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On a parametrix for the hyperbolic mixed problem with diffractive lateral boundary

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§ 1. Introduction and results.

Let Ω be a domain of the closed half space $\overline{R_{+}^{n+1}}=\{x;\ x=(x', x_n),\ x'=(x_0, \cdots, x_{n-1}),\ x_n\geqq 0\}$ containing a neighborhood of a point $x^{0'}=(0, x_1^0, \cdots, x_{n-1}^0)$ in its lateral boundary $\Gamma=\{x;\ x\in\Omega,\ x_n=0\}$ and let $P(x, D)$ be a differential operator of order 2 with C^{∞} coefficients on $\bar{\Omega}$, which is normal hyperbolic with respect to x_0.

Now let us consider the mixed problem in Ω; for some $\delta>0$

$$(1.1)\qquad (P, B)\begin{cases} P(x, D)\, u=0 & \text{in } \Omega \text{ and for } x_n<\delta, \\ B(x, D)\, u=f & \text{in } \Gamma, \\ u=0 & x_0<0 \end{cases}$$

where $[0, \delta]\times\Gamma\subset\Omega$, the given boundary data f vanishes for $x_0<0$, $B(x, D)$ is a differential operator of order 1, and Γ is non-characteristic with respect to B.

Reweiting the principal symbol $P_2(x, \xi)$ of $P(x, D)$ in the following form:

$$(1.2)\qquad P_2(x, \xi)=\left(\xi_n-\lambda(x, \xi')\right)^2-\mu(x, \xi'),\ \xi=(\xi', \xi_n),$$

we assume that Γ is diffractive *i. e.*, that for $(x, \xi)\in T_{\Gamma}^{*}(\Omega)$

$$(1.3)\qquad \left\{\xi_n-\lambda(x, \xi'),\ \mu(x, \xi')\right\}>0 \text{ when } \xi_n=\lambda(x, \xi') \text{ and } \mu(x, \xi')=0$$

where $\{f, g\}$ is the Poisson bracket and then such points $(x, \xi')\in T^{*}(\Gamma)$ with above properties are called diffractive ([8]).

Near a fixed diffractive point $(x^0, \xi^{0'})$, let $\lambda^{\pm}(x, \xi')$ be roots of $P_2(x^0, \xi', \xi_n)=0$ with respect to ξ_n such that for $\xi_0>0$

$$(1.4)\qquad \begin{aligned} &\lambda^{\pm}(x, \xi')=\lambda(x, \xi')\mp\sqrt{\zeta}\,\mu_3^{\frac{1}{2}}(x, \xi'), \\ &\mu_3(x, \xi')>0, \\ &\zeta=\xi_0-\mu_2(x, \xi'')\ \left(\xi''=(\xi_1, \cdots, \xi_{n-1})\right), \\ &\mu_2(x, \xi'') \text{ is real valued}, \\ &\sqrt{1}=1 \text{ and } \sqrt{-1}=-i. \end{aligned}$$

The representation of $\lambda^{\pm}(x, \xi')$ follows from the normal hyperbolicity of P_2 and non-characteristicity of Γ with respect to P. Denoting the principal symbol of B by $B_1(x, \xi)$ and assuming $B_1(x^0, \xi^{0'}, \lambda(x^0, \xi^{0'}))=0$ we can decompose the function $B_1(x, \xi', \lambda^+(x, \xi'))$ as follows:

$$B_1\left(x, \xi', \lambda^+(x, \xi')\right) = R_{\frac{1}{2}}(x, \sqrt{\zeta}, \xi'')\left(\sqrt{\zeta} - D(x, \xi'')\right),$$
$$R(x, \sqrt{\zeta}, \xi'') \neq 0 \text{ and } D(x^0, \xi^{0''}) = 0.$$

Now we suppose in this paper that for some real number $\theta\left(0<\theta<\frac{\pi}{2}\right)$ all of the values of $e^{i\theta}D(x, \xi'')$ lay in a proper cone, with its vertex at the origin, contained in left half of the complex plane, *i. e.*

$$\frac{\pi}{2} + \varepsilon \leqq \arg\left(e^{i\theta}D(x, \xi'')\right) \leqq \frac{3\pi}{2} - \varepsilon \tag{1.5}$$

for some $\varepsilon\,(0<\varepsilon<<1)$. Then we shall prove the following

THEOREM. *Under the conditions described above there exists a parametrix for (P, B) near a diffractive point $(x^0, \xi^{0'})$ whose wave front sets behave themselves similarly to the case of Dirichlet or Neumann boundary value problems.*

The case of Dirichlet and Neumann boundary conditions were considered by Ludwing, Melrose, Taylor and Eskin ([7], [8], [11] and [2]). For more general case, where boundary operator B has real coefficients, it was treated by Ikawa [6] in somewhat different point of view.

On the other hand we obtained in [10] that the problem (P, B) is L^2-well posed only if for $\xi_0>0$

$$\frac{\pi}{2} \leqq \arg\left(D(x', \xi'')\right) \leqq \frac{3}{2}\pi,$$

and from Theorem we have that the above parametrix for $\xi_0>0$ is constructed if

$$\frac{\pi}{2} \leqq \arg\left(D(x', \xi'')\right) \leqq \frac{3}{2}\pi - \varepsilon$$

for a positive $\varepsilon \ll 1$ and for (x', ξ'') near $(x^0, \xi^{0''})$. Furtheoremore from the proof of Theorem we have that the condition (2. 6) described below is also sufficient to obtain such parametrix. Though the conditions (1. 5) and (2. 6) are invariant under transformation of space-variables $(x_1 \cdots x_n)$ preserving unit normal to Ω at Γ, available relations between $D(x', \xi'')$ and $c(x', \xi')$ defined below are not always clear. Moreover if we want to treat mixed problems for a equation of higher order or a system of equations of the first order, we may reduce these problems to ones considered here, but operators P and B must be changed into pseudo-differential ones and then Theorem will be also valid without any change. Here we must emphasize that even if the coefficients of boundary operators are real, the corresponding symbol $D(x', \xi'')$ is not always real and that in general there are essential gaps between the necessary condition to be L^2-well posed for the problem (P, B) and these conditions described in this paper to be able to reduce our problem to a subelliptic interior problem on Γ, even if we restrict ourselves to the case where the Lopatinskii determinant $B_1(x, \xi', \lambda^+(x, \xi'))$ does not vanish for $x' \in \Gamma$ and for $\zeta - i\gamma \neq 0$ with $\gamma \geqq 0$ (see [10]).

To prove Theorem, it seems to be convenient to use phase functions constructed by Ludwig and Eskin ([7] and [2]). Throughout this paper, functions in consideration are assumed to be of C^∞-class without any confusions, a boundary point $(x', 0)$ is simply denoted by x' in symbols and we suppose that $\xi_0 > 0$.

§ 2. Preliminaries.

(i) First of all, for the sake of completeness of our descriptions, following Taylor's and Eskin's considerations we construct a solution φ of the eikonal equation in a conic neighborhood in $T^*(\Omega)$ of $(x^0, \xi^0) \in T^*_\Gamma(\Omega)$

$$(2.1) \qquad \varphi_{x_n} - \lambda\big(x, \varphi_{x'}\big)^2 - \mu(x, \varphi_{x'}) = 0$$

with the following properties:

α) There are real functions $\theta(x, \eta')$ and $\rho(x, \eta')$ with new variables $\eta' = (\eta_0, \eta_1, \cdots, \eta_{n-1})$ with $\eta' \neq 0$, homogeneous in η' of order 1 and $\frac{2}{3}$ respectively such that

$$\varphi_\pm(x, \eta') = \left(\theta \pm \frac{2}{3} \rho^{\frac{3}{2}}\right)(x, \eta')$$

satisfies (2.1) for $\rho \geqq 0$ when (x', x_n) are contained in some neighborhood $U(x^0)$ of x^0 and $\eta' = (\eta_0, \eta'')$ belong to some conic neighborhood of $(\eta_0^0, \eta^{0''})$ with $\eta_0^0 = 0$, $\eta^{0''} = \xi^{0''}$.

β) $\det \left\| \dfrac{\partial^2 \theta}{\partial x_j \partial \eta_k} \right\|_{j,k=0}^{n-1} \neq 0$ for $x_n = 0$, which means that the phase function $\theta(x, \eta') - \langle y', \eta' \rangle$ is non-degenerate for fixed x_n with $\delta \geqq x_n \geqq 0$ for some $\delta > 0$,

$$\rho = \big(\alpha + 0(\alpha^\infty)\big) |\eta'|^{\frac{2}{3}} \qquad \text{for } x_n = 0,$$

where $\alpha = \eta_0 / |\eta'|$ and

$$\frac{\partial \rho}{\partial x_n} > 0 \text{ when } \rho = 0 .$$

Next using $\varphi_\pm$ and solving transport equations we can construct functions $g(x, \eta')$ and $h(x, \eta')$ such that for $\rho \geqq 0$, in particular $\alpha \geqq 0$ (and $\alpha < \alpha_0$ for some $\alpha_0 < 1$) and for $x_n < \delta$:

$$g(x, \eta') \sim \sum_{j=0}^{\infty} g_{-j}(x, \eta') ,$$

$$h(x, \eta') = 0(\alpha^\infty |\eta'|^{-\frac{1}{3}}) ,$$

$$\text{ord}_{\eta'} g_{-j} = -j, \ g_0|_{x_n=0} \neq 0 ,$$

and

$$(2.2) \qquad P(x, D) \int_L (g - \tau h)\, e^{i(\frac{\tau^3}{3} - \rho\tau + \theta)} d\tau = 0(|\eta'|^{-\infty})$$

where L is a complex contour

$$\tau = \begin{cases} |t| e^{-\frac{\pi}{2} i} & t \to -\infty , \\ t e^{\frac{\pi}{6} i} & t \to \infty \end{cases}$$

which is different from Eskin's. By using one of Airy functions defined by

$$A(x) = \int_L e^{i(\frac{\tau^3}{3} - x\tau)} d\tau \Big(= (2\pi)\, e^{\frac{\pi}{3} i} Ai\big(e^{-\frac{2}{3}\pi i}(-x)\big)\Big)$$

([9]) the integral in (2. 2) can be written in the following form:

$$(2.3) \qquad \int_L (g - \tau h)\, e^{i(\frac{\tau^3}{3} - \rho\tau)} d\tau = g A(\rho) - i h A'(\rho)$$

and by the choice of L $A(\rho)$, $A'(\rho)$ have the same asymptotic behaviors, (when $\rho \to \pm\infty$) as in [2], [8]. By applying such asymptotic behaviors and the fact that $A(\alpha|\eta'|^{\frac{2}{3}})$, $A'(\alpha|\eta'|^{\frac{2}{3}})$ do not vanish for real α, the relation (2. 2) can be smoothly extended for negative α such that $-|\eta'|^{-\varepsilon} < \alpha$, where $0 < \varepsilon < \frac{1}{3}$, if we extend $g_{-j}(x, \eta')$ and $h(x, \eta')$ smoothly over such α.

Furthermore for negative α such the $-\frac{1}{2}|\eta'|^{-\varepsilon} > \alpha$, it is shown by Eskin that for $x_n = 0$ setting

$$a_1(x', \eta')\, \alpha = \theta_{x_n}(x', \eta') - \lambda\big(x', \theta_{x'}(x', \eta')\big) ,$$

$$\gamma) \qquad a_2(x', \eta')\, \alpha = \mu\big(x', \theta_{x'}(x', \eta')\big) \big(a_2(x', \eta') > 0\big) \text{ and}$$

$$\theta_1(x, \eta) = i x_n \big((a_2(x', \eta')|\alpha|\big)^{\frac{1}{2}} - x_n a_1(x', \eta')\, \alpha ,$$

the function $(\theta+\theta_1)(x, \eta')$ satisfies the eikonal equation, *i. e.*

$$\left((\theta+\theta_1)_{x_n}-\lambda(x, \theta_{x'})\right)^2-\mu(x, \theta_{x'})=0$$

for $x^n=0$.

By using the above facts, it can be constructed a parametrix for P as follows: choose $\beta_1\in C_0^\infty(R^1)$ and $\beta_2\in C^\infty(R^1)$ such that for some $\alpha_0>0$ $\beta_1(t)=1$ for $|t|<\alpha_0$, $\beta_1(t)=0$ for $|t|>2\alpha_0$ and for some $t_0>0$ $\beta_2(t)=1$ for $t>t_0\gg 1$, $\beta_2(t)=0$ for $t<\frac{1}{2}t_0$. Let $\chi(\eta')=\beta_1\left(\left|\frac{\eta'}{|\eta'|}-\frac{\eta^{0'}}{|\eta^{0'}|}\right|\right)\beta_2(|\eta'|)$ where $\eta^{0'}=(o, \eta^{0''})$ such that $\xi^{0'}=\theta_{x'}(x^0, \eta^{0'})$. Moreover choose χ_1 and $\chi_{-1}\in c^\infty(R^1)$ such that

$$\chi_1(t)=1 \quad \text{for } t>2c,\ \chi_1(t)=0 \quad \text{for } t<c\,;$$

$$\chi_{-1}(t)=1 \quad \text{for } t<-2c,\ \chi_{-1}(t)=0 \quad \text{for } t>-c \text{ and}$$

$$\chi_0(t)=\left(1-\chi_1(t)\right)\left(1-\chi_{-1}(t)\right), \qquad \text{where } c>0\,.$$

Then we define

$$\begin{aligned}GV(x)=&\iint_L\left(g(x, \eta')-\tau h(x, \eta')\right)e^{i(\frac{\tau^3}{3}-\rho\tau+\theta)}d\tau\times\\ \times&\left(A(\alpha|\eta'|^{\frac{2}{3}})^{-1}\right)\left(\chi_1(\alpha|\eta'|^{\epsilon})+\chi_0(\alpha|\eta'|^{\epsilon})\right)\chi(\eta')\,\hat{V}(\eta')\,d\eta'\,,\\ +&\int e^{i(\theta+\theta_1)(x,\eta')}\,d(x, \eta')\,\chi_{-1}(\alpha|\eta'|^{\epsilon})\,\chi(\eta')\,\hat{V}(\eta')\,d\eta'\,,\end{aligned}\tag{2. 4}$$

which is denoted by

$$=G_1\left((\chi_1+\chi_0)\,\chi V\right)+G_2(\chi_{-1}\chi V)\,,$$

where $V\in\mathcal{E}'(R^n)$ and $\hat{V}$ is a Fourier transform of V and where $d(x, \eta')$ is equal to $g_0(x, \eta')$ for $x_n=0$ and $g_0(x', \eta')$ may be assumed to be real. Taking account of the asymptotic behaviors, β) and γ) we may conclude that $G|_{x_n=0}$ is a Fourier integral operator with symbol class $S^0_{\frac{1}{3},0}$ and the phase function $\varphi_0(x, \eta')$ homogeneous in η' of order 1 such that

$$\varphi_0(x, \eta')=\begin{cases}\theta(x, \eta')-\frac{2}{3}\rho(x, \eta')^{\frac{3}{2}}+\frac{2}{3}\alpha^{\frac{3}{2}}|\eta'| & \text{for } \alpha\geqq 0\,,\\ \theta(x, \eta') & \text{for } \alpha<0\,.\end{cases}$$

(ii) Before we consider boundary conditions with respect to GV, we shall show the following lemmas which are used in the next section.

LEMMA 2. 1. *For real z both of real and imaginary parts of $A'(z)/A(z)$ are negative.*

(PROOF) Let ω be $e^{\frac{2}{3}\pi i}$, then from the definition of A we see that

$$(2\pi e^{\frac{\pi}{3}i})^{-1} A(z) = Ai\left(\omega^2(-z)\right).$$

Therefore using Miller's formula ([9]) we obtain that

$$(2\pi e^{\frac{\pi}{3}i})^{-1} A(z) = -\frac{1}{2}\omega\left\{Ai(-z)+i\,Bi(-z)\right\},$$

and then that

$$(2\pi e^{\frac{\pi}{3}i})^{-1} A'(z) = \frac{1}{2}\omega\left\{Ai'(-z)+i\,Bi'(-z)\right\},$$

from which it follows that

$$A'(z)/A(z) = (-1)\left\{Ai(-z)^2+Bi(-z)^2\right\}^{-1}\Big[\left\{Ai'(-z)\,Ai(-z) + Bi'(-z)\,Bi(-z)\right\}+i\left\{Bi'(-z)\,Ai(-z)-Ai'(-z)\,Bi(-z)\right\}\Big].$$

Here the derivative of the last term in [ ] is zero since Ai and Bi are both Airy functions and in fact

$$Bi'(-z)\,Ai(-z)-Ai'(-z)\,Bi(-z) = \frac{1}{\pi} \quad ([9]).$$

On the other hand the first term in [ ] is the derivative of the function $(Ai(-z)^2+Bi(-z)^2)/2$ which is an increasing function with respect to $(-z)$ and does not vanish for all $z\in R$. Thus our assertion holds for such z.

LEMMA 2.2. *Set* $z=\alpha|\eta'|^{\frac{2}{3}}$, *then*

$$A'(\alpha|\eta'|^{\frac{2}{3}})/A(\alpha|\eta'|^{\frac{2}{3}}) = -\left\{|z|^{-1}\left(\frac{1}{4}+O(|z|^{-3})\right)+i|z|^{\frac{1}{2}}\left(1+O(|z|^{-3})\right)\right\}$$

or

$$= (-1)\,|z|^{\frac{1}{2}}\left(1+O(|z|^{-\frac{3}{2}})\right),$$

when $\alpha|\eta'|^{\frac{2}{3}}\gg 1$ *or* $-\alpha|\eta'|^{\frac{2}{3}}\gg 1$ *respectively. Here by* $O(|z|^k)$ *we means that*

$$O(|z|^k) = \sum_{\nu=0}^{\infty}|z|^k(c_\nu|z|^{-\frac{3}{2}\nu}), \text{ and}$$

$$c_\nu \text{ is real for } \nu=0, 1, 2, \cdots$$

which are uniformly asymptotic expansions when $|z|\to\infty$ *and can be differentiated term-by-term.*

(PROOF) Denoting t by $e^{-\frac{2}{3}\pi i}(-\alpha|\eta'|^{\frac{2}{3}})$, from the definition of A, we see that

$$A(\alpha|\eta'|^{\frac{2}{3}}) = \pi^{\frac{1}{2}} e^{\frac{\pi}{3}i} e^{-\frac{2}{3}t^{\frac{3}{2}}}(a_0t^{-\frac{1}{4}}+a_1t^{-\frac{1}{4}-\frac{3}{2}}+\cdots)$$

for $|\alpha|\ |\eta'|^{\frac{2}{3}}\gg 1$, where $a_0\neq 0$ and a_i are real. Therefore we see that

$$A'(\alpha|\eta'|^{\frac{2}{3}})=\pi^{\frac{1}{2}}e^{\frac{\pi}{3}i}e^{-\frac{2}{3}t^{\frac{3}{2}}}e^{-\frac{2}{3}\pi i}\left(a_0t^{\frac{1}{4}}+\left(a_1+\frac{a_0}{4}\right)t^{-\frac{1}{4}-1}+\cdots\right),$$

from which

$$A'(\alpha|\eta'|^{\frac{2}{3}})/A(\alpha|\eta'|^{\frac{2}{3}})$$
$$=e^{-\frac{2}{3}\pi i}\left(t^{\frac{1}{2}}+\frac{1}{4}t^{\frac{1}{2}-\frac{3}{2}}+\cdots\right)$$

for $|t|\gg1$. Then, since $\arg t=\frac{\pi}{3}$ for $\alpha>0$ and $\arg t=-\frac{2}{3}\pi$ for $\alpha<0$, we have that our assertions are valid.

Here we remark that in the case where $\alpha<0$ the imaginary part of $A'(\alpha|\eta'|^{\frac{2}{3}})/A(\alpha|\eta'|^{\frac{2}{3}})$ is neglected, since it is decreasing exponentially when $|\alpha|\ |\eta'|^{\frac{2}{3}}\to\infty$. Furthermore we see easily that the asymptotic expansions are valid for any z such that $|z|>z_0>0$.
But later on we shall use Lemma 2. 1 only for small $|z|$ and Lemma 2. 2 for sufficiently large $|z|$.

Furthermore we remark the following

LEMMA 2. 3. *The following three conditions are equivalent*:
1) $\alpha=0$,
2) *Setting* $\zeta(x,\eta')=\theta_{x_0}-\sqrt{\rho}\rho_{x_0}-\mu_2(x',\theta_{x''}-\sqrt{\rho}\rho_{x''})$, $\zeta(x',\eta')=0$ *and*
3) $\theta_{x_n}-\sqrt{\rho}\rho_{x_n}$ *is a real double root of* $P_2(x,\theta_{x'}-\sqrt{\rho}\rho_{x'},\xi_n)=0$ *with respect to* ξ_n.

(PROOF) From the method of construction of θ and ρ ([2]) we see that for $\alpha=0$ $\theta_{x_n}=\lambda(x',\theta_{x'})$ and $\mu(x',\theta_{x'})=0$ which means 3) is valid. Moreover from (1. 4) it follows that

$$\mu(x',\xi')=\zeta(x',\xi')\,\mu_3(x',\xi')$$

and therefore that $\zeta(x,\eta')=0$ if and only if 3) is true. Finally also the one of construction of ρ and θ implies that $\partial\zeta(x',\eta')/\partial\alpha\neq0$ for $(x',\eta')=(x^0,\eta^0{}')$ when $|\eta'|=1$. Thus we obtain that if $\zeta(x,\eta')=0$, then $\alpha=0$. The proof is complete. Here remark that $\mu_2(x',\xi'')$ is just the symbol $\theta(x',\xi'')$ in [10].

Finally we show the following

LEMMA 2. 4. *Let* $B_1(x',\xi)=\xi_n-\lambda(x',\xi')+c(x',\xi')$. *Then for* $\alpha=0$ $c(x',\theta_{x'})$ *is rewritten in the following form*:

$$c(x',\theta_{x'})(x',o,\eta'')=\left\{\mu_3^{\frac{1}{2}}(x',\theta_{x'})-R_{\frac{1}{2}}^{(0)}(x',\theta_{x'})\,D(x',\theta_{x''})\right\}\times D(x',\theta_{x''})(x',o,\eta''),\tag{2. 5}$$

where $R_{\frac{1}{2}}^{(0)}(x',\theta_{x'})$ is defined below.

(PROOF) Let $\zeta(x', \xi')=\zeta_0-\mu_2(x', \xi'')$. Considering $\{x', \zeta, \xi''\}$ as new variables, by $a(x', \zeta, \xi'')$ here we denote the function $a(x', \xi_0, \xi'')$. Then from our assumption and (1.4) it follows that

$$B_1\left(x', \xi', \lambda^+(x', \xi')\right)=R_{\frac{1}{2}}(x', \sqrt{\zeta}, \xi'')\left(\sqrt{\zeta}-D(x', \xi'')\right)$$
$$=-\sqrt{\zeta}\mu_3^{\frac{1}{3}}(x', \xi'')+c(x', \zeta, \xi'').$$

Therefore expanding both sides of the above equation with respect to $\sqrt{\zeta}$ and comparing with the first and the second terms in ones, by analiticities of the above functions in $\sqrt{\zeta}$ we see that

$$c(x', o, \xi'')=-R_{\frac{1}{2}}(x', o, \xi'')\, D(x', \xi'') \text{ and}$$
$$-\mu_3^{\frac{1}{3}}(x', o, \xi'')=R_{\frac{1}{2}}(x', o, \xi'')-R_{\frac{1}{2}}^{(0)}(x', o, \xi'')\, D(x', \xi'').$$

Thus using Lemma 2.3 and setting $\xi'=\theta_{x'}(x', \eta')$ we see that our assertion is valid.

COROLLARY. *For the same θ as in* (1.5)

$$\frac{\pi}{2} \leqq \arg\left(e^{i\theta} c(x', \theta_{x'})(x', o, \eta'')\right) \leqq \frac{3\pi}{2}. \tag{2.6}$$

Since $\mu_3^{\frac{1}{3}}(x', \theta_{x'})>0$ and $D(x^0, \theta_{x''}(x^0, \eta^{0'}))=0$, we have that our assertion is true if we restrict ourselves to a smaller conic neighborhood of $(x^0, \eta^{0'})$. Here we must mention that the term $c(x', \theta_x')(x', \eta')-c(x', \theta_{x'})(x', o, \eta'')$ does not play any essential role in the next section (Compare with that in [10]).

§ 3. Proof of Theorem.

(i) We first calculate the symbol of $BG_1((\chi_1+\chi_0)\chi V)|_{x_n=0}$:

$$B\left(G_1\left((\chi_1+\chi_0)\chi V\right)\Big|_{x_n=0}\right)(x')$$
$$=\iint_L \Big(B_1\left(x', (\theta_x-\tau\rho_x)_0\right)\left(g_0(x', \eta')-\tau h_0(x', \eta')\right)+$$
$$+B_0(x', \eta', \tau)\Big)\, e^{i(\frac{\tau^3}{3}-\rho_+ +\theta)}\, d\tau\, \frac{(\chi_1+\chi_0)(\alpha|\eta'|^{\varepsilon})\,\chi(\eta')}{A(\alpha|\eta'|^{\frac{2}{3}})}\, \hat{V}(\eta')\, d\eta'$$

where $(\theta_x+\tau\rho_x)_0=(\theta_x+\tau\rho_x)|_{x_n=0}$, $\rho_0=\rho|_{x_n=0}$, $\theta_0=\theta|_{x_n=0}$ and the last term $B_0(x', \eta', \tau)$ is a sum of symbols homogeneous in (η', τ) of order $\leqq 0$.

Here we say that a function $B(x, \eta', \tau)$ is homogeneous in (η', τ') of order m if the following holds:

$$B(x, k\eta', k^{\frac{1}{3}}\tau)=k^m B(x, \eta', \tau)$$

for any $k>0$.

Furthermore we have

$$(3.1)\quad \begin{aligned} &B_1\big(x', (\theta_x-\rho_x\tau)_0\big)(g_0-\tau h_0) \\ &\quad = c_1(x', \eta')+c_2(x', \eta')\,\tau+(\tau^2-\rho_0)\,b(x', \eta', \tau) \end{aligned}$$

and taking $\tau=0$, $\alpha=0$,

$$c_1(x', \eta') = B_1\big(x', (\theta_x)_0\big)\,g_0 \qquad \text{for } \alpha=0\,.$$

Differentiating (3.1) with respect to τ and taking $\tau=0$, $\alpha=0$, we obtain

$$c_2(x', \eta') = -(\rho_{x_n})_0\frac{\partial B_1}{\partial \xi_n}\big(x', (\theta_x)_0\big)\,g_0 - B_1\big(x', (\theta_x)_0\big)\,h_0$$

for $\alpha=0$. Because $\partial b/\partial\tau(x', \eta', \tau)$ is homogeneous in (η', τ) of order 0, $(\tau^2-\rho_0)\,b(x', \eta', \tau)$ is considered as a lower order term $\in S^0_{\frac{1}{3},0}$. Thus we can write $BG_1((\chi_1+\chi_0)\,\chi V)|_{x_n=0}$ in the following form:

$$\begin{aligned} &\int\Big\{d_1(x', \eta')\,A(\rho_0)/A(\alpha|\eta'|^{\frac{2}{3}})+id_2(x', \eta')\,A'(\rho_0)/A(\alpha|\eta'|^{\frac{2}{3}})\Big\}\times \\ &\qquad \times e^{i\theta_0}(\chi_1+\chi_0)\,(\alpha|\eta'|\varepsilon)\,\chi(\eta')\,\tilde{V}(\eta')\,d\eta'\,. \end{aligned}$$

Here

$$d_1(x', \eta') = B_1\big(x', (\theta_x)_0\big)\,g_0+O(\alpha|\eta'|) \qquad \mathrm{mod}\,(S^0_{1,0})\,,$$

$$d_2(x', \eta') = -\big((\rho_{x_n})_0\,g_0+O(\alpha|\eta'|^{\frac{2}{3}}) \qquad \mathrm{mod}\,(S^0_{1,0})\,,$$

$$A(\rho_0)/A(\alpha|\eta'|^{\frac{2}{3}}) = \big(1+O(\alpha^\infty)\big)e^{i(-\frac{2}{3}\rho_0^{\frac{3}{2}}+\frac{2}{3}|\alpha|^{\frac{3}{2}}|\eta'|)}$$

if $\alpha|\eta'|^{\frac{2}{3}}>c_1>0$, and

$$A(\rho_0)/A(\alpha|\eta'|^{\frac{2}{3}}) = 1+O(\alpha^\infty)$$

if $|\alpha|\ |\eta'|^{\frac{2}{3}}<2c_1$, the same relations hold for $A'(\rho_0)/A'(\alpha|\eta'|^{\frac{2}{3}})$ and all of which are of $S_{1,0}$-class. Furthermore the proof of Lemma 2.3 implies that

$$\begin{aligned} B_1\big(x', (\theta_x)_0\big) &= (\theta_{x_n})_0-\lambda\big(x', (\theta_{x'})_0\big)+c\big(x', (\theta_{x'})_0\big) \\ &= c\big(x', (\theta_{x'})_0\big)+O(\alpha|\eta'|)\,. \end{aligned}$$

Therefore using β) (i) in § 2 we obtain that the principal symbol of the amplitude of the above integral is contained in the following symbol:

$$(3.2)\quad \begin{aligned} &c\big(x', (\theta_{x'})_0\big)\,g_0+O(\alpha|\eta'|) \\ &\quad +(-i)\,\big((\rho_{x_n})_0\,g_0+O(\alpha|\eta'|^{\frac{2}{3}})\big)\,A'(\alpha|\eta'|^{\frac{2}{3}})/A(\alpha|\eta'|^{\frac{2}{3}}) \end{aligned}$$

where all of functions except $A'(\alpha|\eta'|^{\frac{2}{3}})/A(\alpha|\eta'|^{\frac{2}{3}})$ are of $S_{1,0}$-class.

Now let Φ be an elliptic Fourier integral operator such that

$$(\Phi V)(x') = \int e^{i(\varphi_0(x',\eta')-\langle y',\eta'\rangle)} a(x',\eta')\, V(\eta')\, dy'\, d\eta'$$

where $a(x',\eta') \in S^0_{1,0}(R^n \times R^n \backslash 0)$ and is positive. Then we can define the inverse elliptic Fourier integral operator ϕ^{-1} such that

$$\phi\phi^{-1}V \equiv V \qquad (\mathrm{mod}\ C^\infty)$$

for any $V \in \mathcal{E}'(\Gamma)$ whose $WF(V)$ belongs to a conic neighborhood of $(x^0, \xi^{0\prime})$. Then by the canonical transformation with the generating function $\varphi_0(x',\eta') - \langle y',\eta'\rangle$ a conical neighborhood Σ of the point $(x^0, \xi^{0\prime})$ is transformed to a conic neighborhood Σ' of the point (y^0, η^0) such that $|\alpha| < 2^{-1}\alpha_0 \ll 1$ for $(y, \eta') \in \Sigma'$ and

$$\varphi_{0x'}(x^0, \eta^{0\prime}) = \xi^{0\prime} \text{ and } \varphi_{0\eta'}(x^0, \eta^{0\prime}) = y^0 .$$

Thus we obtain that the pseudo-differential operator

$$\phi^{-1}\Big(BG_1\big((\chi_1+\chi_0)\,\chi V\big)\Big|_{x_n=0}\Big)$$

has an amplitude with the principal symbol transformed by the canonical transformation from (3.2). Let $\{x'(y,\eta'), \xi'(y,\eta')\}$ be the canonical transform from Σ' to Σ, then the principal symbol may be considered as the function obtained from (3.2) replacing x' by $x'(y,\eta')$ whose terms we shall denote by the same notations without confusion.

Furthermore from β) (i) in § 2 we see that

$$\Big((\rho_{x_n})_0\, g_0 + O(\alpha|\eta'|^{\frac{2}{3}})\Big)\Big/|\eta'|^{\frac{2}{3}}$$

does not vanish if $|\alpha| \leqq \alpha_0 \ll 1$. Therefore extending suitably the above symbol in the whole space $R^n \times R^n \backslash 0$ we obtain a pseudo-differential operator Q whose principal symbol is the inverse of the above function. Then $Q\Phi^{-1}((BG_1(\chi_1+\chi_0)\,\chi V)|_{x_n=0})$ has the principal symbol contained in the following

$$e_1\big(x', (\theta_{x'})_0\big) + O(\alpha|\eta'|)$$
$$+(-i)|\eta'|^{\frac{2}{3}} A'(\alpha|\eta'|^{\frac{2}{3}})/A(\alpha|\eta'|^{\frac{2}{3}}),$$

where $e_1(x', (\theta_{x'})_0)$ satisfies also (2.6). Let w be the function

$$\left(\frac{1}{2\pi}\right)^n \int e^{i\langle y,\eta'\rangle}(\chi_1+\chi_0)\,(\alpha|\eta'|^{\epsilon})\,\chi(\eta')\,\hat{V}(\eta')\,d\eta'$$

where $V \in C_0^\infty(\pi\Sigma')$. Then extending symbols suitably in the whole space $R^n \times (R^n \backslash 0)$, we have that

$$\begin{aligned}(3.3)\qquad &\mathrm{Re}\left(-e^{i\theta}Q\Phi^{-1}\left(BG(w)|_{x_n=0}\right), w\right)\\ &=\left(\left(\int e^{i\langle y,\eta'\rangle}\left(e_2(y,\eta')+O(\alpha|\eta'|)+\mathrm{Re}\left(ie^{i\theta}|\eta'|^{\frac{2}{3}}A'(\alpha|\eta'|^{\frac{2}{3}})/A(\alpha|\eta'|^{\frac{2}{3}})\right)\right)\cdot\right.\right.\\ &\left.\hat{w}d\eta', w\right)+O(\|w\|_0^2)\end{aligned}$$

where $e_2(y, \eta')\geqq 0$, $e_2(y, \eta')\in S^1_{1,0}$ and the last term is caused by a commutator. Since by Lemma 2.1 and 2.2 we see that

$$\begin{aligned}(3.4)\qquad &\mathrm{Re}\left\{ie^{i\theta}|\eta'|^{\frac{2}{3}}A'(\alpha|\eta'|^{\frac{2}{3}})/A(\alpha|\eta'|^{\frac{2}{3}})\right\}\\ &\geqq k_1|\eta'|^{\frac{2}{3}}(|\alpha|\,|\eta'|^{\frac{2}{3}})^{\frac{1}{2}} \text{ or } k_2|\eta'|^{\frac{2}{3}},\end{aligned}$$

when $|\alpha|\,|\eta'|^{\frac{2}{3}}\geqq c_1>0$ or $|\alpha|\,|\eta'|^{\frac{2}{3}}<2c_1$ respectively. Hence it implies that Re { } in (3.4) is larger than $k_3|\eta'|^{\frac{2}{3}}$ for $\alpha\geqq -2c|\eta'|^{-\epsilon}$. Here and hereafter in this section we denote some positive numbers by k_j ($j=1, 2, \cdots, 6$) and c_j ($j=1, 2, \cdots, 11$).

Now we have that (3.3) $\geqq c_2\|w\|^2_{\frac{1}{3}}$ when we take t_0 sufficiently large in the definition β_2 in § 2. For, we apply first the Gårding sharp form to the term containing $e_2(y, \eta')$. Next to treat the term $O(\alpha|\eta'|)$, from (3.4), taking α_0 sufficiently small, we see that for $k_4\gg 1$

$$\frac{k_3}{2}\left(|\eta'|^{\frac{2}{3}}\gamma(\alpha|\eta'|^{\frac{1}{3}})\right)+k_4|\alpha|\,|\eta'|\left(1-\gamma(\alpha|\eta'|^{\frac{1}{3}})\right)\gg|\alpha|\,|\eta'|$$

if $\alpha|\eta'|\varepsilon\geqq -2c$,
where the left hand side $a(\eta')$ in the above inequality is of $S_{1-\frac{1}{3},0}$-class and is less than the half of the real part in (3.4). Here γ is a function such that $\gamma(t)=1$ if $|t|\leqq\delta_1$ and $\gamma(t)=0$ if $|t|\geqq 2\delta_1$, for $\delta_1\gg 1$. Therefore also using Gårding sharp form we obtain

$$\begin{aligned}&\mathrm{Re}\left(\left(\int e^{i\langle y,\eta'\rangle}\left(a(\eta')+O(\alpha|\eta'|)\right)\hat{w}d\eta', w\right)\right.\\ &\geqq -c_3\|w\|^2_{\frac{1}{6}}.\end{aligned}$$

Thus we see that desired inequality is valid.

Furthermore from the above inequality it follows that

$$\left\|Q\Phi^{-1}\left(BG_1(w)|_{x_n=0}\right)\right\|_{-\frac{1}{3}}\geqq c_4\|w\|_{\frac{1}{3}}$$

from which we see by a usual method that

$$\left\|Q\Phi^{-1}\left(BG_1(w)|_{x_n=0}\right)\right\|_0\geqq c_5\|w\|_{\frac{2}{3}}$$

and finally we obtain that for sufficiently large t_0 and α_0^{-1}

$$\left\|\phi^{-1}\left(BG_1(w)|_{x_n=0}\right)\right\|_0 \geqq c_6\|w\|_{\frac{2}{3}}\,. \tag{3. 5}$$

(ii) Next we shall treat $BG_2(\chi_{-1}xV)$. Form γ), (i) in § 2 we see that the principal symbol of $BG_2(w)|_{x_n=0}$ has the following form:

$$\begin{aligned} B_1\left(x', \left((\theta+\theta_1)_x\right)_0\right) &= B_1\left(x', (\theta_{x'})_0, \lambda^+\left(x', (\theta_{x'})_0\right)\right) \\ &= \lambda^+\left(x', (\theta_{x'})_0\right) - \lambda\left(x', (\theta_{x'})_0\right) + c\left(x', (\theta_{x'})_0\right) \\ &= -\sqrt{\zeta}\,\mu_3\left(x', (\theta_{x'})_0\right)^{\frac{1}{2}} + c\left(x', (\theta_{x'})_0\right) \end{aligned}$$

where

$$\mu_3\left(x', (\theta_{x'})_0\right) > 0\,, \quad \zeta = (\theta_{x_0})_0 - \mu_2\left(x', (\theta_{x''})_0\right),$$

$$c\left(x', (\theta_{x'})_0\right) = c\left(x', (\theta_{x'})_0\right)\Big|_{\alpha=0} + O(\alpha|\eta'|)\,,$$

$$|\alpha|^{\frac{1}{2}} \in S^0_{1-\epsilon,0}\,, \quad \sqrt{\zeta} \in S^0_{1-\epsilon,\epsilon}\,,$$

$$\mu_3^{\frac{1}{2}}\left(x', (\theta_{x'})_0\right) \in S^{\frac{1}{2}}_{1,0}\,, \quad D\left(x', (\theta_{x''})_0\right) \in S^{\frac{1}{2}}_{1,0}$$

and

$$|\eta'|^{\frac{1}{2}}\sqrt{\zeta} \sim k_5|\alpha|^{\frac{1}{2}}|\eta'| \geqq ck_5|\eta'|^{1-\frac{\epsilon}{2}}\,,$$

since $|\alpha|\;|\eta'|^{\epsilon} > c$.
Thus by the same way as in (i) considering $\mathrm{Re}\,(-e^{i\theta}\Phi^{-1}(BG_2(w)|_{x_n=0}), w)$, we have that for $w = \chi_{-1}\chi V$ and for sufficiently large t_0 and α_0^{-1}

$$\left\|\Phi^{-1}\left(BG_2(w)|_{x_n=0}\right)\right\|_0 \geqq c_7\|w\|_{1-\frac{\epsilon}{2}} \geqq c_8\|w\|_{\frac{2}{3}}\,. \tag{3. 6}$$

(iii) Take t_0 and α_0^{-1} sufficiently large and add (3. 5) and (3. 6). Then by modifying χ_j such that $(\chi_1+\chi_0)^2+\chi_{-1}^2=1$, we see that

$$\begin{aligned} \left\|\phi^{-1}\left(BG(V)|_{x_n=0}\right)\right\|_0^2 &\geqq c_9\left(\left\|(\chi_1+\chi_0)\,\chi V\right\|_{2/3}^2 + \|\chi_{-1}\chi V\|_{2/3}^2\right) \\ &\quad - c_{10}\|\chi V\|_{1/3}^2 \geqq c_{11}\|\chi V\|_{2/3}^2 \end{aligned} \tag{3. 7}$$

is valid for $V \in c_0^\infty(\pi\Sigma')$, since $\chi_1+\chi_0$ and $\chi_{-1} \in S^0_{1-\epsilon,0} \subset S^0_{\frac{2}{3},0}$. Furthermore it is easily seen that for the adjoint operator of $\phi^{-1}(BG(\cdot)|_{x_n=0})$ the analogous estimate as above is true, for to derive the estimate (3. 7) the relation (1. 5) and Lemma 2. 1 and 2. 2 were essential and therefore we merely consder to take complex conjugates to the terms appeared in that.

Finally adding an elliptic pseudo-differential operator and a compact one with symbol $\in S^{-\infty}_{1,0}$ to $\phi^{-1}(BG(\cdot)|_{x_n=0})$ and using a priori estimates as above, we have that the equation for $V \in \mathcal{E}'(\pi\Sigma')$

$$\phi^{-1}(BGV|_{x_n=0}) = \phi^{-1}f \qquad (\mathrm{mod}\ C^\infty)$$

is solved for given $f \in \mathcal{E}'(\pi\Sigma)$ with $WF(f) \subset \Sigma$ such that $WF(V) \subset WF(\phi^{-1}f)$.

To show the last assertion mentioned above, we remark first that for some neighborhood of a point (y, η') with $|\alpha| > 0$

$$K = \left(A'(\alpha|\eta'|^{\frac{2}{3}})/A(\alpha|\eta'|^{\frac{2}{3}})\right)|\eta'|^{\frac{2}{3}} \in S^1_{1,0}$$

and furthermore from the definition of $e_2(y, \eta')$ and (2. 5) we see that for a point (y, η') such that $\alpha=0$ but $B_1(x'(y, \eta'), \theta_x(x'(y, \eta'), \eta')) \neq 0$, $e_2(y, \eta') > 0$ and hence $Q\Phi^{-1}(BG_i(\cdot)|_{x_n=0})$ ($i=1$ or 2) is elliptic in some neighborhood of (y, η'). Finally we see that the other point $(y_1, \eta^{1'}) \in \Sigma^1$, the symbol of the commutator of K and a function $f(y)$ $(\in C_0^\infty(U(y^1)))$belongs to $S^{\frac{1}{3}}_{\frac{1}{3},0}$. For, the main symbol of $[K(D), f(y)]$ is $D_{\eta'}K(\eta') \cdot \frac{\partial}{\partial y} f(y)$ and $K(\eta') \in S^{\frac{2}{3}}_{\frac{1}{3},0}$ if $|\alpha||\eta'|^{\frac{2}{3}} < 2c_1$. Furthermore if $|\alpha|\,|\eta'|^{\frac{2}{3}} > c_1$, the principal symbol of $D_{\eta'}K(\eta') = (-i)$ $D_{\eta'}(|\alpha|^{\frac{1}{2}}|\eta'|) = (-i)\, D_{\eta'}(|\eta_0|^{\frac{1}{2}}|\eta'|^{\frac{1}{2}})$ whose absolute value $\leqq k_6|\eta'|^{-\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}} \leqq k_6|\eta'|^{\frac{1}{3}}$. Thus applying a priori-estimates as in (3. 7) and using cut off functions, by a standard method we see that the extension of $\phi^{-1}(BG(\cdot)|_{x_n=0})$ mentioned before is hypoelliptic.

Thus we obtain the desired micro-local solution of (1. 1) modulo C^∞-functions near the given point (x^0, ξ^0).

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