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On the torsion theoretic support of a module

By John A. BEACHY

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Goldman in [8] studied modules ${}_RX$ for which the localization X_σ has finite length in the quotient category determined by a torsion radical σ. He showed that for this to occur a necessary requirement is that the set of prime torsion radicals π such that $X_\pi \neq 0$ has only finitely many maximal elements; it is an open question whether this condition holds for a finitely generated module over any Noetherian ring. It was shown in [2] that the condition holds for the module ${}_RR$ if R is a ring with Krull dimension. This result will be used in section 1 below to show that any finitely generated module over a fully bounded, Noetherian ring has only finitely many maximal elements in its support. It will also be used to show that if R is a ring with Krull dimension, then R is Artinian if and only if every prime torsion radical is maximal, extending Theorem 5.10 of [8].

For a finitely generated module X over a commutative, Noetherian ring R, it is well-known that the following conditions hold: (i) a prime ideal belongs to the support of X if and only if it contains an associated prime ideal of X; (ii) for any multiplicative set S of R, the associated prime ideals of the localization X_S correspond to the associated prime ideals of X which do not meet S; (iii) for any multiplicative set S, if X is S-torsion, then so is the injective envelope $E(X)$ of X. Fully bounded modules are defined in section 2, and then in section 3 it is shown that any finitely generated, fully bounded, Artin-Rees module over a Noetherian ring satisfies the above conditions, with multiplicative sets replaced by torsion radicals and prime ideals replaced by prime torsion radicals. Cahen has shown in [4] that every module over a Noetherian ring satisfies the torsion theoretic form of condition (iii) if and only if every finitely generated module satisfies condition (i). This can be generalized to show that (with appropriate formulations) conditions (i), (ii) and (iii) are equivalent for any fixed module over a ring with Krull dimension. As Golan has observed in [7], any finite dimensional module which satisfies condition (i) must have only finitely many maximal elements in its support, since it has only finitely many associated primes.

Throughout the paper, R will denote an associative ring with identity, and R-Mod will denote the category of unital left R-modules. The reader is referred to Stenström [10] for any undefined terms. A left exact subfunctor σ of the identity on R-Mod will be called a torsion radical if $\sigma(X/\sigma X)=0$ for all modules ${}_RX$. If σ is a torsion radical, then ${}_RX$ is called σ-torsion if $\sigma X=X$ and σ-torsionfree if $\sigma X=0$, and these two classes of modules define the torsion theory associated with σ. A submodule $Y\subseteq X$ is called σ-dense if X/Y is σ-torsion and σ-closed if X/Y is σ-torsionfree; the σ-closure of Y in X is the intersection of all σ-closed submodules of X which contain Y.

For a torsion radical σ and a module ${}_RX$, the module of quotients $Q_\sigma(X)=X_\sigma$ is defined as the σ-closure of $X/\sigma X$ in its injective envelope. This defines the quotient functor Q_σ from R-Mod into the quotient category R-Mod$/\sigma$ determined by σ, where R-Mod$/\sigma$ is the full subcategory of R-Mod defined by all modules X which are σ-torsionfree and σ-closed in their injective envelope. There is a one-to-one correspondence between the σ-closed submodules of X and the subobjects of X_σ in R-Mod$/\sigma$.

A torsion radical σ is larger than a torsion radical τ, denoted $\sigma\geq\tau$, if $\sigma X\supseteq\tau X$ for all modules ${}_RX$. A torsion radical μ is said to be maximal if $\mu\neq 1$ (the identity functor) and for any torsion radical σ, $\sigma\geq\mu$ implies $\sigma=\mu$ or $\sigma=1$. The largest torsion radical σ for which ${}_RM$ is σ-torsionfree is given by $\sigma X=\mathrm{rad}_{E(M)}\,X$, where $E(M)$ is the injective envelope of M and $\mathrm{rad}_{E(M)}\,X$ is the intersection of all kernels of R-homomorphisms from X to $E(M)$. Note that $\mathrm{rad}_{E(M)}\geq\sigma$ if and only if M is σ-torsionfree.

A nonzero module ${}_RM$ is called monoform if for each submodule $N\subseteq M$, every homomorphism $f\colon N\to M$ is either zero or a monomorphism. This is equivalent to the condition that every nonzero submodule of M is σ-dense in M, for $\sigma=\mathrm{rad}_{E(M)}$, which shows that M is monoform if and only if M_σ is a simple object in R-Mod$/\sigma$. A torsion radical π is said by Goldman to be prime if $\pi=\mathrm{rad}_{E(M)}$ for a monoform module M; this occurs if and only if R-Mod$/\pi$ has an injective cogenerator which is the injective envelope of a simple object, and so if π is prime, then R-Mod$/\pi$ has only one isomorphism class of simple objects. In particular, if M is simple in R-Mod, then $\mathrm{rad}_{E(M)}$ is prime, and in fact $\mathrm{rad}_{E(M)}$ is minimal in the set of prime torsion radicals. (If N is monoform and $\mathrm{rad}_{E(N)}\leq\mathrm{rad}_{E(M)}$, then N must contain a submodule isomorphic to M, so $E(N)\simeq E(M)$ since any monoform module is uniform.)

For a module ${}_RX$, ass (X) will denote the set of prime torsion radicals which are defined by a monoform submodule of X, and supp (X) will denote the set of prime torsion radicals π for which $X_\pi\neq 0$. For a torsion radical σ, σ-supp$(X)=\{\pi\in\text{supp}(X)|\pi\geq\sigma\}$. If π is a prime torsion radical defined by a monoform module M and $\pi\in\sigma$-supp(X), then $X_\pi\neq 0$ implies $\pi X\neq X$, and so there exists a nonzero homomorphism $f\colon X\to E(M)$, which shows that $X/\ker(f)$ contains a submodule isomorphic to a nonzero submodule of M, and thus $\pi\in\text{ass}(X/Y)$ for a σ-closed submodule $Y\subseteq X$. Note that if Y is an essential submodule of X, then ass$(Y)=$ass(X), since any nonzero submodule of a monoform module is again monoform. A nonzero module M is called prime if Ann$(N)=$Ann(M) for each nonzero submodule $N\subseteq M$; in this case Ann(M) must be a prime ideal. The set of prime ideals P such that $P=$Ann(M) for a prime submodule M of X will be denoted by ASS(X); the set of prime ideals P such that $P\in$ASS(X/Y) for some submodule $Y\subseteq X$ will be denoted by SUPP(X).

The Krull dimension of a module ${}_RX$ (see [9]) will be denoted by K-dim(X), and is defined by transfinite recursion, as follows: if X is Artinian, then K-dim$(X)=0$; if α is an ordinal and K-dim$(X)\not<\alpha$, then K-dim$(X)=\alpha$ if there is no infinite descending chain $X=X_0\supset X_1\supset\cdots$ of submodules X_i such that for $i=1, 2, \cdots$ K-dim$(X_{i-1}/X_i)\not<\alpha$. The Krull dimension of the ring R is defined to be K-dim$({}_RR)$. Any left Noetherian ring has Krull dimension, and many of the properties of Noetherian rings hold more generally for rings with Krull dimension. In particular, if N denotes the prime radical of R, then N is nilpotent and R/N is semiprime Goldie. This can be used to show that if R has Krull dimension, then every proper torsion radical of R-Mod is contained in a maximal torsion radical, and the maximal torsion radicals are precisely those defined by the minimal prime ideals of R [2]. (Any semiprime Goldie ring satisfies the condition [3], and the lattice of torsion radicals for R is isomorphic to that of R/N, since N is nilpotent.) If ${}_RX$ has Krull dimension, then every nonzero submodule of X contains a monoform submodule [9, Theorem 2.1 and Corollary 2.5], and if R has Krull dimension, then every nonzero submodule of X contains a prime submodule [9, Theorem 8.3]. Thus any indecomposable injective module over a ring with Krull dimension defines a prime torsion radical, and both ass(X) and ASS(X) are nonempty for any nonzero module over a ring with Krull dimension.

If R is left Noetherian, it is said to be left fully bounded if for each prime ideal P of R, every essential left ideal of the ring R/P contains a nonzero two-sided ideal. It is well-known that this condition holds if and only if the map which assigns to each indecomposable injective left R-module its associated prime ideal is a bijection between the set of isomorphism classes of indecomposable injective modules and the set of prime ideals of R. In this case, if ${}_R M$ is monoform, then $\mathrm{rad}_{E(M)}=\mathrm{rad}_{E(R/P)}$ for $\{P\}=\mathrm{ASS}\,(M)$, so that prime torsion radicals just correspond to prime ideals. It is also well-known that a left Noetherian ring R is left fully bounded if for each finitely generated module ${}_R X$ there exist elements $x_1, x_2, \cdots, x_n \in X$ such that $\mathrm{Ann}\,(X)=\mathrm{Ann}\,(x_1, x_2, \cdots, x_n)$. Recently Cauchon [5] has shown that the converse holds (his result will be generalized in section 2).

§ 1. On the maximal elements of supp (X)

Goldman has shown that the localization X_σ of a finitely generated module ${}_R X$ over a left Noetherian ring has a composition series in R-Mod$/\sigma$ if and only if σ-supp (X) is finite and every element in σ-supp (X) is maximal in σ-supp (X). This raises the following question [8, p. 329]: when does supp (X) have finitely many maximal elements, for a finitely generated module X? Using the fact that supp (R) has finitely many maximal elements when R is left Noetherian, Goldman has shown that a left Noetherian ring is left Artinian if and only if every prime torsion radical of R-Mod is maximal. Theorem 1. 1 extends this result to rings with Krull dimension, giving an easy proof which does not depend on the machinery of Goldman's paper.

THEOLEM (1. 1) *If R has Krull dimension on the left, then R is left Artinian if and only if every prime torsion radical of R-Mod is maximal.*

PROOF. If R is left Artinian, then each prime torsion radical of R-Mod is defined by a simple module, so it is minimal in the set of prime torsion radicals.

Conversely, assume that every prime torsion radical of R-Mod is maximal. Since R-Mod has only finitely many maximal torsion radicals, by assumption it must have only finitely many prime torsion radicals. In particular, there must be only finitely many isomorphism classes of simple modules, defining torsion radicals $\{\mu_i\}_{i=1}^n$. If ${}_R M$ is monoform, then $\bigcap_{i=1}^n \mu_i(M)=0$ since the direct sum of the injective envelopes of simple modules from each isomorphism class is a cogenerator for R-Mod. Thus $\mu_i(M)=0$ for some i, since M is uniform and $\mu_i(M)\neq 0$ for all i implies $\bigcap_{i=1}^n \mu_i(M)\neq 0$. Thus $\mathrm{rad}_{E(M)}\geq \mu_i$, which implies by assumption that $\mathrm{rad}_{E(M)}=\mu_i$, so M contains a minimal submodule. This shows that every nonzero left R-module contains a minimal submodule, and for a ring with Krull dimension this implies that R is left Artinian. In fact, if $A_1\supseteq A_2\supseteq\cdots$ is a descending chain of left ideals of R, let $A=\bigcap_{i=1}^\infty A_i$. Then R/A has an essential socle which must be finitely generated, since any module with

Krull dimension has finite uniform dimension, and it is easy to check that $A=A_n$ for some n.

LEMMA (1. 2) *If ${}_RX$ can be embedded in a finite direct sum of copies of ${}_RY$, then σ-supp$(X)\subseteq\sigma$-supp(Y), for any torsion radical σ.*

PROOF. Assume that there exists an embedding $f: X\to Y^n$ for some n. If Y is π-torsion for a prime torsion radical π, then Y^n is π-torsion, and so is X, since the class of π-torsion modules is closed under direct sums and submodules. Thus if $\pi\in\sigma$-supp(X), then $\pi\geq\sigma$ and $\pi X\neq X$, so $\pi Y\neq Y$ and $\pi\in\sigma$-supp(Y).

THEOREM (1. 3) *Let ${}_RX$ be a module with Krull dimension for which there exist elements $x_1, x_2, \cdots, x_n\in X$ such that $Ann(X)=Ann(x_1, \cdots, x_n)$. Then supp$(X)$ has only finitely many maximal elements, which are defined by the prime ideals minimal over $Ann(X)$.*

PROOF. If $\mathrm{Ann}(X)=\mathrm{Ann}(x_1, \cdots, x_n)$, then there exists an "embedding" f: $R/\mathrm{Ann}(X)\to X^n$ defined by $f(r)=(rx_1, rx_2, \cdots, rx_n)$ for $r\in R/\mathrm{Ann}(X)$. By the preceding lemma, $\mathrm{supp}(R/\mathrm{Ann}(X))\subseteq\mathrm{supp}(X)$. Since X is an $R/\mathrm{Ann}(X)$-module, the reverse inclusion also holds. Since ${}_RX$ has Krull dimension, the existence of the embedding implies that $R/\mathrm{Ann}(X)$ has Krull dimension, so $\mathrm{supp}(X)=\mathrm{supp}(R/\mathrm{Ann}(X))$ has only finitely many maximal elements, which are defined by the minimal prime ideals of $R/\mathrm{Ann}(X)$.

COROLLARY (1. 4) *Let ${}_RX$ be finitely generated. Then supp(X) has only finitely many maximal elements in any one of the following cases.*

(a) *R is commutative and has Krull dimension.*

(b) *R is left fully bounded and left Noetherian.*

(c) *R is left and right Noetherian and X is torsionless.*

PROOF. (c) Recall that a module X is called torsionless if for each $0\neq x\in X$ there exists $f\in\mathrm{Hom}_R(X, R)$ such that $f(x)\neq 0$. If X is torsionless, let A be the sum in R of all homomorphic images of X. Then if R is right Noetherian it must satisfy the descending chain condition for left annihilators, so there must exist elements $a_1, \cdots, a_n\in A$ with $\mathrm{Ann}(A)=\mathrm{Ann}(a_1, \cdots, a_n)$. By the definition of A, $a_i=\sum_{j=1}^{k}f_{ij}(x_{ij})$ for $x_{ij}\in X$ and $f_{ij}\in\mathrm{Hom}_R(X, R)$. If $rx_{ij}=0$ for all i, j, then $ra_i=0$ for all i, and thus $rA=0$, so if $rx\neq 0$ for some $0\neq x\in X$, then since X is torsionless there exists $f: X\to R$ with $rf(x)=f(rx)\neq 0$, a contradiction. Thus X satisfies the condition of Theorem 1. 3, with $\mathrm{Ann}(X)=\mathrm{Ann}(\{x_{ij}\})$.

§2. Fully bounded modules

PROPOSITION (2. 1) *Assume that R has Krull dimension on the left. The* *following conditions are equivalent for a module $_RX$ and a torsion radical σ.*

(1) *Every prime torsion radical in σ-supp(X) is defined by its associated prime ideal.*

(2) *If Y is a σ-closed submodule of X and $M \subseteq X/Y$ is monoform and prime, then there exist elements $m_1, \cdots, m_n \in M$ such that $Ann(M) = Ann(m_1, \cdots, m_n)$.*

(3) *If Y is a σ-closed submodule of X, then every nonzero finite dimensional submodule of X/Y contains an essential submodule W with $Ann(W) = Ann(w_1, \cdots, w_n)$ for some elements $w_1, \cdots, w_n \in W$.*

PROOF. If π is a prime torsion radical defined by a monoform module M, then $\mathrm{ASS}(M) = \{P\}$ for some prime ideal P, since R has Krull dimension, and P is uniquely determined by π.

(1)$\Rightarrow$(2). Let $Y \subseteq X$ be σ-closed, and let $M \subseteq X/Y$ be monoform and prime, with $\mathrm{Ann}(M) = P$. Then M defines a prime torsion radical $\pi \geq \sigma$, so by assumption π is defined by $E(R/P)$. Therefore M_π must be isomorphic to U_π for some uniform left ideal U of R/P, and we may assume that U is isomorphic to a submodule of M. Since R/P is a prime Goldie ring, it satisfies the descending chain condition for left annihilators, so there must exist $u_1, \cdots, u_n \in U$ with $P = \mathrm{Ann}(u_1, \cdots, u_n)$. If $m_1, \cdots, m_n$ are the images of $u_1, \cdots, u_n$ under the monomorphism from U to M, then $\mathrm{Ann}(M) = \mathrm{Ann}(m_1, \cdots, m_n)$ since M is prime.

(2)$\Rightarrow$(3). If $Y \subseteq X$ is σ-closed, then each nonzero finite dimensional submodule of X/Y contains an essential direct sum $M_1 \oplus \cdots \oplus M_n$ of monoform, prime submodules, since R has Krull dimension. We can apply condition (2) to each submodule M_i.

(3)$\Rightarrow$(1). If $\pi \in \sigma$-supp(X), then π is defined by a monoform, prime module $M \subseteq X/Y$ for some $Y \subseteq X$. Since $\pi \geq \sigma$, we have $\sigma M = 0$, and then since $M \cap \sigma(X/Y) = 0$ we can assume that Y is σ-closed. By assumption M contains a nonzero submodule W with $\mathrm{Ann}(W) = \mathrm{Ann}(w_1, \cdots, w_n)$ for some $w_1, \cdots, w_n \in W$. Since M is prime, $P = \mathrm{Ann}(W) = \mathrm{Ann}(M)$ is prime, and the induced embedding of R/P into M^n implies that $E(R/P)$ must be isomorphic to a direct summand of $E(M)^n$. Applying the Krull-Remak-Schmidt-Azumaya theorem shows that $E(M)$ is isomorphic to a direct summand of $E(R/P)$, and so π is defined by $E(R/P)$.

PROPOSITION (2.2) *The class of modules which satisfy the conditions of Proposition 2.1 is closed under formation of submodules, factor modules, direct sums and (group) extensions.*

PROOF. If $0 \to Y \to X \to W \to 0$ is an exact sequence of R-modules, then σ-supp$(X) = \sigma$-supp$(Y) \cup \sigma$-supp(W). Furthermore, σ-supp$(\oplus_{\alpha \in A} X_\alpha) = \cup_{\alpha \in A}$

σ-supp(X_α) for any direct sum $\oplus_{\alpha\in A} X_\alpha$.

DEFINITION (2.3) The module ${}_RX$ is said to be fully bounded if for each prime ideal $P\in \mathrm{SUPP}(X)$, every essential left ideal of the ring R/P contains a nonzero two-sided ideal.

THEOREM (2.4) *Assume that R is left Noetherian. The following conditions are equivalent for any finitely generated module ${}_RX$.*

(1) *X is fully bounded.*

(2) *X satisfies the conditions of Proposition 2.1. for every torsion radical σ.*

(3) *For each prime ideal $P\in SUPP(X)$, R/P is left fully bounded.*

PROOF. (1)$\Rightarrow$(2). If $M\subseteq X/Y$ is monoform and prime, with $\mathrm{Ann}(M)=P$, then by assumption M is finitely generated and R/P is left bounded. If $\mathrm{Hom}_R(N, R/P)=0$ for all $N\subseteq M$, then M is a singular R/P-module and so $\cap_{i=1}^n \mathrm{Ann}(m_i)/P$ is essential in R/P, for the generators $m_1, \cdots, m_n$ of M. This contradicts the fact that $\mathrm{Ann}(M)=P$, since by assumption there exists an ideal I such that $P\subsetneqq I\subseteq \cap_{i=1}^n \mathrm{Ann}(m_i)$, and $\mathrm{Ann}(M)$ must contain I. We must therefore have $\mathrm{Hom}_R(N, R/P)\neq 0$ for some $N\subseteq M$. As in the proof of Corollary 1.4 (c), there exist elements $x_1, \cdots, x_n\in N$ such that $\mathrm{Ann}(x_1, \cdots, x_n)=P$, so condition (2) of Proposition 2.1 is satisfied.

(2)$\Rightarrow$(3). Let $P\in \mathrm{SUPP}(X)$, with $P=\mathrm{Ann}(W)$ for some monoform, prime module $W\subseteq X/Y$. By condition (2) of Proposition 2.1, we can assume that there exist elements $w_1, \cdots, w_n\in W$ such that $P=\mathrm{Ann}(w_1, \cdots, w_n)$, so that there is an embedding $R/P\to W^n$. Let E be an indecomposable, injective R/P-module, and let π be the prime torsion radical of R-Mod which is defined by E. Then by Lemma 1.2, $\mathrm{supp}(R/P)\subseteq \mathrm{supp}(W)\subseteq \mathrm{supp}(X)$, so π is defined by its associated prime ideal since X satisfies condition (1) of Proposition 2.1, and the associated prime ideal contains P. This establishes the necessary bijection between isomorphism classes of indecomposable injective R/P-modules and prime ideals containing P, so R/P must be left fully bounded.

(3)$\Rightarrow$(1). This follows immediately from the definition.

COROLLARY (2.5) *If R is left Noetherian, then the following conditions are equivalent for any finitely generated module ${}_RX$.*

(1) *$R/Ann(X)$ is fully bounded.*

(2) *For each sub-factor module $W\subseteq X/Y$ there exist elements $w_1, \cdots, w_n\in W$ such that $Ann(W)=Ann(w_1, \cdots, w_n)$.*

(3) *X is fully bounded and there exist elements $x_1, \cdots, x_n\in X$ such that $Ann(X)=Ann(x_1, \cdots, x_n)$.*

COROLLARY (2. 6) *Let R be left Noetherian and let ${}_RX$ be a finitely generated module which is σ-torsion for the torsion radical σ. If X is fully bounded, then every prime ideal in $SUPP(X)$ is σ-dense in R.*
PROOF. If $P \in SUPP(X)$, then R/P can be embedded in a σ-torsion module, so P must be σ-dense.

§3. On the relationship between supp(X) and ass(X)

Throughout this section it will be assumed that each nonzero left R-module contains a monoform submodule, so that $\mathrm{ass}(X) \neq \phi$ if $X \neq 0$.

A submodule $Y \subseteq X$ is said to be essentially closed in X if Y has no proper essential extension in X. Note that in this case, if W is maximal in the set of submodules of X which intersect Y trivially, then the image of W in X/Y is an essential submodule. A torsion radical σ is called stable if the class of σ-torsion modules is closed under injective envelopes. This happens if and only if for each module ${}_RX$, the σ-torsion submodule σX is essentially closed in X [10, Chapter VI, Proposition 7. 1].

PROPOSITION (3. 1) *For a module ${}_RX$ and torsion radical σ, the following conditions are equivalent.*

(1) *For each submodule $Y \subseteq X$, $\mathrm{ass}(Y_\sigma) = \{\pi \in \mathrm{ass}(Y) \mid \pi \geq \sigma\}$.*

(2) *σX is essentially closed in X.*

PROOF. (1)$\Rightarrow$(2). If σX is essential in $Y \subseteq X$, then $\mathrm{ass}(Y) = \mathrm{ass}(\sigma X)$, so by assumption $\mathrm{ass}(Y_\sigma) = \{\pi \in \mathrm{ass}(\sigma X) \mid \pi \geq \sigma\} = \phi$. This implies that $Y_\sigma = 0$, so $Y = \sigma Y = \sigma X$.

(2)$\Rightarrow$(1). Let $Y \subseteq X$, and let $W \subseteq Y$ be maximal in the set of submodules whose intersection with σY is trivial. Then the image of W in $Y/\sigma Y$ is essential, so $\mathrm{ass}(Y_\sigma) = \mathrm{ass}(Y/\sigma Y) = \mathrm{ass}(W) \subseteq \mathrm{ass}(Y)$. If $\pi \in \mathrm{ass}(Y_\sigma)$ is defined by a monoform submodule $M \subseteq W$, then $\sigma M = 0$ implies $\pi \geq \sigma$. Conversely, if $\pi \in \mathrm{ass}(Y)$ and $\pi \geq \sigma$, then π is defined by a monoform submodule $M \subseteq Y$ with $M \cap \sigma Y = 0$. Thus mapping both M and W into $Y/\sigma Y$ shows that M contains a submodule isomorphic to a submodule of W, so $\pi \in \mathrm{ass}(W) = \mathrm{ass}(Y_\sigma)$.

COROLLARY (3. 2) *The torsion radical σ is stable if and only if $\mathrm{ass}(X_\sigma) = \{\pi \in \mathrm{ass}(X) \mid \pi \geq \sigma\}$ for every module ${}_RX$.*

PROPOSITION (3. 3) *The following conditions are equivalent for any module ${}_RX$ and any torsion radical σ.*

(1) *For each submodule $Y \subseteq X$, $\pi \in \sigma\text{-}\mathrm{supp}(Y) \Longleftrightarrow \sigma \leq \pi \leq \tau$ for some $\tau \in \mathrm{ass}(Y)$.*

(2) *ρX is essentially closed in X for all torsion radicals $\rho \geq \sigma$.*

(3) *πX is essentially closed in X for all $\pi \in \sigma$-supp(X).*

(4) *For each submodule $Y \subseteq X$ and each $\pi \in \sigma$-supp(X), ass$(Y_\pi) = \{\tau \in$ ass$(Y) | \tau \geq \pi\}$.*

PROOF. (1)$\Rightarrow$(2). Suppose that $\rho \geq \sigma$ and ρX is an essential submodule of $Y \subseteq X$. Then $\rho X \subsetneqq Y$ implies that there exists $\pi \in$ ass$(Y/\rho X) \subseteq \rho$-supp$(Y) \subseteq$ σ-supp(Y). By assumption, $\pi \leq \tau$ for some $\tau \in$ ass$(Y) =$ ass(ρX). This is a contradiction since $\tau \geq \pi \geq \rho$.

(2)$\Rightarrow$(3). This follows immediately from the definition of σ-supp(X).

(3)$\Rightarrow$(4). This follows from Proposition 3.1.

(4)$\Rightarrow$(1). If $\pi \in \sigma$-supp(Y), then $Y_\pi \neq 0$, so there exists $\pi \in$ ass(Y_π), with $\tau \geq \pi$ since Y_π is π-torsionfree. Then by assumption $\tau \in$ ass(Y). The converse is obvious.

Several results can be deduced immediately from Proposition 3.3. The torsion radical σ is stable if every prime torsion radical $\pi \geq \sigma$ is stable. Every torsion radical of R-Mod$/\sigma$ is stable if every σ-torsionfree module satisfies the conditions of Proposition 3.3. The conditions of Proposition 3.3 hold for every finite dimensional module and every torsion radical if and only if for any indecomposable injective module ${}_RE$ and any prime torsion radical π, either $\pi E = 0$ or $\pi E = E$.

Let R be left Noetherian. Then a two-sided ideal I of R is said to have the AR-property with respect to a finitely generated module ${}_RX$ if for each submodule $Y \subseteq X$ and each integer $n > 0$, there is an integer $h(n) > 0$ such that $I^{h(n)}X \cap Y \subseteq I^nY$. If I has the AR-property with respect to ${}_RR$, it is simply said to have the AR-property. A finitely generated module ${}_RX$ is called an Artin-Rees module if every ideal of R has the AR-property with respect to X. Note that if I has the AR-property with respect to X, then it also has the property with respect to any submodule or factor module of X. The first assertion is easily verified; to check the second, suppose that $Y/W \subseteq X/W$. For each n there exists $h(n)$ with $I^{h(n)}X \cap Y \subseteq I^nY$, and so $I^{h(n)}(X/W) \cap (Y/W) = (I^{h(n)}X + W)/W \cap (Y/W) \subseteq ((I^{h(n)}X \cap Y) + W)/W \subseteq (I^nY + W)/W = I^n(Y/W)$.

PROPOSITION (3.4) *Let R be left Noetherian, let I be any ideal of R, and let σ be the torsion radical defined by the powers of I. Then I has the AR-property with respect to a finitely generated module ${}_RX$ if and only if $\sigma(X/Y)$ is essentially closed in X/Y, for every factor X/Y of X.*

PROOF. Since R is left Noetherian, $\sigma X = \{x \in X | I^n x = 0$ for some $n\}$ defines a torsion radical of R-Mod. If $\sigma(X/Y)$ is essentially closed in X/Y for every factor of a fixed module X, then suppose that $n > 0$ and $Y \subseteq X$ are given. Since $I^nY \cap Y = I^nY$, the set of submodules W of X such that

$W\cap Y=I^nY$ is nonempty, and the set must have a maximal element since we assume that X is finitely generated and therefore Noetherian. If W is maximal in this set, then X/W must be an essential extension of $Y+W/W\simeq Y/I^nY$, which is σ-torsion. By assumption X/W must be σ-torsion, and then since X is finitely generated we must have $I^m(X/W)=0$ for some $m>0$, that is, $I^mX\subseteq W$, so that $I^mX\cap Y\subseteq W\cap Y=I^nY$.

Conversely, suppose that I has the AR-property with respect to X. Then since I has the same property with respect to every factor module of X, it is sufficient to show that σX is essentially closed in X. If σX is essential in $Y\subseteq X$, then $I^n(\sigma X)=0$ since σX is finitely generated, so for some m, $I^mY\cap\sigma X\subseteq I^n(\sigma X)=0$. Thus $I^mY=0$ and Y is σ-torsion, so we may conclude that $Y=\sigma X$.

It follows from Proposition 3.4 that an ideal I has the AR-property if and only if the associated torsion radical σ is stable. (If I has the AR-property, $\sigma X=X$, and $y\in E(X)$, then Ry is essential over $\sigma Ry=Ry\cap\sigma X$, so $\sigma Ry=Ry$ and thus $\sigma E(X)=E(X)$.) From this it follows, since R is left Noetherian, that I has the AR-property if and only if for any indecomposable injective module ${}_RE$, either $\sigma E=0$ or $\sigma E=E$.

PROPOSITION (3.5) *If R is left Noetherian, then the following conditions are equivalent for a finitely generated module ${}_RX$.*

(1) *X is an Artin-Rees module.*

(2) *For each submodule $Y\subseteq X$, $\sigma(X/Y)$ is essentially closed in X/Y for any bounded torsion radical σ.*

PROOF. (1)$\Rightarrow$(2). Let σ be a bounded torsion radical, that is, assume that every σ-dense left ideal contains a σ-dense two-sided ideal. It is sufficient to show that σX is essentially closed in X, so suppose that σX is essential in $Y\subseteq X$. Then since X is Noetherian, there exist submodules $\{Y_i\}_{i=1}^n$ of Y such that Y/Y_i is uniform, $\cap_{i=1}^n Y_i=0$, and Y is an essential submodule of the subdirect sum $\oplus_{i=1}^n Y/Y_i$. If P_i is the associated prime ideal of Y/Y_i, with $P_i=\mathrm{Ann}(M_i)$ for $M_i\subseteq Y/Y_i$, then $P_i^{m(i)}(Y/Y_i)=0$ for some $m(i)>0$, since X is an Artin-Rees module [10, Chapter VII, Proposition 4.3]. Now $P_i=\mathrm{Ann}(W_i)$ for $W_i=M_i\cap\sigma(Y/Y_i)$, since W_i is nonzero (because $\oplus_{i=1}^n Y/Y_i$ is an essential extension of σX). Furthermore, P_i is the largest ideal which annihilates the generators of W_i, so P_i must be σ-dense because σ is bounded (the intersection of finitely many left annihilators of elements of a σ-torsion module is always σ-dense). Thus $D=\prod_{i=1}^n P_i^{m(i)}$ is a σ-dense ideal with $DY=0$, so Y is σ-torsion and $Y=\sigma X$.

(2)$\Rightarrow$(1). This follows from Proposition 3.4, since for any ideal I, the torsion radical defined by powers of I is bounded.

THEOREM (3.6) *Let R be left Noetherian. If ${}_RX$ is a finitely generated, fully bounded, Artin-Rees module, then the following conditions hold.*

(a) $\pi \in supp(X) \Longleftrightarrow \pi \leq \tau$ *for some* $\tau \in ass(X)$.

(b) *For any torsion radical* σ, $ass(X_\sigma) = \{\pi \in ass(X) | \pi \geq \sigma\}$.

(c) *For any torsion radical* σ, σX *is essentially closed in* X.

PROOF. By Proposition 3.3 it is sufficient to show that condition (c) holds. The proof is exactly the same as the proof that $(1) \Rightarrow (2)$ in Proposition 3.5, with the one exception that the prime ideals P_i must be shown to be σ-dense by using Corollary 2.6.

Actually, Theorem 3.6 could be stated in a much stronger form. Any submodule or homomorphic image of X must also satisfy conditions (a)—(c), and condition (a) could be stated in terms of σ-supp(X) for any torsion radical σ. Since X is assumed to be fully bounded, supp(X) could also be replaced by SUPP(X).

COROLLARY (3.7) *If ${}_RX$ is a finitely generated, fully bounded, Artin-Rees module over a left Noetherian ring, then supp(X) has only finitely many maximal elements.*

PROOF. The maximal elements of supp(X) must belong to ass(X), which is finite.

COROLLARY (3.8) *If ${}_RX$ is a finitely generated, Artin-Rees module over a fully bounded left Noetherian ring, then any prime ideal minimal over Ann(X) is an associated prime ideal of X.*

PROOF. If P is a prime ideal minimal over Ann(X), then P defines a maximal element of supp(X), by Theorem 1.3. By Theorem 3.6, P must belong to ASS(X).

If R is left Noetherian and left fully bounded, then it can be shown that every torsion radical of R-Mod is bounded. Thus Propositions 3.5 and 3.3 can be used to characterize Artin-Rees modules over such rings. If ${}_RX$ is finitely generated, then it is an Artin-Rees module if and only if for each module $W \subseteq X/Y$, $\pi \in \text{supp}(W) \Longleftrightarrow \pi \leq \tau \in \text{ass}(W)$. As the following example shows, it is not sufficient to only require the condition on supp(X). Let F be a field and let R be the ring of lower triangular matrices $\left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\right\}$ over F. Let $A = \left\{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right\}$ and let $B = \left\{\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}\right\}$. Then R-Mod has two prime torsion radicals, defined by the simple modules A/B and B, and if $X = A \oplus A/B$, then supp(X) satisfies the desired condition, but X is not an Artin-Rees module because $AB = 0$ while $A^n A \cap B = A \cap B = B$ for all $n > 0$. As was shown by Goldman [8, p. 338], supp(A) does not satisfy the desired condition, even though A is a direct summand of X.

The conditions "fully bounded" and "Artin-Rees" are independent. As shown above, a module may be fully bounded but not Artin-Rees. On the other hand, any simple module S is obviously an Artin-Rees module, but it is fully bounded if and only if $R/\mathrm{Ann}(S)$ is simple Artinian.

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McGill University
Montreal, Quebec
Canada

and

Northern Illinois University
DeKalb, Illinois
U. S. A.