Hokkaido Mathematical Journal. Vol. 6 (1977) p. 260-272

Hyperinvariant subspaces for contractions of class $C_{\cdot 0}$

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(Received August 2, 1976)

1. Introduction

Let T be a bounded operator on a separable Hilbert space $\mathfrak{H}$. A subspace $\mathfrak{L}$ of $\mathfrak{H}$ is said to be *hyperinvariant* for T if $\mathfrak{L}$ is invariant for every operator that commutes with T. In [2] the hyperinvariant subspaces for a unilateral shift were determined, and those for an isometry in [1]. Recall that T is said to be of class $C_{\cdot 0}$ if T is a contraction (i. e., $\|T\| \leqq 1$) and $T^{*n} \longrightarrow 0$ (strongly) as $n \longrightarrow \infty$. Hence a unilateral shift is of class $C_{\cdot 0}$. Let T be of class $C_{\cdot 0}$. Then it necessarily follows that

$$\delta_* \equiv \dim(1-TT^*)\mathfrak{H} \geqq \dim(1-T^*T)\mathfrak{H} \equiv \delta$$

(see [6]). In the case of $\delta_*=\delta<\infty$, in an earlier paper [8] we established a canonical isomorphism between the lattice of hyperinvariant subspaces for T and that for the *Jordan model* of T. In this paper we extend this result to the case of $\delta<\delta_*<\infty$. For an operator T of this class we shall present complete description of the hyperinvariant subspaces $\mathfrak{N}$ with the property that every subspace of $\mathfrak{N}$ hyperinvariant for T is hyperinvariant for the restricted operator $T|\mathfrak{N}$. The author wishes to express his gratitude to Prof. T. Ando for his constant encouragement.

2. Preliminaries

Let θ be an $n \times m$ ($\infty > n \geqq m$) matrix over H^∞ on the unit circle. Such a matrix θ is called *inner* if $\theta(z)$ is isometry a. e. on the unit circle. For such an inner function θ a Hilbert space $\mathfrak{H}(\theta)$ and an operator $S(\theta)$ are defined by

(1) $\quad \mathfrak{H}(\theta) = H_n^2 \ominus \theta H_m^2$ and $S(\theta)h = P_\theta(Sh)$ for h in $\mathfrak{H}(\theta)$,

where H_n^2 is the Hardy space of n-dimensional (column) vector valued functions, P_θ is the projection from H_n^2 onto $\mathfrak{H}(\theta)$, and S is the simple unilateral shift, that is, $(Sh)(z)=zh(z)$. A contraction T of class $C_{\cdot 0}$ with $\delta_*=n$ and $\delta=m$ is unitarily equivalent to an $S(\theta)$ of this type [7]. Thus in the sequel we may discuss $S(\theta)$ in place of T.

For a completely non unitary contraction T, it is possible to define

$\phi(T)$ for every function ϕ in H^∞. In particular, for $S(\theta)$ given above $\phi(S(\theta))$ can be equivalently defined by the following:

$$\phi\big(S(\theta)\big)\,h = P_\theta \phi h \quad \text{for } h \text{ in } \quad \mathfrak{H}(\theta) \qquad \text{(see [5], [7])}.$$

If there is a function ϕ such that $\phi(T)=0$, then T is said to be of class C_0. T of class $C_{\cdot 0}$ with $\delta \leqq \delta_* < \infty$ is of class C_0 if and only if $\delta=\delta_*$ [7].

Suppose T_1 is a bounded operator on $\mathfrak{H}_1$ and T_2 a bounded operator on $\mathfrak{H}_2$. If there exists a *complete injective family* $\{X_\alpha\}$ from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ (i. e., for each α, X_α is an one to one bounded operator from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ and $\bigvee_\alpha X_\alpha \mathfrak{H}_1 = \mathfrak{H}_2$) such that for each α $X_\alpha T_1 = T_2 X_\alpha$, then we write $T_1 \overset{\text{ci}}{\prec} T_2$. If $T_1 \overset{\text{ci}}{\prec} T_2$ and $T_2 \overset{\text{ci}}{\prec} T_1$, then T_1 and T_2 are said to be *completely injection-similar,* and denote by $T_1 \overset{\text{ci}}{\sim} T_2$ [6].

An $n \times m$ $(n \geqq m)$ *normal* inner matrix N' over H^∞ is, by definition, of the form:

$$N' = \left[\begin{array}{cccc} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_m \\ \hline 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} n-m \end{array} \tag{2}$$

where, for each i, ψ_i is a scalar inner function and a divisor of its succesor. Then

$$S(N') = S(\psi_1) \oplus \cdots \oplus S(\psi_m) \oplus \underbrace{S \cdots \oplus S}_{n-m}$$

is called a *Jordan operator.*

Let θ be an $n \times m$ $(\infty > n \geqq m)$ inner matrix over H^∞ and N a *corresponding* normal matrix, i. e., N is the $n \times m$ normal inner matrix of the form (2), where $\psi_1, \psi_2 \cdots, \psi_m$ are the "*invariant factors*" of θ, that is,

$$\psi_k = \frac{d_k}{d_{k-1}} \qquad \text{for} \quad k = 1, 2, \cdots, m,$$

where $d_0=1$ and d_k is the largest common inner divisor of all the minors of order k. In this case, Nordgren [4] has shown that there exist pairs of matrices Δ_i, Λ_i and Δ'_i, Λ'_i $(i=1, 2)$ satisfying

$$\Delta_i \theta = N \Lambda_i, \tag{3}$$

$$\theta \Lambda'_i = \Delta'_i N, \tag{3$'$}$$

$$(\det \Lambda_i)\,(\det \Lambda'_i) \wedge d_m = 1, \tag{4}$$

(5) $$(\det \Delta_1)(\det \Delta_1') \wedge (\det \Delta_2)(\det \Delta_2') = 1,$$

(5)′ $$(\det \Lambda_1)(\det \Lambda_1') \wedge (\det \Lambda_2)(\det \Lambda_2') = 1,$$

where $x \wedge y$ denotes the largest common inner divisor of scalar function x and y in H^∞. Setting

(6) $$X_i = P_N \Delta_i | H(\theta) \quad \text{and}$$

(6)′ $$Y_i = P_\theta \Delta_i' | H(N) \quad \text{for} \quad i = 1, 2,$$

$\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ are complete injective families satisfying the following relations:

(7) $$X_i S(\theta) = S(N) X_i \quad \text{and}$$

(8) $$S(\theta) Y_i = Y_i S(N) \quad \text{for} \quad i = 1, 2.$$

This implies $S(\theta) \overset{ci}{\sim} S(N)$ (cf. [6]).

To every subspace $\mathfrak{L}$ of $\mathfrak{H}(\theta)$, invariant for $S(\theta)$, there corresponds an unique factorization $\theta = \theta_2 \theta_1$ of θ such that θ_1 is an $k \times m$ inner matrix and θ_2 is an $n \times k$ inner matrix $(n \geqq k \geqq m)$ satisfying

$$\mathfrak{L} = \theta_2 \{H_k^2 \ominus \theta_1 H_m^2\} = \theta_2 H_k^2 \ominus \theta H_m^2.$$

In this case $S(\theta)|\mathfrak{L}$ and $P_{\mathfrak{L}^\perp} S(\theta)|\mathfrak{L}^\perp$ are unitarily equivalent to $S(\theta_1)$ and $S(\theta_2)$, respectively. For this discussion see [7].

Let M be an $m \times m$ normal inner matrix over H^∞. Then, in [8], we showed that, in order that a factorization $M = M_2 M_1$ corresponds to a subspace hyperinvariant for $S(M)$, it is necessary and sufficient that both M_1 and M_2 are $m \times m$ normal inner matrices.

3. Jordan operator

Let $N = \begin{bmatrix} M \\ 0 \end{bmatrix}$ be an $n \times m$ normal inner matrix over H^∞, that is, M is an $m \times m$ normal inner matrix over H^∞. Then $S(N)$ on $\mathfrak{H}(N)$ are identified with

$$S(M) \oplus S_{n-m} \quad \text{on} \quad \mathfrak{H}(M) \oplus H_{n-m}^2,$$

where $(S_{n-m} h)(z) = zh(z)$ for h in H_{n-m}^2.

Let $\mathfrak{N}$ be a hyperinvariant subspace for $S(N)$. Then it is clear that $\mathfrak{N}$ is decomposed to the direct sum,

$$\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2,$$

where $\mathfrak{N}_1$ is a subspace of $\mathfrak{H}(M)$, hyperinvariant for $S(M)$, and $\mathfrak{N}_2$ is a subspace of H_{n-m}^2, hyperinvariant for S_{n-m}. In this case we have the following lemma.

LEMMA 1. *For $\mathfrak{N}_1$ and $\mathfrak{N}_2$ which are hyperinvariant for $S(M)$ and S_{n-m}, respectively, in order that the direct sum $\mathfrak{N}=\mathfrak{N}_1\oplus\mathfrak{N}_2$ is hyperinvariant for $S(N)$, it is necessary and sufficient that $\mathfrak{N}_2=\{0\}$ or there exists an inner function ϕ such that $\mathfrak{N}_2=\phi H^2_{n-m}$ and $\mathfrak{N}_1\supseteqq\phi(S(M))\,\mathfrak{H}(M)$.*

PROOF. An operator $X=\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22}\end{bmatrix}$ commutes with $S(N)$, *if* and only if Y_{ij} satisfy the following conditions:

$$Y_{11}S(M)=S(M)\,Y_{11},\quad Y_{12}S_{n-m}=S(M)Y_{12},$$
$$Y_{21}S(M)=S_{n-m}Y_{21}\quad\text{and}\quad Y_{22}S_{n-m}=S_{n-m}Y_{22}.$$

Since $S(M)^n\longrightarrow 0$ as $n\longrightarrow\infty$ and S_{n-m} is isometry, we have $Y_{21}=0$. Thus if $\mathfrak{N}_2=\{0\}$, then it follows that $X\mathfrak{N}\subseteqq\mathfrak{N}$ for every X commuting $S(N)$. By the lifting theorem (cf. [5], [7]), a bounded operator Y_{12} from H^2_{n-m} to $H(M)$ intertwines S_{n-m} and $S(M)$, if and only if there is an $m\times(n-m)$ matrix Ω over H^∞ such that $Y_{12}=P_M\Omega$. Thus, if $\mathfrak{N}_2=\phi H^2_{n-m}$ and $\mathfrak{N}_1\supseteqq\phi(S(M))\,\mathfrak{H}(M)$ for some inner function ϕ, then we have

$$\begin{aligned} X\mathfrak{N} &= (Y_{11}\mathfrak{N}_1+Y_{12}\phi H^2_{n-m})\oplus Y_{22}\phi H^2_{n-m} \\ &\subseteqq(\mathfrak{N}_1+P_M\Omega\phi H^2_{n-m})\oplus\phi H^2_{n-m} \\ &\subseteqq(\mathfrak{N}_1+P_M\phi H^2_{m})\oplus\phi H^2_{n-m} \\ &=\left(\mathfrak{N}_1+\phi\left(S(M)\right)\mathfrak{H}(M)\right)\oplus\phi H^2_{n-m} \\ &\subseteqq\mathfrak{N}_1\oplus\phi H^2_{n-m}=\mathfrak{N} \end{aligned}$$

for every X commuting with $S(N)$.

Conversely suppose $\mathfrak{N}=\mathfrak{N}_1\oplus\mathfrak{N}_2$ is hyperinvariant for $S(N)$, and $\mathfrak{N}_2\neq\{0\}$. Then by [2] there exists an inner function ϕ such that $\mathfrak{N}_2=\phi H^2_{n-m}$. Let Ω_i $(i=1,2,\cdots,m)$ be the $m\times(n-m)$ matrix such that the (j,k)-th entry of Ω_i is 1 for $(j,k)=(i,1)$ and 0 for $(j,k)\neq(i,1)$. Setting

$$X_i=\begin{bmatrix} 0 & Y_i \\ 0 & 0\end{bmatrix}\quad\text{and}\quad Y_i=P_M\Omega_i,$$

each X_i commutes with $S(N)$, hence we have $\mathfrak{N}_1\supseteqq\sum_{i=1}^{n}Y_i\phi H^2_{n-m}=P_M\phi H^2_m=\phi(S(M))\,\mathfrak{H}(M)$. This completes the proof.

THEOREM 1. *In order that a factorization $N=N_2N_1$ of N into the product of an $n\times k$ inner matrix N_2 and an $k\times m$ inner matrix N_1 ($n\geqq k\geqq m$) corresponds to a hyperinvariant subspace $\mathfrak{N}$ for $S(N)$, it is necessary and sufficient that N_1 and N_2 are normal matrices satisfying* (i) *or* (ii):

(i) $k=m$,

(ii) *$k=n$ and N_2 has the form* $\begin{bmatrix} M_2 & 0 \\ 0 & \phi 1_{n-m} \end{bmatrix}$

PROOF. First, assume that $k=m$, and both N_1 and N_2 are normal inner matrices. Then, setting $N_2=\begin{bmatrix} M_2' \\ 0 \end{bmatrix}$, it follows that $N_2\{H_m^2 \ominus N_1 H_m^2\}=M_2'\{H_m^2 \ominus N_1 H_m^2\}$ is hyperinvariant for $S(M)$ (see [8]). Therefore, by Lemma 1, it is hyperinvariant for $S(N)$. Next, assume that N_1 and N_2 are normal matrices satisfying (ii). Set $N_1=\begin{bmatrix} M_1 \\ 0 \end{bmatrix}$. Then we have

$$\mathfrak{N}=N_2\{H_n^2 \ominus N_1 H_m^2\}=M_2\{H_m^2 \ominus M_1 H_m^2\} \oplus \phi H_{n-m}^2 .$$

Normality of M_1 and M_2 implies that $M_2\{H_m^2 \ominus M_1 H_m^2\}$ is hyperinvariant for $S(M)$. On the other hand, normality of N_2 implies $M_2 H_m^2 \supseteqq \phi H_m^2$, and hence we have

$$M_2 H_m^2 \ominus M H_m^2 \supseteqq \phi(S(M)) \mathfrak{H}(M) .$$

Thus from Lemma 1 we deduce that $\mathfrak{N}$ is hyperinvariant for $S(N)$.

Conversely, first, assume that $\mathfrak{N}=\mathfrak{N}_1 \oplus \{0\}$ is hyperinvariant for $S(N)$, and $N=N_2N_1$ is the factorization corresponding to $\mathfrak{N}$. Since $S(N)|\mathfrak{N}=S(M)|\mathfrak{N}_1$ is of class C_0, $S(N_1)$ is of class C_0 (cf. 2). This implies that N_1 is an $m\times m$ inner matrix, that is, $k=m$. Setting $N_2=\begin{bmatrix} M_2 \\ \Gamma \end{bmatrix}$, where M_2 is an $m\times m$ matrix and Γ an $(n-m)\times m$ matrix, we have

$$M=M_2N_1,\ \mathfrak{N}_1=M_2\{H_m^2 \ominus N_1 H_m^2\} \quad \text{and} \quad \Gamma H_m^2=\{0\} .$$

Since $\Gamma=0$ and N_2 is inner, it follows that M_2 is inner. Thus the hyperinvariance of $\mathfrak{N}_1$ corresponding to $M=M_2N_1$ implies that M_2 and N_1 are $m \times m$ normal inner matrices. Next assume that $\mathfrak{N}=\mathfrak{N}_1 \oplus \phi H_{n-m}^2$ and $\mathfrak{N}_1 \supseteqq \phi(S(M)) \mathfrak{H}(M)$. Clearly we have

$$P_{\mathfrak{N}^\perp} S(N)\big|\mathfrak{N}^\perp=P_{\mathfrak{N}_1^\perp} S(M)\big|\mathfrak{N}_1^\perp \oplus S(\phi 1_{n-m}) ,$$

where $\mathfrak{N}_1^\perp$ denotes the orthogonal complement of $\mathfrak{N}_1$ in $\mathfrak{H}(M)$. Since the right-hand operator is of class C_0 (page 129 of [7]), $S(N_2)$ is of class C_0. This implies that N_2 is an $n\times n$ matrix; i.e., $k=n$. To the hyperinvariant subspace $\mathfrak{N}_1$ for $S(M)$ there corresponds a factorization $M=M_2M_1$, where M_1 and M_2 are $m\times m$ normal inner matrices. Thus setting $N_2'=\begin{bmatrix} M_2 & 0 \\ 0 & \phi 1_{n-m} \end{bmatrix}$ and $N_1'=\begin{bmatrix} M_1 \\ 0 \end{bmatrix}$, it is clear that $N=N_2'N_1'$ and $\mathfrak{N}=N_2'\{H_n^2 \ominus N_1' H_m^2\}$. From the uniqueness of the factorization of N into product of two inner matrices corresponding to (hyper) invariant subspace $\mathfrak{N}$, only this factorization $N=N_2'N_1'$ corresponds to $\mathfrak{N}$, that is, $N_2=N_2'$ and $N_1=N_1'$. Since

$$M_2\{H_m^2\ominus M_1 H_m^2\}=\mathfrak{L}_1\supseteqq\phi\big(S(M)\big)\,\mathfrak{L}(M)=P_M\phi H_m^2,$$

we have $M_2H_m^2\supseteqq\phi H_m^2$; this implies that every entry of M_2 is a divisor of ϕ. Therefore N_2 is an $n\times n$ normal inner matrix. Hence N_1 and N_2 are normal inner matrices satisfying (ii).

4. Lattice isomorphism

Let θ be an $n\times m$ inner matrix and N be the corresponding normal inner matrix. Set

$$\alpha(\mathfrak{L})=\bigvee_Z\{Z\mathfrak{L}:\ ZS(\theta)=S(N)\,Z\} \tag{9}$$

and

$$\beta(\mathfrak{N})=\bigvee_W\{W\mathfrak{N}:\ WS(N)=\ S(\theta)\,W\} \tag{10}$$

for each subspace $\mathfrak{L}$ and $\mathfrak{N}$ hyperinvariant for $S(\theta)$ and $S(N)$, respectively, where $\vee\mathfrak{L}_i$ denotes the minimum subspace including all $\mathfrak{L}_i$. Since $S(\theta)\overset{ci}{\sim}S(N)$, it is clear that $\alpha(\mathfrak{L})$ is the non trivial hyperinvarinat subspace for $S(N)$, if $\mathfrak{L}$ is non trivial.

Lemma 2. *If $\theta=\theta_2\theta_1$ is the factorization corresponding to a non trivial hyperinvariant subspace $\mathfrak{L}$ for $S(\theta)$, then θ_1 is an $m\times m$ inner matrix, or θ_2 is an $n\times n$ inner matrix.*

Proof. Let $S(\theta)=\begin{bmatrix}T_1 & * \\ 0 & T_2\end{bmatrix}$ and $S(N)=\begin{bmatrix}S_1 & * \\ 0 & S_2\end{bmatrix}$ be the triangulations corresponding to $\mathfrak{H}(\theta)=\mathfrak{L}\oplus\mathfrak{L}^\perp$ and $\mathfrak{H}(N)=\alpha(\mathfrak{L})\oplus\alpha(\mathfrak{L})^\perp$, respectively. Theorem 1 implies that S_1 or S_2 is of class C_0. First, suppose $u(S_1)=0$ for some u in H^∞. For the bounded operator X_1 given by (6) and every f in $\mathfrak{L}$, in virtue of (3), it follows that

$$X_1u(T_1)f=X_1u\big(S(\theta)\big)f=P_N\Delta_1P_\theta uf=P_N\Delta_1uf,$$
$$=P_Nu\Delta_1f=u\big(S(N)\big)\,X_1f=0.$$

Since X_1 is an injection, we have $u(T_1)f=0$, which implies that T_1 is of class C_0, that is, θ_1 is an $m\times m$ inner matrix. Next suppose S_2 is of class C_0, hence so is S_2^*. For Y_i given by (6)$'$ and every Z such that $ZS(\theta)=S(N)\,Z$, in virtue of (8), Y_iZ commutes with $S(\theta)$; this implies $Y_iZ\mathfrak{L}\subseteqq\mathfrak{L}$ and hence $Y_i\alpha(\mathfrak{L})\subseteqq\mathfrak{L}$. Thus we have $Y_i^*\mathfrak{L}^\perp\subseteqq\alpha(\mathfrak{L})^\perp$. From this and (8), for each

h in $\mathfrak{L}^{\perp}$, it follows that

$$Y_i^* T_2^* h = S_2^* Y_i^* h \qquad \text{for} \quad i=1, 2.$$

From this we can deduce that

$$Y_i^* u(T_2^*) h = u(S_2^*) Y_i^* h \quad \text{for every } u \text{ in } H^\infty,$$

(see [7] chap 3). Since $Y_1 \mathfrak{H}(N) \vee Y_2 \mathfrak{H}(N) = \mathfrak{H}(\theta)$, we have $u(T_2^*)=0$ for u satisfying $u(S_2^*)=0$. Therefore θ_2 is an $n \times n$ inner matrix. This completes the proof.

The following theorem implies that the mapping $\alpha: \mathfrak{L} \longrightarrow \alpha(\mathfrak{L})$ is isomorphism from the lattice of hyperinvariant subspaces for $S(\theta)$ onto that for $S(N)$, and its inverse is given by $\beta: \mathfrak{N} \longrightarrow \beta(\mathfrak{N})$.

THEOREM 2. *For X_i and Y_i $(i=1, 2)$ given by* (6) *and* (6)′,

$$\alpha(\mathfrak{L}) = X_1 \mathfrak{L} \vee X_2 \mathfrak{L}, \quad \textit{and} \quad \beta \cdot \alpha(\mathfrak{L}) = \mathfrak{L}, \tag{11}$$

$$\beta(\mathfrak{N}) = Y_1 \mathfrak{N} \vee Y_2 \mathfrak{N} \quad \textit{and} \quad \alpha \cdot \beta(\mathfrak{N}) = \mathfrak{N}, \tag{12}$$

where $\mathfrak{L}$ and $\mathfrak{N}$ are arbitrary hyperinvariant subspaces for $S(\theta)$ and $S(N)$, respectively.

PROOF. Let $\theta = \theta_2 \theta_1$ and $N = N_2 N_1$ be the factorizations of θ and N corresponding to $\mathfrak{L}$ and $\alpha(\mathfrak{L})$, respectively. Then the proof of Lemma 2 implies that both θ_1 and N_1 are $k \times m$ matrices and both θ_2 and N_2 are $n \times k$ matrices, where $k=n$ or $k=m$. Since $X_i \mathfrak{L} \subseteqq \alpha(\mathfrak{L})$ and $Y_i \alpha(\mathfrak{L}) \subseteqq \mathfrak{L}$, it clearly follows that

$$\Delta_i \theta_2 H_k^2 \subseteqq N_2 H_k^2 \quad \text{and} \quad \Delta_i' N_2 H_k^2 \subseteqq \theta_2 H_k^2,$$

which guarantee the existence of $k \times k$ matirces A_i and B_i over H^∞ satisfying

$$\Delta_i \theta_2 = N_2 A_i \quad \text{and} \quad \Delta_i' N_2 = \theta_2 B_i. \tag{13}$$

This and (3) implies that

$$A_i \theta_1 = N_1 \Lambda_i \quad \text{and} \quad B_i N_1 = \theta_1 \Lambda_i'. \tag{13)$'$}$$

By (13) we have

$$\Delta_i' \Delta_i \theta_2 = \theta_2 B_i A_i, \tag{14}$$

and by (13)′

$$B_i A_i \theta_1 = \theta_1 \Lambda_i' \Lambda_i. \tag{14)$'$}$$

Thus, if $k=n$, then det A_i is a divisor of (det Δ_i) (det Δ_i'), and if $k=m$ then det A_i is a divisor of (det Λ_i) (det Λ_i'). To prove the first relation of (11), suppose that

$$f \in \alpha(\mathfrak{L}) \ominus \{X_1\mathfrak{L} \vee X_2\mathfrak{L}\}.$$

Then f is orthogonal to $\Delta_1\theta_2 H_k^2 \vee \Delta_2\theta_2 H_k^2$. On the other hand $f \in \alpha(\mathfrak{L})$ implies the existence of g belonging to $H_k^2 \ominus N_1 H_m^2$ such that $f = N_2 g$. Thus for every h in H_k^2, we have for $i = 1, 2$

$$(15) \qquad 0 = (f, \Delta_i\theta_2 h) = (N_2 g, N_2 A_i h) = (g, A_i h).$$

If $k = n$, then, by (5) and Beurling's theorem

$$A_i H_n^2 \supseteqq (\det A_i)\, H_m^2 \supseteqq (\det \Delta_i)\ (\det \Delta_i')\, H_n^2$$

induce $A_1 H_n^2 \vee A_2 H_n^2 = H_n^2$ and hence $g = 0$. If $k = m$, then it follows that from (13) and (4) det N_1 is a divisor of d_m, and that $A_i H_m^2 \supseteqq (\det \Delta_i)$ $(\det \Delta_i')\, H_m^2$; this implies, by (4), $N_1 H_m^2 \vee A_i H_m^2 = H_m^2$. Consequently we have $g = 0$. Thus we showed that if $k = n$, then $\alpha(\mathfrak{L}) = X_1\mathfrak{L} \vee X_2\mathfrak{L}$, and if $k = m$, then $\alpha(\mathfrak{L}) = \overline{X_1\mathfrak{L}} = \overline{X_2\mathfrak{L}}$. The rest is proved in a similar way. Thus we can conclude the proof.

COROLLARY 1. *Let θ be an $n \times m$ $(n > m)$ inner matrix over H^∞. Then for any non constant scalar inner function ϕ, $\overline{\phi(S(\theta))\, \mathfrak{H}(\theta)}$ is a non trivial hyperinvariant subspace for $S(\theta)$.*

PROOF. Since $\{X_1, X_2\}$ is a complete injective family, it is clear that

$$\overline{\alpha\big(\phi\big(S(\theta)\big)\,\mathfrak{H}(\theta)\big)} = \overline{\phi\big(S(N)\big)\,\mathfrak{H}(N)}.$$

The following relation:

$$\mathfrak{H}(M) \oplus \phi H_{n-m}^2 \supseteqq \phi\big(S(N)\big)\,\mathfrak{H}(N) \supseteqq \{0\} \oplus \phi H_{n-m}^2$$

implies that $\overline{\phi(S(N))\, \mathfrak{H}(N)}$ is trivial and hence so $\overline{\phi(S(\theta))\, \mathfrak{H}(\theta)}$ is by Theorem 2.

COROLLARY 2. *$K\phi(S(\theta)) = \{h \in \mathfrak{H}(\theta) : \phi(S(\theta))\, h = 0\}$ is a non trivial hyperinvariant subspace for $S(\theta)$ if and only if $\phi \wedge d_m \neq 1$.*

PROOF. It is clear that $K\phi(S(\theta))$ is hyperinvariant for $S(\theta)$ and

$$\alpha\big(K\phi\big(S(\theta)\big)\big) = K\phi\big(S(N)\big) = K\phi\big(S(M)\big) \oplus \{0\}.$$

Since, by the definition, we have $d_m = \det M$, we must show that

$$K\phi\big(S(M)\big) = \{0\} \quad \text{if and only if} \quad \phi \wedge (\det M) = 1.$$

But this results have already been proved in [3].

5. Restricted operators

For an arbitrary subspace $\mathfrak{L}$ of $\mathfrak{H}(\theta)$ we define the subspace $\alpha'(\mathfrak{L})$ of

$\mathfrak{H}(N)$ by

(15) $$\alpha'(\mathfrak{L}) = X_1\mathfrak{L} \vee X_2\mathfrak{L}.$$

Similarly define the subspace $\beta'(\mathfrak{N})$ of $\mathfrak{H}(\theta)$ by

(16) $$\beta'(\mathfrak{N}) = Y_1\mathfrak{N} \vee Y_2\mathfrak{N} \quad \text{for a subspace } \mathfrak{N} \text{ of } \mathfrak{H}(N).$$

Then by Theorem 2 $\alpha'(\mathfrak{L}) = \alpha(\mathfrak{L})$ if $\mathfrak{L}$ is hyperinvariant for $S(\theta)$.

THEOREM 3. *Let $\mathfrak{L}$ be a hyperinvariant subspace for $S(\theta)$. If $\mathfrak{L}'$ is a subspace of $\mathfrak{L}$, hyperinvariant for $S(\theta)|\mathfrak{L}$, then $\alpha'(\mathfrak{L}')$ is a subspace of $\alpha'(\mathfrak{L})$, hyperinvariant for $S(N)|\alpha'(\mathfrak{L})$ and $\beta'(\alpha'(\mathfrak{L}')) = \mathfrak{L}'$.*

PROOF. Let $\theta = \theta_2\theta_1$ and $N = N_2N_1$ be the factorization of θ and N corresponding to $\mathfrak{L}$ and $\alpha'(\mathfrak{L}) = \alpha(\mathfrak{L})$, respectively.

$$\mathfrak{L} = \theta_2\{H_k^2 \ominus \theta_1 H_m^2\}$$

implies that $\theta_2|\mathfrak{H}(\theta_1)$ is unitary from $\mathfrak{H}(\theta_1)$ onto $\mathfrak{L}$. Hence, in virtue of

$$\left(S(\theta)\big|\mathfrak{L}\right)\left(\theta_2\big|\mathfrak{H}(\theta_1)\right) = \left(\theta_2\big|\mathfrak{H}(\theta_1)\right)\left(S(\theta_1)\right),$$

it follows that $(\theta_2|\mathfrak{H}(\theta_1))^{-1}\mathfrak{L}'$ is hyperinvariant for $S(\theta_1)$. Now for A_i and B_i given by (13), from (14) or (14)'. $(\det A_i)$ $(\det B_i)$ is a divisor of $(\det \Delta_i)$ $(\det \Delta_i')$ or $(\det \Lambda_i)$ $(\det \Lambda_i')$. Thus by (5) or (5)' we have

(17) $$(\det A_1)\,(\det B_1) \wedge (\det A_2)\,(\det B_2) = 1.$$

It is easy to show that for $X_i' = P_{N_1}A_i|\mathfrak{H}(\theta_1)$,

$$X_1'\left(\theta_2\big|\mathfrak{H}(\theta_1)\right)^{-1}\mathfrak{L}' \vee X_2'\left(\theta_2\big|\mathfrak{H}(\theta_1)\right)^{-1}\mathfrak{L}'$$

is hyperinvariant for $S(N_1)$, by making use of (13)', (4) and (17), as we have shown Theorem 2 by making use of (3), (4), (5) and (6). Since $N_2|\mathfrak{H}(N_1)$ is unitary from $\mathfrak{H}(N_1)$ onto $\alpha'(\mathfrak{L}) = \alpha(\mathfrak{L})$,

$$\left(S(N)\big|\alpha(\mathfrak{L})\right)\left(N_2\big|\mathfrak{H}(N_1)\right) = \left(N_2\big|\mathfrak{H}(N_1)\right)S(N_1)$$

implies that

$$\begin{aligned}
&N_2\left(X_1'\left(\theta_2\big|\mathfrak{H}(\theta_1)\right)^{-1}\mathfrak{L}' \vee X_2'\left(\theta_2\big|\mathfrak{H}(\theta_1)\right)^{-1}\mathfrak{L}'\right)\\
&= N_2\left(P_{N_1}A_1\left(\theta_2\big|\mathfrak{H}(\theta_1)\right)^{-1}\mathfrak{L}' \vee P_{N_1}A_2\left(\theta_2\big|\mathfrak{H}(\theta_1)\right)^{-1}\mathfrak{L}'\right)\\
&= P_N N_2 A_1\left(\theta_2\big|\mathfrak{H}(\theta_1)\right)^{-1}\mathfrak{L}' \vee P_N N_2 A_2\left(\theta_2\big|\mathfrak{H}(\theta_1)\right)^{-1}\mathfrak{L}'\\
&= P_N\Delta_1\theta_2\left(\theta_2\big|\mathfrak{H}(\theta_1)\right)^{-1}\mathfrak{L}' \vee P_N\Delta_2\theta_2\left(\theta_2\big|\mathfrak{H}(\theta_1)\right)^{-1}\mathfrak{L}'\\
&= P_N\Delta_1\mathfrak{L}' \vee P_N\Delta_2\mathfrak{L}' = X_1\mathfrak{L}' \vee X_2\mathfrak{L}' = \alpha'(\mathfrak{L}')
\end{aligned}$$

is hyperinvariant for $S(N)|\alpha'(\mathfrak{L})$. $\beta'(\alpha'(\mathfrak{L}')) = \mathfrak{L}'$ is proved by the same way as Theorem 2. Thus we complete the proof.

The same argument as the proof of Theorem 3 yields.

THEOREM 3′. *Let $\mathfrak{N}$ be a hyperinvariant subspace for $S(N)$. If $\mathfrak{N}'$ is a subspace of $\mathfrak{N}$, hyperinvariant for $S(N)|\mathfrak{N}$, then $\beta'(\mathfrak{N}')$ is a subspace of $\beta'(\mathfrak{N})$, hyperinvariant for $S(\theta)|\beta'(\mathfrak{N})$, and $\alpha'(\beta'(\mathfrak{N}'))=\mathfrak{N}'$.*

THEOREM 4. *Let $\mathfrak{L}$ be a subspace hyperinvariant for $S(\theta)$. Then $\mathfrak{L}'$ is a subspace of $\mathfrak{H}(\theta)$, hyperinvariant for $S(\theta)$, if it is a subspace of $\mathfrak{L}$, hyperinvariant for $S(\theta)|\mathfrak{L}$.*

PROOF. Set $\alpha'(\mathfrak{L}')=\mathfrak{N}'$ and $\alpha'(\mathfrak{L})=\alpha(\mathfrak{L})=\mathfrak{N}$. Theorem 3 implies that $\mathfrak{N}'$ is hyperinvariant for $S(N)|\mathfrak{N}$. Let $N=N_2N_1$ be the factorization of N corresponding to $\mathfrak{N}$. Then $(N_2|\mathfrak{H}(N_1))^{-1}\mathfrak{N}'$ is a subspace of $\mathfrak{H}(N_1)$, hyperinvariant for $S(N_1)$. Since N_1 is a $k\times m$ ($k=n$ or $k=m$) normal inner matrix over H^∞, by Theorem 1 there is an $l\times m$ normal inner matrix N_1' and an $k\times l$ normal inner matrix N_2' such that

$$N_1=N_2'N_1' \quad \text{and} \quad \left(N_2\middle|\mathfrak{H}(N_1)\right)^{-1}\mathfrak{N}'=N_2'\{H_l^2\ominus N_1'H_m^2\},$$

where $n\geqq k\geqq l\geqq m$, and $l=m$ or $l=n$. It is easy to show that N_2N_2' and N_1' satisfy the condition (i) or the condition (ii) of Theorem 1; this implies that

$$\mathfrak{N}'=N_2N_2'\{H_l^2\ominus N_1'H_m^2\}$$

is hyperinvariant for $S(N)$. Thus

$$\beta(\mathfrak{N}')=\beta'(\mathfrak{N}')=\beta'\left(\alpha'(\mathfrak{L}')\right)=\mathfrak{L}'$$

is hyperinvariant for $S(\theta)$. Thus we conclude the proof.

Now, we determine a particular hyperinvariant subspace $\mathfrak{L}_*$ for $S(\theta)$ by the following relation:

$$\mathfrak{L}_*=\left\{h\in\mathfrak{H}(\theta):\ S(\theta)^n h\longrightarrow 0 \text{ as } n\longrightarrow\infty\right\} \quad ([7]\ \text{P. } 73).$$

Then, from $\alpha(\mathfrak{L}_*)\subseteqq\mathfrak{H}(M)$ and $\beta(\mathfrak{H}(M))\subseteqq\mathfrak{L}^*$, it follows that $\alpha(\mathfrak{L}_*)=\mathfrak{H}(M)$.

THEOREM 5. *Let $\mathfrak{L}$ be a subspace hyperinvariant for $S(\theta)$. In order that if $\mathfrak{L}'$ is a subspace of $\mathfrak{L}$, hyperinvariant for $S(\theta)$, then $\mathfrak{L}'$ is hyperinvariant for $S(\theta)|\mathfrak{L}$, it is necessary and sufficient that there is a function ϕ in H^∞ such that*

$$\mathfrak{L}=\overline{\phi\left(S(\theta)\right)\mathfrak{H}(\theta)} \quad \text{or} \quad \mathfrak{L}=\overline{\phi\left(S(\theta)\right)\mathfrak{H}(\theta)}\cap\mathfrak{L}_*.$$

PROOF. SUFFICIENCY. Case a: suppose $\mathfrak{L}=\overline{\phi(S(\theta))\mathfrak{H}(\theta)}$ and hence $\alpha(\mathfrak{L})=\overline{\phi(S(N))\mathfrak{H}(N)}$. Let $N=N_2N_1$ be the factorization corresponding to $\alpha(\mathfrak{L})$. Then $N_2=\text{diag}(\phi_1,\cdots,\phi_m,\phi,\cdots,\phi)$, where $\phi_i=\phi\wedge\psi_i$ for $i=1,2,\cdots,m$. Set $\phi=\phi_i u_i$ and $\psi_i=\phi_i v_i$ for $i=1,2,\cdots,m$. Then it follows that for $i=$

$1, 2, \cdots, m-1,$

$$\phi_{i+1}=\phi\wedge\psi_{i+1}=\phi_i u_i\wedge\phi_i v_i\frac{\psi_{i+1}}{\phi_i}=\phi_i\left(u_i\wedge v_i\frac{\psi_{i+1}}{\phi_i}\right).$$

Since $u_i\wedge v_i=1$, this implies that

$$\frac{\phi_{i+1}}{\phi_i}\wedge v_i=1. \tag{18}$$

Let $\mathfrak{L}'$ be a subspace of $\mathfrak{L}$, hyperinvariant for $S(\theta)$. Then there is the factorization $N_1=N_2'N_1'$, where N_1' is a $k\times m$ inner matrix and N_2' is an $n\times k$ inner matrix, such that $\alpha(\mathfrak{L}')=N_2N_2'\{H_k^2\ominus N_1'H_m^2\}$ (see [7] P. 291). The hyperinvariance of $\alpha(\mathfrak{L}')$ implies that N_2N_2' and N_1' are normal inner matrices satisfying (i) or (ii) of Theorem 1. First, assume (i). Then N_1' is an $m\times m$ normal inner matrix and hence N_2' is an $n\times m$ inner matrix. From the normalities of N_2N_2' and N_2, we can deduce that N_2' has the form $\begin{bmatrix}M'\\0\end{bmatrix}$, where $M'=\operatorname{diag}(t_1, t_2, \cdots, t_m)$. Since $\phi_i t_i$ is a divisor of ψ_i, it follows that t_i is a divisor of v_i and, by (18), $\frac{\phi_{i+1}}{\phi_i}\wedge t_i=1$. Then normality of N_2N_2' implies that there is an inner function w_i such that $w_i=\frac{\phi_{i+1}t_{i+1}}{\phi_i t_i}$. From $t_i w_i=\frac{\phi_{i+1}}{\phi_i}$ t_{i+1}, it follows that t_i is a divisor of t_{i+1}. Thus N_2' is normal. Hence N_2^{-1} $\alpha(\mathfrak{L}')=N_2'\{H_m^2\ominus N_1'H_m^2\}$ is hyperinvariant for $S(N_1)$. Therefore $\alpha(\mathfrak{L}')$ is hyperinvariant for $S(N)|\alpha(\mathfrak{L})$. Consequently $\beta'(\alpha(\mathfrak{L}'))=\beta(\alpha(\mathfrak{L}'))=\mathfrak{L}'$ is hyperinvariant for $S(\theta)|\mathfrak{L}$. Next assume that N_2N_2' and N_1' satisfy (ii). Then we have $N_2'=\operatorname{diag}(t_1, \cdots, t_m, t, \cdots, t)$, for inner functions $t_1, t_2, \cdots, t_m$ and t. It is proved as above that t_i is a divisor of t_{i+1} for $i=1, 2, \cdots, m-1$. Since ϕ_m t_m is a divisor of ϕt, t_m is a divisor of $u_m t$. On the other hand since t_m is a divisor of v_m and $v_m\wedge u_m=1$, t_m is a divisor of t. Thus it follows that N_2' is normal. Consequently in the same way as above we can deduce that $\mathfrak{L}'$ is hyperinvariant for $S(\theta)|\mathfrak{L}$.

Case b: suppose $\mathfrak{L}=\overline{\phi(S(\theta))\mathfrak{H}(\theta)}\cap\mathfrak{L}_*$. Then by Corollary 1 and $\alpha(\mathfrak{L}_*)$ $=\mathfrak{H}(M)$ we have

$$\alpha(\mathfrak{L})=\overline{\phi\big(S(N)\big)\mathfrak{H}(N)}\cap\mathfrak{H}(M)=\overline{\phi\big(S(M)\big)\mathfrak{H}(M)},$$

because α is a lattice isomorphism. Let $N=N_2N_1$ be the factorization corresponding to $\alpha(\mathfrak{L})$. Then it follows that

$$N_2=\begin{bmatrix}M_2\\0\end{bmatrix}\quad\text{with}\quad M_2=\operatorname{diag}(\phi_1, \phi_2, \cdots, \phi_m),$$

where $\phi_i=\phi\wedge\psi_i$ for $i=1, 2, \cdots, m$. Let $\mathfrak{L}'$ be a subspace of $\mathfrak{L}$, hyperinvariant for $S(\theta)$, and $N_1=N_2'N_1'$ be the factorization of N_1 such that $N=$

$(N_2N_2')\,N_1'$ is the factorization of N corresponding to $\alpha'(\mathfrak{L}')=\alpha(\mathfrak{L}')$. The hyperinvariance of $\alpha(\mathfrak{L}')$ for $S(N)$ implies that N_2N_2' and N_1' are normal inner matrices satidfying (i). In the same way as Case a it follows that N_2' is an $m\times m$ normal inner matrix. Therefore it is simple to show that $\mathfrak{L}'$ is hyperinvariant for $S(\theta)|\mathfrak{L}$.

NECESSITY. Let $\mathfrak{L}$ be the hyperinvariant subspace for $S(\theta)$ such that $\mathfrak{L}'$ is hyperinvariant for $S(\theta)|\mathfrak{L}$, if $\mathfrak{L}'$ is a subspace of $\mathfrak{L}$, hyperinvariant for $S(\theta)$. Then, for every subspace $\mathfrak{N}'$ of $\alpha(\mathfrak{L})$ such that $\mathfrak{N}'$ is hyperinvariant for $S(N)$, it follows from Theorem 3 that $\beta(\mathfrak{N}')=\beta'(\mathfrak{N}')$ is hyperinvariant for $S(\theta)|\mathfrak{L}$. Hence, by Theorem 3, $\mathfrak{N}'=\alpha'(\beta'(\mathfrak{N}'))$ is hyperinvariant for $S(N)|\alpha(\mathfrak{L})$. Let $N=N_2N_1$ be the factorization corresponding to $\alpha(\mathfrak{L})$. Then N_2 and N_1 are normal inner matrices.

Case a': assume that N_1 and N_2 have the form:

$N_1=\mathrm{diag}\,(\xi_1,\xi_2,\cdots,\xi_m)$ and

$$N_2=\begin{bmatrix}M_2\\0\end{bmatrix}\quad\text{with}\quad M_2=\mathrm{diag}\,(\eta_1,\eta_2,\cdots,\eta_m).$$

Then it follows that η_i and ξ_i satisfy (18), that is $\frac{\eta_{i+1}}{\eta_i}$ and ξ_i are relatively prime. In fact, if it were not true, then we have

$$\omega\equiv\frac{\eta_{i+1}}{\eta_i}\wedge\frac{\xi_j}{\xi_{j-1}}\neq 1\quad\text{for some}\quad j:\ 1\leqq j\leqq i,\ \xi_0=1.$$

Set

$$M_2'=\mathrm{diag}\,(\eta_1,\cdots,\eta_{j-1},\eta_j\omega,\eta_{j+1}\omega,\cdots,\eta_i\omega,\eta_{i+1},\cdots,\eta_m)$$

$$N_1'=\mathrm{dag}\left(\xi_1,\cdots,\xi_{j-1},\frac{\xi_j}{\omega},\frac{\xi_{j+1}}{\omega},\cdots,\frac{\xi_i}{\omega},\xi_{i+1},\cdots,\xi_m\right)$$

and $N_2'=\begin{bmatrix}M_2'\\0\end{bmatrix}$. It is clear that $\mathfrak{N}'\equiv N_2'\{H_m^2\ominus N_1'H_m^2\}$ is a subspace of $\alpha(\mathfrak{L})$. Since N_1' and N_2' are normal inner matrices, by Lemma 1 $\mathfrak{N}'$ is hyperinvariant for $S(N)$. However,

$$\left(N_2\middle|\mathfrak{H}(N_1)\right)^{-1}N_2'\mathfrak{H}(N_1')=\mathrm{diag}\,(1,\cdots,1,\omega,\cdots,\omega,1,\cdots 1)\,\mathfrak{H}(N_1')$$

implies that $\mathfrak{N}'$ is not hyperinvariant for $S(N)|\alpha(\mathfrak{L})$. Thus we have $\frac{\eta_{i+1}}{\eta_i}<\xi_i=1$. Since ξ_i is a divisor of ξ_{i+1}, it follows that

$$\eta_m\wedge\psi_i=\eta_m\wedge(\eta_i\xi_i)=\eta_i\left(\frac{\eta_m}{\eta_i}\wedge\xi_i\right)=\eta_i.$$

Thus we have

$$\alpha(\mathfrak{L})=\overline{\eta_m\left(S(M)\right)\mathfrak{H}(M)}=\overline{\eta_m\left(S(N)\right)\mathfrak{H}(N)}\cap\mathfrak{H}(M).$$

Consequently $\mathfrak{L}=\overline{\eta_m\,(S(\theta))\,\mathfrak{H}\,(\theta)}\cap\mathfrak{L}_*$.

Case b': assume that N_1 and N_2 are normal inner matrices satisfying (ii). In this case, we can show

$$\mathfrak{L} = \overline{\phi\big(S(\theta)\big)\mathfrak{H}(\theta)} \quad \text{for some } \phi \text{ in } H^\infty$$

in the same way as Case a'. Thus we complete the proof of Theorem 5.

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