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A commutativity theorem for left s-unital rings

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Throughout R will represent a ring, and N the set of all nilpotents of R. Following [5], R is called a left s-unital ring if for each $x\in R$ there exists an element e such that $ex=x$. As was shown in [5, Theorem 1], if F is a finite subset of a left s-unital ring R then there exists $e\in R$ such that $ex=x$ for all $x\in F$.

In this note, we consider the following conditions:

1) For each $x\in R$ there exists a positive integer n such that $x-x^{n+1}\in N$.

1′) For each $x\in R$ there exist positive integers m and n such that $x^{m}-x^{m+n}\in N$.

1″) For each $x\in R$ there exist a positive integer n and an element x' in the subring $[x]$ generated by x such that $x^{n}=x^{n+1}x'$.

1‴) For each $x\in R$ there exists an element $x'\in[x]$ such that $x-x^{2}x'\in N$.

2) $x-y\in N$ and $y-z\in N$ imply that $x^{2}=z^{2}$ or $xy=yx$.

2_{*}) $x-y\in N$ implies that $x^{2}=y^{2}$ or both x and y are contained in the centralizer $V_{R}(N)$ of N in R.

2*) $x-y\in N$ implies that $x^{2}=y^{2}$ or $xy=yx$.

In general, $2_{*})\Rightarrow 2)\Rightarrow 2^{*})$, and $1)\Rightarrow 1')\Rightarrow 1'')\Leftarrow 1''')$. Moreover, $1'')\Rightarrow 1''')$. In fact, for any $x'\in[x]$ we have $(x-x^{2}x')^{n}=(x^{n}-x^{n+1}x')-(x^{n}-x^{n+1}x')x''$ with some $x''\in[x]$.

Recently, in [1, Theorem 2], we proved that if a left s-unital ring R satisfies 1) and 2) then R is commutative. More recently, in [3, Theorem 2], D. L. Outcalt and A. Yaqub have proved that if a ring R with a left identity satisfies 1″) and 2_{*}) then R is commutative. It is the purpose of this note to present the following theorem which includes [1, Theorem 2] as well as [3, Theorem 2].

THEOREM. *Let R be a left s-unital ring satisfying* 2). *Then each of* 1), 1′), 1″), 1‴) *implies others and that R is commutative.*

The next easy lemma is included in [1, Lemma 1].

LEMMA 1. *Assume that R satisfies* 2*).

(1) $x^{2}\in V_{R}(N)$ *for each $x\in R$, especially every idempotent is central.*

(2) *N is an ideal.*

COROLLARY 1. (1) *If R contains* 1 *and satisfies* 2) *then N is a commutative ideal.*

(2) *Let f be a ring homomorphism of R onto R^*. If R satisfies* $1''')$ *and* 2), *then so does R^*.*

PROOF. (1) is evident from the proof of [1, Lemma 3].

(2) To our end, it suffices to prove that $f(N)$ coincides with the set N^* of all nilpotents in R^*. Let x^* be an element of R^* with $x^{*r}=0$. Choose an element $x\in R$ with $f(x)=x^*$, and an element $x'\in[x]$ with $x-x^2x'\in N$. Then $x-x^{r+1}x'^r=(x-x^2x')+xx'(x-x^2x')+\cdots+(xx')^{r-1}(x-x^2x')\in N$ by Lemma 1 (2). Hence, $x^*\in f(N)$, proving $N^*=f(N)$.

LEMMA 2. *Assume that a left s-unital ring R satisfies* $1'')$. *Then every finite subset F of R is contained in a finitely generated subring with a left identity.*

PROOF. There exists an element c such that $cx=x$ for all $x\in F$. Choose an element $d\in[c]$ such that $c^n=c^{n+1}d$ for some positive integer n. Then $e=c^nd^n$ is an idempotent and $ec^n=c^n$. Hence, $ex=ec^nx=c^nx=x$ for all $x\in F$, whence it follows that e is a left identity of the subring $[e, F]$.

COROLLARY 2 (cf. [4, Corollary 3. 5]). *If R satisfies* $2^*)$, *then* $1''')$ *implies* 1).

PROOF. Since N is an ideal (Lemma 1 (2)), it suffices to prove the assertion for the case $N=0$. Then R is evidently s-unital, and we may assume further that R contains 1 (Lemmas 2 and 1 (1)). By $1''')$, there exists an integer k such that $2=4k$ (in R). As is well known, R is a subdirect sum of subdirectly irreducible rings $R_\lambda(\lambda\in\Lambda)$. Noting that R_λ contains no non-trivial idempotents, we can easily see that each non-zero element of R_λ is a unit. Hence, each R_λ is a division ring. Evidently, the characteristic of each R_λ is a divisor of $4k-2$, so that one can easily see that for each $x\in R$ there exists a positive integer n such that $x-x^{n+1}=0$.

Now, we are ready to complete the proof of our theorem.

PROOF OF THEOREM. Since 1), $1')$, $1'')$ and $1''')$ are equivalent by Corollary 2, it suffices to prove that 1) implies the commutativity of R, which is [1, Theorem 2] itself. However, for the sake of completeness, we shall give here a somewhat elementary proof. By Lemmas 2 and 1 (1), an arbitrary finite subset of R is contained in some finitely generated subring with identity. Henceforth, in virtue of Corollary 1 (2), we may restrict our attention to the case R is a finitely generated subdirectly irreducible ring with 1. Then, noting that R contains no non-trivial idempotents, we see that each element of R is either a nilpotent or a unit. By Jacobson's theorem (to which an elementary proof is given in [2]), R/N is a finite field of characteristic p. By Corollary 1 (1), the ideal N is commutative. Now, we shall show that N is contained in the center of R. Suppose there exist $r \in R$ and $s \in N$ such that $rs \neq sr$. Since $2r=(1+r)^2-1-r^2 \in V_R(N)$ by Lemma 1, there holds $2(rs-rs)=0$. If $p \neq 2$ then $p(rs-sr)=(pr)s-s(pr)=0$ and $2(rs-sr)=0$ yield a contradiction $rs-sr=0$. If $p=2$ then $r^{2^k}-r \in N$ for some positive integer k. This together with $r^2s-sr^2=0$ gives a contradiction $0=(r^{2^k}-r)s-s(r^{2^k}-r)=sr-rs$. Hence, N is contained in the center of R. Combining this with the fact that the multiplicative group of the finite field R/N is cyclic, we readily see that R is commutative.

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