives* mod n *such that the union* $\{T, -T\}$ *is a complete set of residues prime to n. Then the* $\phi(n)/2$ *real numbers* $\cot(\pi a/n)$, $a\in T$ *are linearly independent over* $\boldsymbol{Q}$, *where* $\phi(n)$ *is the Euler totient function.*

Proof. Let D be the set of all Dirichlet characters to the modulus n. For a map

$$F:\ (\boldsymbol{Z}/n\boldsymbol{Z})^{\times}\longrightarrow\boldsymbol{C}$$

from the multiplicative group $(\boldsymbol{Z}/n\boldsymbol{Z})^{\times}$ of the residue class ring $\boldsymbol{Z}/n\boldsymbol{Z}$ to the complex field $\boldsymbol{C}$, we define the Fourier transform by

$$\hat{F}(\chi)=\frac{1}{\phi(n)}\sum_{\substack{a(\mathrm{mod}\ n)\\(a,n)=1}}F(a)\bar{\chi}(a)\qquad(\chi\in D).$$

Then the inversion formula

$$F(a)=\sum_{\chi\in D}\hat{F}(\chi)\chi(a)\qquad(a\in Z,\ (a,n)=1)$$

holds.

We define

$$H(a)=-\frac{1}{n}\sum_{x=1}^{n-1}e^{-2\pi iax/n}\log(1-e^{2\pi ix/n})\qquad(a\in Z).$$

The formulas (6) and (16) in Lehmer [7] yield

$$\hat{H}(\chi_0)=\frac{1}{n}\sum_{p|n}\frac{\log p}{p-1},$$

where χ_0 is the principal character to the modulus n.
For $\chi \neq \chi_0$ we have easily

$$\hat{H}(\chi)=\frac{1}{\phi(n)}L(1,\bar{\chi}).$$

Hence the inversion formula yields

$$(1)\qquad H(a)=\frac{1}{n}\sum_{p|n}\frac{\log p}{p-1}+\frac{1}{\phi(n)}\sum_{\chi\neq\chi_0}L(1,\bar{\chi})\,\chi(a)\qquad (a\in Z,\ (a,n)=1).$$

The formulas (6) and (12) in Lehmer [7] yield

$$\frac{\pi}{n}\cot\left(\frac{\pi a}{n}\right)=H(a)-H(-a)\qquad (a\in Z,\ a\not\equiv 0\ (\mathrm{mod}\ n)).$$

From this and (1),

$$(2)\qquad \frac{\pi}{n}\cot\left(\frac{\pi a}{n}\right)=\frac{1}{\phi(n)}\sum_{\chi\neq\chi_0}(\chi(a)-\chi(-a))\,L(1,\bar{\chi})$$
$$=\frac{2}{\phi(n)}\sum_{\chi(-1)=-1}\chi(a)\,L(1,\bar{\chi})\qquad (a\in Z,\ (a,n)=1).$$

Let ζ denote a primitive n-th root of unity. Then the Galois group of $Q(\zeta)$ over Q is given by the mappings $\sigma_a:\ \zeta\longrightarrow\zeta^a$ $(a\in S)$, where S is a complete set of residues prime to n.

We set

$$f(x)=\frac{1}{i}\cot\left(\frac{\pi x}{n}\right).$$

Clearly, $f(b)$ belongs to $Q(\zeta)$ for any integer b and $f(b)^{\sigma_a}=f(ab)$.

Suppose that there exist $C_b\in Q$ such that

$$\sum_{b\in T}C_b f(b)=0.$$

Then applying the mappings $\sigma_{\bar{a}}$ $(a\in T)$, we get

$$\sum_{b\in T}C_b f(\bar{a}b)=0,$$

where $\bar{a}$ is defined by $\bar{a}a\equiv 1\ (\mathrm{mod}\ n)$.
Then by the Frobenius determinant relation (see [6; p. 284]) and (2), we have that

$$\det_{a,b\in T}[f(\bar{a}b)]=\prod_{\chi(-1)=-1}\left(\sum_{a\in T}\bar{\chi}(a)f(a)\right)$$
$$=\prod_{\chi(-1)=-1}\left(\sum_{a\in T}\bar{\chi}(a)\sum_{\psi(-1)=-1}\frac{2n}{\pi i\phi(n)}\,\psi(a)\,L(1,\bar{\psi})\right)$$

$$=\left(\frac{n}{\pi i}\right)^{\frac{\phi(n)}{2}} \prod_{\chi(-1)=-1}\left(\sum_{\psi(-1)=-1} L(1,\bar{\psi}) \sum_{a\in T} \frac{2}{\phi(n)}\,\bar{\chi}(a)\,\psi(a)\right)$$

$$=\left(\frac{n}{\pi i}\right)^{\frac{\phi(n)}{2}} \prod_{\chi(-1)=-1} L(1,\chi) \neq 0.$$

Hence $C_b=0$ for all $b\in T$, as required.

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Tadashige Okada
Hachinohe Institute of Technology
Ohbiraki, Hachinohe
Aomori, Japan