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On the equivalence problems associated with simple graded Lie algebras

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Introduction

In our earlier paper [6] we have settled the equivalence problems associated with the class of simple graded Lie algebras of the first kind (irreducible l-systems in the terminology there). In our recent paper [8] we have also settled the equivalence problems associated with the class of simple graded Lie algebras of contact type, and have applied the results to the geometry of non-degenerate real hypersurfaces of complex manifolds.

The main purpose of the present paper is to settle analogous equivalence problems for the full class of simple graded Lie algebras.

Let $\mathfrak{g}=\sum_{p}\mathfrak{g}_{p}$ be a graded Lie algebra over the field $\boldsymbol{R}$ of real numbers which satisfies the following conditions: 1) $\mathfrak{g}$ is finite dimensional and simple, 2) $\mathfrak{g}_{-1}\neq 0$, and $\mathfrak{m}=\sum_{p<0}\mathfrak{g}_{p}$ is generated by $\mathfrak{g}_{-1}$. Such a graded Lie algebra will be called a simple graded Lie algebra of the μ-th kind, where μ stands for the positive integer with $\mathfrak{g}_{-\mu}\neq 0$ and $\mathfrak{g}_{p}=0$ for all $p<-\mu$. Let us denote by $\mathfrak{G}$ (resp. by $\mathfrak{M}$) the graded Lie algebra $\mathfrak{g}=\sum_{p}\mathfrak{g}_{p}$ (resp. $\mathfrak{m}=\sum_{p<0}\mathfrak{g}_{p}$).

Now to the graded Lie algebra $\mathfrak{G}$ there is associated a homogeneous space G/G' in a natural manner, where the Lie algebra of G is equal to $\mathfrak{g}$, and the Lie algebra of G' to $\mathfrak{g}'=\sum_{p\geqq 0}\mathfrak{g}_{p}$. Let $\tilde{G}$ be the linear isotropy group of the homogeneous space G/G' at the origin o, which may be represented on the vector space $\mathfrak{m}$. Then we extend the group $\tilde{G}$ to a linear Lie group $G^{\#}_{0}$ of $\mathfrak{m}$, and introduce the notion of a $G^{\#}_{0}$-structure of type $\mathfrak{M}$. It should be noted that the two groups $\tilde{G}$ and $G^{\#}_{0}$ coincide if and only if $\mathfrak{G}$ is of the first kind or of contact type. (By definition $\mathfrak{G}$ is of contact type if it is of the second kind and dim $\mathfrak{g}_{-2}=1$.) It should be also noted that a $G^{\#}_{0}$-structure of type $\mathfrak{M}$ admits a strongly regular differential system (in the sense of [7]) as an underlying structure. Furthermore we introduce the notion of a normal connection of type $\mathfrak{G}$, which is defined to be a Cartan connection of type G/G' whose curvature satisfies certain linear relations.

The main theorem (Theorem 2.7) in the present paper is concerned with the equivalence problem for $G^\sharp_0$-structures of type $\mathfrak{M}$, and may be roughly stated as follows: Assume that $\mathfrak{G}$ is the prolongation of $(\mathfrak{M}, \mathfrak{g}_0)$. Then every normal connection of type $\mathfrak{G}$, (P, ω), on a manifold M induces a $G^\sharp_0$-structure of type $\mathfrak{M}$, $(P^\sharp, \xi)$, on M in a natural manner, and the assignment $(P, \omega)\to P^\sharp, \xi)$ gives a one-to-one correspondence up to isomorphisms between the totality of normal connections of type $\mathfrak{G}$ and the totality of $G^\sharp_0$-structures of type $\mathfrak{M}$. We also show that the "harmonic part" $H(K)$ of the curvature K of a normal connection (P, ω) gives a fundamental system of invariants (Theorem 2. 9).

In the forthcoming papers we shall apply our theory to various problems in the geometry as well as the analysis, especially to the geometrizations and integrations of differential equations (cf. Tresse [9] and Cartan [2]). For example, consider a system of ordinary differential equations of the second order :

$$(\mathscr{X}) \qquad y''_i = \phi_i(x, y_1, \cdots, y_{n-1}, y'_1, \cdots, y'_{n-1}), \ 1 \leqq i \leqq n-1 .$$

Then we shall show that to the equation $(\mathscr{X})$ there is associated in an invariant manner a normal connection of type $\mathfrak{G}$, (P, ω), defined on the space $(x, y_1, \cdots, y_{n-1}, y'_1, \cdots, y'_{n-1})$, so that the integrals of the equation $(\mathscr{X})$ may be represented by the geodesics of the connection (P, ω), where $\mathfrak{G}$ is a simple graded Lie algebra of the second kind, and the underlying Lie algebra $\mathfrak{g}$ is isomorphic with the simple Lie algebra $\mathfrak{sl}(n+1, \boldsymbol{R})$. In particular we shall find from this fact that the integration of the equation $(\mathscr{X})$, to a great extent, depends on the structure of the automorphism group of the equation $(\mathscr{X})$ or of the connection (P, ω) .

In § 1 we first recall several known facts on the graded Lie algebra $\mathfrak{G}$, and then define a complex $\{C^{p,q}(\mathfrak{G}), \partial\}$, which is naturally associated with $\mathfrak{G}$, and which generalizes or rather refines the so-called Spencer complex (associated with a simple graded Lie algebra of the first kind). We also show that there is naturally defined the "adjoint operator" $\partial^* : C^{p,q}(\mathfrak{G})\to C^{p+1,q-1}(\mathfrak{G})$ of the operator $\partial : C^{p+1,q-1}(\mathfrak{G})\to C^{p,q}(\mathfrak{G})$. We shall see that the "harmonic theory" for the system $\{C^{p,q}(\mathfrak{G}), \partial, \partial^*\}$ plays an important role in our whole theory. In § 2 we introduce the notions of a $G^\sharp_0$-structure of type $\mathfrak{M}$ and of a normal connection of type $\mathfrak{G}$, and state the main theorem. We also prove Theorem 2. 9, based on the Bianchi identity for the connection (P, ω). § 3∼§ 5 are devoted to the proof of the main theorem. First of all we prove, in § 3, the important fact that every $G^\sharp_0$-structure of type $\mathfrak{M}$, $(P^\sharp, \xi)$, is naturally reduced to a $\tilde{G}$-structure $(\tilde{P}, \tilde{\xi})$ (Theorems 3. 7 and 3. 8). Starting from the $\tilde{G}$-structure $(\tilde{P}, \tilde{\xi})$, we construct, in § 4, a normal connection of type $\mathfrak{G}$, (P, ω), the prolongation of $(P^\sharp, \xi)$, which induces the given

$(P^{\sharp}, \xi)$ (Theorems 4.6 and 4.15). We thus see that the assignment $(P, \omega)\to(P^{\sharp}, \xi)$ is surjective. Here we notice that our method of the prolongation adopted in the present paper is closely related to Cartan's general method of the equivalence ([1]). Finally in § 5 we prove the uniqueness of normal connections of type $\mathfrak{G}$ or the statement that the assignment $(P, \omega)\to(P^{\sharp}, \xi)$ is injective.

Preliminary remarks

1. Let V be a finite dimensional vector space over a field K. As usual $GL(V)$ denotes the general linear group of V, and $\mathfrak{gl}(V)$ the Lie algebra of all endomorphisms of V. V^* denotes the dual space of V, and $\wedge^p(V)$ the space of exterior p-vectors of V. Given another vector space W over K, the tensor product $W\otimes\wedge^p(V^*)$ may be naturally identified with the space $\mathrm{Hom}(\wedge^p(V), W)$, *i.e.*, the space of all linear maps of $\wedge^p(V)$ to W.

2. Graded Lie algebras. Let $\mathfrak{g}$ be a Lie algebra over a field K. Let $(\mathfrak{g}_p)_{p\in\boldsymbol{Z}}$, $\boldsymbol{Z}$ being the group of integers, be a family of subspaces of $\mathfrak{g}$ which satisfies the following conditions:

(GLA. 1) $\qquad \mathfrak{g}=\sum_p \mathfrak{g}_p \qquad$ (direct sum);

(GLA. 2) $\qquad \dim \mathfrak{g}_p<\infty$;

(GLA. 3) $\qquad [\mathfrak{g}_p, \mathfrak{g}_p]\subset\mathfrak{g}_{p+q}$.

Under these conditions we say that the system $\mathfrak{G}=\{\mathfrak{g}, (\mathfrak{g}_p)\}$ or the direct sum $\mathfrak{g}=\sum_p \mathfrak{g}_p$ is a graded Lie algebra over K.

Let $\mathfrak{G}=\{\mathfrak{g}, (\mathfrak{g}_p)\}$ and $\mathfrak{G}'=\{\mathfrak{g}', (\mathfrak{g}'_p)\}$ be two graded Lie algebras. A homomorphism φ of $\mathfrak{g}$ to $\mathfrak{g}'$ as Lie algebras is called a homomorphism of $\mathfrak{G}$ to $\mathfrak{G}'$ if φ preserves the gradings, *i.e.*, $\varphi(\mathfrak{g}_p)\subset\mathfrak{g}'_p$ for all p. A homomorphism φ of $\mathfrak{G}$ to $\mathfrak{G}'$ is called an isomorphism of $\mathfrak{G}$ onto $\mathfrak{G}'$ if φ is bijective. The notion of an isomorphism naturally gives rise to the notions of an automorphism and of a derivation.

3. In the present paper we always assume the differentiability of class C^∞ unless otherwise stated. Let M be a manifold. $T(M)$ denotes the tangent bundle of M. Given a vector field X on M, $\mathscr{L}_X$ denotes the Lie derivation with respect to X.

As to Lie groups and principal fibre boundles we use the standard notations as in [3]. Especially let P be a principal fibre bundle over a base manifold M with a Lie group G as structure group. For $a\in G$, R_a denotes the right translation $P\ni z\to za\in P$. Let $\mathfrak{g}$ be the Lie algebra of G. For

$A \in \mathfrak{g}$, A^* denotes the (vertical) vector field on P induced from the 1-parameter group of right translations, $\{R_{a_t}\}$, where $a_t = \exp tA$.

Let P be as above. Let H be a Lie subgroup of G, and Q a principal fibre bundle over the base space M with structure group H. Then we say that Q is a reduction of P to H if the following conditions are satisfied: i) Q is a submanifold of P; ii) $\varpi = \pi \circ \iota$, where ι denotes the injection of Q to P, and π (resp. ϖ) the projection of P (resp. of Q) onto M; iii) ι gives a bundle homomorphism of Q to P (corresponding to the injective homomorphism of H into G.)

4. Linear group structures. Let V be an n-dimensional vector space over the field $\boldsymbol{R}$ of real numbers, and G a Lie subgroup of $GL(V)$. Let P be a principal fibre bundle over an n-dimensional manifold M with structure group G, and θ a V-valued 1-form on P. Then we say that the pair (P, θ) is a G-structure on M if it satisfies the following conditions:

(LG. 1) Let X be a tangent vector to P. Then $\theta(X)=0$ if and only if X is a vertical vector;

(LG. 2) $R_a{}^*\theta = a^{-1}\theta$, $a \in G$.

Let (P, θ) (resp. (P', θ')) be a G-structure on a manifold M (resp. on M'). A bundle isomorphism φ of P onto P' is called an isomorphism of (P, θ) onto (P', θ') if $\varphi^*\theta' = \theta$.

We shall now remark that our definition of a G-structure is equivalent to the usual one.

Let M be an n-dimensional manifold, and $F(M)$ the frame bundle of M. As usual $F(M)$ may be regarded as the set of all linear isomorphisms of V onto the tangent spaces $T(M)_x$, $x \in M$, which is a principal fibre bundle over the base space M with structure group $GL(V)$. Let $\bar{\theta}$ be the V-valued 1-form on $F(M)$ defined by $\bar{\theta}(X) = z^{-1} \cdot \pi_*(X)$ for all $X \in T(F(M))_z$ and $z \in F(M)$, where π denotes the projection of $F(M)$ onto M.

Now a reduction of the frame bundle $F(M)$ to the group G is usually called a G-structure on M (cf. [5]). Let P be a G-structure in the usual sense on M, and ι the injection of P to $F(M)$. Put $\theta = \iota^*\bar{\theta}$, which is usually called the basic form of P. Then the pair (P, θ) gives a G-structure in our sense on M. Conversely every G-structure in our sense, (P, θ), on M can be obtained in this manner. Indeed there is a unique bundle homomorphism ι of P to $F(M)$ (corresponding to the injective homomorphism of G into $GL(V)$) such that ι induces the identity transformation of M and such that $\theta = \iota^*\bar{\theta}$. Hence P may be regarded as a G-structure in the usual sense on M, and θ as its basic form. Thus we have seen that the two definitions are equivalent.

Finally let H be a Lie subgroup of G. Let (P, θ) (resp. (Q, η)) be a G-structure (resp. an H-structure) on M. Then we say that (Q, η) is a reduction of (P, θ) to H or (P, θ) is an extension of (Q, η) to G if i) Q is a reduction of P to H, and ii) $\eta=\iota^*\theta$, ι being the injection of Q to P, or equivalently if i) $Q\subset P\subset F(M)$, and ii) Q is a submanifold of P. Clearly every H-structure has a unique extension to G.

5. Cartan connections. Let G/G' be the homogeneous space of a Lie group G over its closed subgroup G'. Let $\mathfrak{g}$ (resp. $\mathfrak{g}'$) be the Lie algebra of G (resp. of G'). Let P be a principal fibre bundle over a base manifold M with structure group G', where $\dim M=\dim G/G'$, and let ω be a $\mathfrak{g}$-valued 1-form on P. Then we say that the pair (P, ω) is a Cartan connection of type G/G' on M or ω is a Cartan connection of type G/G' in P if the following conditions are satisfied:

(C. 1) Let X be a tangent vector to P. If $\omega(X)=0$, then $X=0$;

(C. 2) $\omega(A^*)=A$, $A\in\mathfrak{g}'$;

(C. 3) $R^*_a\omega=Ad(a^{-1})\,\omega$, $a\in G'$.

Let (P, ω) (resp. (P', ω')) be a Cartan connection of type G/G' on a manifold M (resp. on M'). A bundle isomorphism φ of P onto P' is called an isomorphism of (P, ω) onto (P', ω') if $\varphi^*\omega'=\omega$.

§ 1. Simple graded Lie algebras

1. 1. Simple graded Lie algebras (cf. [7] and [8]). In the following K means the field $\boldsymbol{R}$ of real numbers or the field $\boldsymbol{C}$ of complex numbers.

Let $\mathfrak{G}=\{\mathfrak{g}, (\mathfrak{g}_p)\}$ be a graded Lie algebra over K. Put

$$\mathfrak{m}=\sum_{p<0}\mathfrak{g}_p\,,$$

being a subalgebra of $\mathfrak{g}$. Then we say that $\mathfrak{G}$ is simple if it satisfies the following conditions:

(SGLA. 1) The Lie algebra $\mathfrak{g}$ is finite dimensional and simple;

(SGLA. 2) $\mathfrak{g}_{-1}\neq 0$, and the Lie algebra $\mathfrak{m}$ is generated by $\mathfrak{g}_{-1}$.

We denote by $\mathfrak{M}$ the (truncated) graded subalgebra $\{\mathfrak{m}, (\mathfrak{g}_p)_{p<0}\}$ of $\mathfrak{G}$. Then (SGLA. 2) means that $\mathfrak{M}$ is a fundamental graded Lie algebra (or simply FGLA) in the sense of [7]. A simple graded Lie algebra $\mathfrak{G}$ is called of the μ-th kind if $\mathfrak{M}$ is of the μ-th kind, *i.e.*, $\mathfrak{g}_{-\mu}\neq 0$ and $\mathfrak{g}_p=0$ for any $p<-\mu$.

Let $\mathfrak{G}$ be a simple graded Lie algebra of the μ-th kind over K. Let B denote the Killing form of the Lie algebra $\mathfrak{g}$. The following two lemmas are well known.

LEMMA 1. 1. *There is a unique element E in the centre of $\mathfrak{g}_0$ such that*

$$\mathfrak{g}_p = \left\{ X \in \mathfrak{g} \,\middle|\, [E, X] = pX \right\} \qquad \textit{for all } p\,.$$

LEMMA 1. 2. (1) *$B(\mathfrak{g}_p, \mathfrak{g}_q)=0$ if $p+q\neq 0$.*

(2) *For every p the restriction of B to $\mathfrak{g}_p\times\mathfrak{g}_{-p}$ is non-degenerate.* In particular it follows that $\mathfrak{g}_p=0$ for all $p>\mu$.

LEMMA 1. 3. *Let $p\geqq 0$.*

(1) $[\mathfrak{g}_{-p+1}, \mathfrak{g}_{-1}]=\mathfrak{g}_{-p}$.

(2) *If $X_p\in\mathfrak{g}_p$ and $[X_p, \mathfrak{g}_{-1}]=0$, then $X_p=0$.*

PROOF. By (SGLA. 2) we have $[\mathfrak{g}_{-p+1}, \mathfrak{g}_{-1}]=\mathfrak{g}_{-p}$ for all $p\geqq 2$. Since $E\in\mathfrak{g}_0$, we have $[\mathfrak{g}_0, \mathfrak{g}_{-1}]=\mathfrak{g}_{-1}$. Furthermore we easily see that $\mathfrak{a}=[\mathfrak{g}_1, \mathfrak{g}_{-1}]+\sum_{i\neq 0}\mathfrak{g}_i$ is a (graded) ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple and $\mathfrak{a}\supset\mathfrak{g}_{-1}\neq 0$, it follows that $\mathfrak{a}=\mathfrak{g}$ and hence $[\mathfrak{g}_1, \mathfrak{g}_{-1}]=\mathfrak{g}_0$. We have thus proved (1). Using (1), we have $B([X_p, \mathfrak{g}_{-1}], \mathfrak{g}_{-p+1})=B(X_p, \mathfrak{g}_{-p})=0$. This fact together with Lemma 1. 2 gives $X_p=0$, which proves (2).

Consider the derivation algebra Der $(\mathfrak{M})$ of the FGLA, $\mathfrak{M}$. By Lemma 1. 3, $\mathfrak{g}_0$ may be naturally identified with a subalgebra of Der $(\mathfrak{M})$, and $\mathfrak{G}$ naturally with a graded subalgebra of the prolongation of $(\mathfrak{M}, \mathfrak{g}_0)$. (For the definition of the prolongation, see [7].) Let us now define an ideal $\mathfrak{h}_0$ of $\mathfrak{g}_0$ by

$$\mathfrak{h}_0 = \left\{ X \in \mathfrak{g}_0 \,\middle|\, [X, \mathfrak{g}_p] = 0 \quad \text{for all } p \leqq -2 \right\},$$

which may be naturally regarded as a subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$.

LEMMA 1. 4. *The following three statements are mutually equivalent:*

(1) *$\mathfrak{G}$ is the prolongation of $(\mathfrak{M}, \mathfrak{g}_0)$.*

(2) *The subspace $\mathfrak{h}_0$ of $\mathfrak{gl}(\mathfrak{g}_{-1})$ is of finite type, i. e., the k-th prolongation $\mathfrak{h}_0^{(k)}$ of $\mathfrak{h}_0$ vanishes for some k.*

(3) *The first prolongation $\mathfrak{h}_0^{(1)}$ of $\mathfrak{h}_0$ vanishes.*

The proof of this fact is quite similar to the proof of Lemma 3. 4 of [8], and therefore it is omitted.

LEMMA 1. 5. (1) *The case $K=\boldsymbol{R}$. There is an involutive automorphism σ of $\mathfrak{g}$ having the following properties:*

1) $\sigma\mathfrak{g}_p=\mathfrak{g}_{-p}$;

2) *$B(X, \sigma X)<0$ for $X\neq 0$.*

(2) *The case $K=\boldsymbol{C}$. There is an involutive automorphism σ of $\mathfrak{g}$ as a Lie algebra over $\boldsymbol{R}$ having the following properties:*

1) $\sigma\mathfrak{g}_p=\mathfrak{g}_{-p}$;

2) *$\sigma(\lambda X)=\bar{\lambda}\sigma X$ for $\lambda\in\boldsymbol{C}$ and $X\in\mathfrak{g}$, and hence the bilinear form $B(X, \sigma Y)$ $(X, Y\in\mathfrak{g})$ is hermitian;*

3) $B(X, \sigma X)<0$ *for* $X\neq 0$.

Although this fact is known, we shall give an outline of the proof for completeness. (The following proof is due to Dr. Kaneda.) First consider the case $K=\boldsymbol{C}$. Since $\mathrm{ad}(E)$ is a semi-simple endomorphism of $\mathfrak{g}$, there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $E\in\mathfrak{h}$. Let $\mathfrak{h}_0$ be the real part of $\mathfrak{h}$, *i.e.*, $\mathfrak{h}_0$ is the (real) subspace of $\mathfrak{h}$ consisting of all $H\in\mathfrak{h}$ such that $\alpha(H)$ is real for any non-zero root α (associated with $\mathfrak{h}$). Since the eigenvalues of $\mathrm{ad}(E)$ are all integers, it follows that $E\in\mathfrak{h}_0$. However we know that there is a compact real form $\mathfrak{u}$ of $\mathfrak{g}$ such that $\mathfrak{h}_0\subset\sqrt{-1}\,\mathfrak{u}$ (cf. [10]). Hence $E\in\sqrt{-1}\,\mathfrak{u}$, which proves our assertion. Next consider the case $K=\boldsymbol{R}$. In this case we may assume that the complexification $\mathfrak{g}^c$ of $\mathfrak{g}$ is simple. (Suppose that $\mathfrak{g}^c$ is not simple. Then $\mathfrak{g}$ is endowed with a complex structure, so that $\mathfrak{G}$ becomes a simple graded Lie algebra over $\boldsymbol{C}$. Thus the problem is reduced to the case $K=\boldsymbol{C}$.) As above there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $E\in\mathfrak{h}$, and E is in the real part $\mathfrak{h}_0$ of $\mathfrak{h}^c$. However we know that there is a compact real form $\mathfrak{u}$ of $\mathfrak{g}^c$ such that $\mathfrak{g}=\mathfrak{g}\cap\mathfrak{u}+\mathfrak{g}\cap\sqrt{-1}\,\mathfrak{u}$ and such that $\mathfrak{h}_0\subset\sqrt{-1}\,\mathfrak{u}$ (cf. [10]). Hence $E\in\mathfrak{g}\cap\sqrt{-1}\,\mathfrak{u}$, which proves our assertion.

LEMMA 1.6. *Let* $1\leqq p\leqq\mu-1$.

(1) $[\mathfrak{g}_{-p-1}, \mathfrak{g}_1]=\mathfrak{g}_{-p}$.

(2) *If* $X_p\in\mathfrak{g}_p$ *and* $[X_p, \mathfrak{g}_{-p-1}]=0$, *then* $X_p=0$.

PROOF. Since $\mathfrak{m}$ is generated by $\mathfrak{g}_{-1}$, we see from Lemma 1.5 that the subalgebra $\sum_{i>0}\mathfrak{g}_i$ of $\mathfrak{g}$ generated by $\mathfrak{g}_1$. For any $i\geqq-\mu$, we define a subspace $\mathfrak{b}_i$ of $\mathfrak{g}_i$ inductively as follows: $\mathfrak{b}_{-\mu}=\mathfrak{g}_{-\mu}$ and $\mathfrak{b}_i=[\mathfrak{b}_{i-1}, \mathfrak{g}_1]$ if $i>-\mu$. From the remark above we see that the sum $\mathfrak{b}=\sum_{i\geqq-\mu}\mathfrak{b}_i$ is a (graded) ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple and $\mathfrak{b}\supset\mathfrak{g}_{-\mu}\neq 0$, it follows that $\mathfrak{b}=\mathfrak{g}$. Hence $[\mathfrak{g}_{-p-1}, \mathfrak{g}_1]=\mathfrak{g}_{-p}$ for any $1\leqq p\leqq\mu-1$, proving (1). (2) follows easily from (1) and LEMMA 1.2.

1.2. The homogeneous space G/G'. Let $\mathfrak{G}$ be a simple graded Lie algebra of the μ-th kind over K.

Consider the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ coincides with the Lie algebra $\mathrm{ad}(\mathfrak{g})$ of all inner derivations of $\mathfrak{g}$, and the assignment $X\rightarrow\mathrm{ad}(X)$ gives an isomorphism of $\mathfrak{g}$ onto $\mathrm{ad}(\mathfrak{g})$. We shall identify $\mathfrak{g}$ and $\mathrm{ad}(\mathfrak{g})$ through this isomorphism. Thus we have

$$\mathrm{Ad}(a)\,X=aX\,, \qquad a\in\mathrm{Aut}(\mathfrak{g})\,, \qquad X\in\mathfrak{g}\,.$$

We denote by G_0 the automorphism group $\mathrm{Aut}(\mathfrak{G})$ of the graded Lie algebra $\mathfrak{G}$. It is easy to see that the Lie algebra of G_0 is $\mathfrak{g}_0$ ([8], Lemma 2.4). For any p we define a subspace $\mathfrak{f}^{(p)}$ of $\mathfrak{g}$ by $\mathfrak{f}^{(p)}=\sum_{i\geqq p}\mathfrak{g}_i$. Then the family $\mathfrak{F}=\{\mathfrak{f}^{(p)}\}$ gives a filtration of the Lie algebra $\mathfrak{g}$. We denote by G' the automorphism group of the filtred Lie algebra $(\mathfrak{g}, \mathfrak{F})$, *i. e.*, G' is the subgroup of $\mathrm{Aut}(\mathfrak{g})$ consisting of all $a\in \mathrm{Aut}(\mathfrak{g})$ which satisfy $\mathrm{Ad}(a)\,\mathfrak{f}^{(p)}=\mathfrak{f}^{(p)}$ for all p. Then it can be easily shown that the Lie algebra of G' is

$$\mathfrak{g}' = \mathfrak{f}^{(0)} = \sum_{i\geqq 0} \mathfrak{g}_i$$

([8], Lemma 2. 5).

LEMMA 1. 7 ([8], Lemma 2. 6). *Every element a of G' can be written uniquely in the form*:

$$a = b\cdot \exp X_1 \cdots \exp X_\mu ,$$

where $b\in G_0$ and $X_p\in \mathfrak{g}_p$.

Let $\mathrm{Aut}(\mathfrak{g})^0$ denote the connected component of the identity of $\mathrm{Aut}(\mathfrak{g})$. Then we define an open subgroup G of $\mathrm{Aug}(\mathfrak{g})$ by

$$G = \mathrm{Aut}(\mathfrak{g})^0\cdot G' = \mathrm{Aut}(\mathfrak{g})^0\cdot G_0 ,$$

G' being a closed subgroup of G, we have the homogeneous space G/G'. Clearly G/G' is connected. Here we notice that if $K=\boldsymbol{C}$, then G, G' and G_0 are all complex Lie groups, and G/G' is a complex manifold.

We denote by ρ the linear isotropy representation of G' on the tangent space $T(G/G')_o$ to G/G' at the origin o. We have

$$\mathfrak{g} = \mathfrak{m}+\mathfrak{g}' \qquad \text{(direct sum)},$$

and hence, as usual, $T(G/G')_o$ may be identified with the vector space $\mathfrak{m}$. This being said, the representation $\rho: G'\to GL(\mathfrak{m})$ may be described as follows:

$$\rho(a)\, X \equiv \mathrm{Ad}(a)\, X \qquad (\mathrm{mod}\ \mathfrak{g}'),$$

where $a\in G'$ and $X\in \mathfrak{m}$. The homomorphism $\rho: G'\to GL(\mathfrak{m})$ naturally induces a Lie algebra homomorphism $\mathfrak{g}'\to \mathfrak{gl}(\mathfrak{m})$, which we denote by the same letter ρ. We have

$$\rho(X)\, Y \equiv [X, Y] \qquad (\mathrm{mod}\ \mathfrak{g}'),$$

where $X\in \mathfrak{g}'$ and $Y\in \mathfrak{m}$.

By using Lemmas 1. 2 and 1. 3 we can easily prove the following

LEMMA 1. 8. *The natural representation of G_0 on $\mathfrak{g}_{-1}$ is faithful.*

LEMMA 1. 9. *The kernel of the homomorphism $\rho: G'\to LG(\mathfrak{m})$ is* $\exp \mathfrak{g}_\mu$.

This fact follows easily from Lemmas 1. 6, 1. 7 and 1. 8.

Finally we note that the action of G on G/G' is effective (cf. [8], Lemma 3. 7), and that the space G/G' is compact (cf. [8], Lemma 3. 8).

1. 3.[*)] The operators ∂ and ∂^*. Let $\mathfrak{G}$ be a simple graded Lie algebra of the μ-th kind over K.

Let us consider the subalgebra $\mathfrak{m}=\sum_{p<0}\mathfrak{g}_p$ of $\mathfrak{g}$. We remark that $\mathfrak{g}$ may be regarded as a (left) $\mathfrak{m}$-module with respect to the representation $\mathfrak{m}\ni X\rightarrow \mathrm{ad}(X)\in\mathfrak{gl}(\mathfrak{m})$. Accordingly, as is well known, to the $\mathfrak{m}$-module $\mathfrak{g}$ there is associated a complex

$$\cdots\longrightarrow C^q(\mathfrak{G})\xrightarrow{\partial} C^{q+1}(\mathfrak{G})\longrightarrow\cdots$$

as follows: $C^q(\mathfrak{G})$ is defined to be the space

$$\mathfrak{g}\otimes\wedge^q(\mathfrak{m}^*)=\mathrm{Hom}\left(\wedge^q(\mathfrak{m}),\mathfrak{g}\right),$$

and the operator ∂ is defined by the following formula:

$$(\partial c)(X_1\wedge\cdots\wedge X_{q+1})=\sum_i(-1)^{i+1}\left[X_i, c(X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge X_{q+1})\right]$$
$$+\sum_{i<j}(-1)^{i+j}c\left([X_i,X_j]\wedge X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge\hat{X}_j\wedge\cdots\wedge X_{q+1}\right),$$

where $c\in C^q(\mathfrak{G})$, and $X_1,\cdots,X_{q+1}\in\mathfrak{m}$.

Let σ be an involutive automorphism of $\mathfrak{g}$ having the properties in Lemma 1.5. Then we define an inner product (,) in $\mathfrak{g}$ by

$$(X,Y)=-B(X,\sigma Y),\qquad X,\ Y\in\mathfrak{g}.$$

It should be noted that if $K=\boldsymbol{C}$, then the inner product (,) is hermitian. The inner product (,) in $\mathfrak{g}$ naturally induces an inner product in $C^q(\mathfrak{G})$. Namely let $\{e_1,\cdots,e_m\}$ be an orthonormal basis of $\mathfrak{g}$: $(e_i,e_j)=\delta_{ij}$. Then

$$(c,c')=\frac{1}{q!}\sum_{i_1,\cdots,i_q}\left(c(e_{i_1}\wedge\cdots\wedge e_{i_q}),\ c'(e_{i_1}\wedge\cdots\wedge e_{i_q})\right),$$

where $c,\ c'\in C^q(\mathfrak{G})$.

Let us now calculate the adjoint operator ∂^* of ∂. Let $\{e_1,\cdots,e_m\}$ be a basis of $\mathfrak{g}$. By Lemma 1.2 the space $\sum_{p>0}\mathfrak{g}_p$ may be regarded as the dual space $\mathfrak{m}^*$ of $\mathfrak{m}=\sum_{p<0}\mathfrak{g}_p$, and hence there is a unique basis $\{e^*_1,\cdots,e^*_m\}$ of $\mathfrak{m}^*=\sum_{p>0}\mathfrak{g}_p$ such that $B(e_i,e^*_j)=\delta_{ij}$. Then we define an operator $\partial^*: C^{q+1}(\mathfrak{G})\rightarrow C^q(\mathfrak{G})$ by the following formula:

$$(\partial^*c)(X_1\wedge\cdots\wedge X_q)=\sum_j\left[e^*_j, c(e_j\wedge X_1\wedge\cdots\wedge X_q)\right]$$
$$+\frac{1}{2}\sum_{i,j}(-1)^{i+1}c\left([e^*_j,X_i]_-\wedge e_j\wedge X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge X_q\right),$$

*) The discussions in this paragraph are closely related to the studies of Lie algebra cohomology done in Kostant [4].

where $c \in C^q(\mathfrak{G})$, $X_1, \cdots, X_q \in \mathfrak{m}$, and $[e^*_j, X_i]_-$ denotes the $\mathfrak{m}$-component of $[e^*_j, X_i]$ with respect to the decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{g}'$. As is easily seen, $\partial^* c$ does not depend on the choice of the basis $\{e_1, \cdots, e_m\}$.

LEMMA 1.10. *The operator ∂^* defined above is the adjoint operator of ∂ with respect to the inner product $(\ ,\ )$, that is,*

$$(\partial c, c') = (c, \partial^* c'), \qquad c \in C^q(\mathfrak{G}), \qquad c' \in C^{q+1}(\mathfrak{G}).$$

PROOF. We shall prove this lemma in the case where $q=1$. (The general case can be similarly dealt with.) Let $\{e_1, \cdots, e_m\}$ be an orthonormal basis of $\mathfrak{m}$. Then we have $e^*_i = -\sigma e_i$, and hence

$$\begin{aligned}
[e_i, e_j] &= \sum_k B\big([e_i, e_j], e^*_k\big)\, e_k = \sum_k B\big(e_i, [e_j, e^*_k]\big)\, e_k \\
&= \sum_k \overline{B\big(\sigma e_i, [\sigma e_j, \sigma e^*_k]\big)}\, e_k = -\sum_k \overline{B\big(e^*_i, [e^*_j, e_k]\big)}\, e_k \\
&= -\sum_k \overline{B\big(e^*_i, [e^*_j, e_k]_-\big)}\, e_k .
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
(\partial c, c') &= -\frac{1}{2} \sum_{i,j} B\big((\partial c)(e_i \wedge e_j), \sigma c'(e_i \wedge e_j)\big) \\
&= -\frac{1}{2} \sum_{i,j} B\Big(\big[e_i, c(e_j)\big], \sigma c'(e_i \wedge e_j)\Big) \\
&\quad + \frac{1}{2} \sum_{i,j} B\Big(\big[e_j, c(e_i)\big], \sigma c'(e_i \wedge e_j)\Big) \\
&\quad + \frac{1}{2} \sum_{i,j} B\Big(c\big([e_i, e_j]\big), \sigma c'(e_i \wedge e_j)\Big) \\
&= -\sum_{i,j} B\Big(c(e_j), \sigma\Big(\big[e^*_i, c'(e_i \wedge e_j)\big]\Big)\Big) \\
&\quad - \frac{1}{2} \sum_{j,k} B\Big(c(e_k), \sigma c'\big([e^*_j, e_k]_- \wedge e_j\big)\Big) \\
&= (c, \partial^* c'),
\end{aligned}$$

which proves Lemma 1.10.

As usual the operator

$$\Delta = \partial^* \partial + \partial \partial^* : C^q(\mathfrak{G}) \longrightarrow C^q(\mathfrak{G})$$

will be called the Laplacian, and a form $c \in C^q(\mathfrak{G})$ will be called harmonic if $\Delta c = 0$. Clearly c is harmonic if and only if $\partial c = \partial^* c = 0$. We denote by $H^q(\mathfrak{G})$ the space of all harmonic forms in $C^q(\mathfrak{G})$. Then we have the orthogonal decomposition

$$C^q(\mathfrak{G}) = H^q(\mathfrak{G}) + \Delta C^q(\mathfrak{G}) .$$

The group G_0 linearly acts on the space $C^q(\mathfrak{G})$ through the map $G_0 \times C^q(\mathfrak{G}) \ni (a, c) \to c^a \in C^q(\mathfrak{G})$ defined as follows:

$$(c^a)(X_1 \wedge \cdots \wedge X_q) = \mathrm{Ad}(a^{-1})\, c\Big(\mathrm{Ad}(a)\, X_1 \wedge \cdots \wedge \mathrm{Ad}(a)\, X_q\Big) ,$$

where $X_1, \cdots, X_q \in \mathfrak{m}$. We can easily prove the following.

LEMMA 1.11. *The action of G_0 on $C^q(\mathfrak{G})$ is compatible with the operators ∂ and ∂^*, that is,*

$$(\partial c)^a = \partial(c^a) ,$$

$$(\partial^* c)^a = \partial^*(c^a) , \qquad c \in C^q(\mathfrak{G}) , \qquad a \in G_0 .$$

Now the group G' linearly acts on the space $C^q(\mathfrak{G})$ through the map $G' \times C^q(\mathfrak{G}) \ni (a, c) \to c^a \in C^q(\mathfrak{G})$ defined as follows:

$$(c^a)(X_1 \wedge \cdots \wedge X_q) = \mathrm{Ad}(a^{-1})\, c\Big(\rho(a)\, X_1 \wedge \cdots \wedge \rho(a)\, X_q\Big) .$$

LEMMA 1.12. *The action of G' on $C^q(\mathfrak{G})$ is compatible with the operator ∂^*, that is,*

$$\partial^*(c^a) = (\partial^* c)^a , \qquad c \in C^q(\mathfrak{G}) , \qquad a \in G' .$$

PROOF. We shall prove this lemma in the case where $q=2$. (The general case can be similarly dealt with.) For any $X \in \mathfrak{m}$ we have

$$\begin{aligned}\Big(\partial^*(c^a)\Big)(X) &= \sum_j \Big[e^*_j, (c^a)(e_j \wedge X)\Big] + \frac{1}{2} \sum_j (c^a)\Big([e^*_j, X]_- \wedge e_j\Big) \\ &= \sum_j \mathrm{Ad}(a^{-1}) \Big[\mathrm{Ad}(a)\, e^*_j,\ c\Big(\rho(a)\, e_j \wedge \rho(a)\, X\Big)\Big] \\ &\quad + \frac{1}{2} \sum_j \mathrm{Ad}(a^{-1})\, c\Big(\rho(a)\, [e^*_j, X]_- \wedge \rho(a)\, e_j\Big) .\end{aligned}$$

We have $\mathrm{Ad}(a)\, e^*_j \in \mathfrak{m}^*$ and

$$\begin{aligned}B\Big(\rho(a)\, e_i,\ \mathrm{Ad}(a)\, e^*_j\Big) &= B\Big(\mathrm{Ad}(a)\, e_i,\ \mathrm{Ad}(a)\, e^*_j\Big) \\ &= B(e_i, e^*_j) = \delta_{ij} .\end{aligned}$$

Hence putting $e'_j = \rho(a)\, e_i$, we obtain $e'^*_i = \mathrm{Ad}(a)\, e^*_i$. Furthermore we can easily show that

$$\rho(a)\, [e^*_j, X]_- = \Big[e'^*_j, \rho(a)\, X\Big]_- .$$

Therefore it follows that

$$
\begin{aligned}
\left(\partial^*(c^a)\right)(X) &= \sum_j \mathrm{Ad}(a^{-1})\left[e'^*_j, c\left(e'_j \wedge \rho(a)\, X\right)\right] \\
&\quad + \frac{1}{2}\sum_j \mathrm{Ad}(a^{-1})\, c\left(\left[e'^*_j, \rho(a)\, X\right]_- \wedge e'^*_j\right) \\
&= (\partial^* c)^a(X),
\end{aligned}
$$

which proves Lemma 1. 12.

The group G' acting on the space $C^q(\mathfrak{G})$, the Lie algebra $\mathfrak{g}'$ acts on $C^q(\mathfrak{G})$ through the map $\mathfrak{g}' \times C^q(\mathfrak{G}) \ni (X, c) \to c^X \in C^q(\mathfrak{G})$ defined as follows:

$$
c^X = \frac{d}{dt}(c^{a}{}_t)_{t=0} \qquad (a_t = \exp t\, X).
$$

Clearly we have

$$
\begin{aligned}
(c^X)(X_1 \wedge \cdots \wedge X_q) &= -\left[X, c(X_1 \wedge \cdots \wedge X_q)\right] \\
&\quad + \sum_i c\left(X_1 \wedge \cdots \wedge \rho(X)\, X_i \wedge \cdots \wedge X_q\right),
\end{aligned}
$$

where $X_1, \cdots, X_q \in \mathfrak{m}$.

1. 4. The spaces $C^{p,q}(\mathfrak{G})$. Since $\mathfrak{m} = \sum_{j<0} \mathfrak{g}_j$, the space $\wedge^q(\mathfrak{m}^*)$ is decomposed as follows:

$$
\wedge^q(\mathfrak{m}^*) = \sum_{r_1, \cdots, r_q < 0} \mathfrak{g}^*_{r_1} \wedge \cdots \wedge \mathfrak{g}^*_{r_q}.
$$

For any q an i we define a subspace $\wedge^q_i(\mathfrak{m}^*)$ by

$$
\wedge^q_i(\mathfrak{m}^*) = \sum_{\substack{r_1 + \cdots + r_q = i, \\ r_1, \cdots, r_q < 0}} \mathfrak{g}^*_{r_1} \wedge \cdots \wedge \mathfrak{g}^*_{r_q}.
$$

Here we promise that $\wedge^0_0(\mathfrak{m}^*) = K$, and that $\wedge^q_i(\mathfrak{m}^*)$ vanishes in the following three cases: i) $q > 0$ and $i > -q$; ii) $q < 0$; iii) $q = 0$ and $i \neq 0$. Clearly we have

$$
\wedge^q(\mathfrak{m}^*) = \sum_i \wedge^q_i(\mathfrak{m}^*) \qquad \text{(direct sum)}.
$$

Consequently since $\mathfrak{g} = \sum_j \mathfrak{g}_j$, the space $\mathfrak{g} \otimes \wedge^q(\mathfrak{m}^*)$ is (orthogonally) decomposed as follows:

$$
\mathfrak{g} \otimes \wedge^q(\mathfrak{m}^*) = \sum_{i,j} \mathfrak{g}_j \otimes \wedge^q_i(\mathfrak{m}^*).
$$

These being prepared, we define, for any p and q, a subspace $C^{p,q}(\mathfrak{G})$ of $C^q(\mathfrak{G}) = \mathfrak{g} \otimes \wedge^q(\mathfrak{m}^*)$ by

$$
C^{p,q}(\mathfrak{G}) = \sum_j \mathfrak{g}_j \otimes \wedge^q_{j-p-q+1}(\mathfrak{m}^*).
$$

In particular we have

$$C^{p,0}(\mathfrak{G}) = \mathfrak{g}_{p-1} ,$$
$$C^{p,1}(\mathfrak{G}) = \sum_{j<p} \mathfrak{g}_j \otimes \mathfrak{g}^*{}_{j-p} .$$

Clearly we have the orthogonal decomposition

$$C^q(\mathfrak{G}) = \sum_p C^{p,q}(\mathfrak{G}) .$$

We can easily verify the following

LEMMA 1.13. (1) $\partial C^{p,q}(\mathfrak{G}) \subset C^{p-1,q+1}(\mathfrak{G})$.

(2) $\partial^* C^{p,q}(\mathfrak{G}) \subset C^{p+1,q-1}(\mathfrak{G})$.

Accordingly we obtain the two complexes $\{C^{p,q}(\mathfrak{G}), \partial\}$ and $\{C^{p,q}(\mathfrak{G}), \partial^*\}$. The complex $\{C^{p,q}(\mathfrak{G}), \partial\}$ is a generalization or rather a refinement of the so-called Spencer complex associated with a simple graded Lie algebra of the first kind.

By Lemma 1.13 we have $\Delta C^{p,q}(\mathfrak{G}) \subset C^{p,q}(\mathfrak{G})$. We denote by $H^{p,q}(\mathfrak{G})$ the space of all harmonic forms in $C^{p,q}(\mathfrak{G})$. Then the space $C^{p,q}(\mathfrak{G})$ is orthogonally decomposed as follows:

$$C^{p,q}(\mathfrak{G}) = H^{p,q}(\mathfrak{G}) + \Delta C^{p,q}(\mathfrak{G}) ,$$

and the space $H^q(\mathfrak{G})$ as follows:

$$H^q(\mathfrak{G}) = \sum_p H^{p,q}(\mathfrak{G}) .$$

The derivation algebra $\mathrm{Der}(\mathfrak{M})$ of the FGLA, $\mathfrak{M}$ may be regarded as a subspace of $C^{0,1}(\mathfrak{G}) = \sum_{j<0} \mathfrak{g}_j \otimes \mathfrak{g}^*{}_j$. This being said, we remark that $\mathrm{Der}(\mathfrak{M})$ is the kernel of the map $\partial : C^{0,1}(\mathfrak{G}) \to C^{-1,2}(\mathfrak{G})$, which means that

$$H^{0,1}(\mathfrak{G}) \cong \mathrm{Der}(\mathfrak{M})/\partial \mathfrak{g}_0 .$$

LEMMA 1.14. *$\mathfrak{G}$ is the prolongation of $(\mathfrak{M}, \mathfrak{g}_0)$ if and only if, for every $p \geqq 1$, the sequence*

$$0 \longrightarrow \partial \mathfrak{g}_p \longrightarrow C^{p,1}(\mathfrak{G}) \xrightarrow{\partial} C^{p-1,2}(\mathfrak{G})$$

is exact.

This fact is clear from the definition of the prolongation given in [7]. In terms of the spaces $H^{p,1}(\mathfrak{G})$, Lemma 1.14 means that $\mathfrak{G}$ is the prolongation of $(\mathfrak{M}, \mathfrak{g}_0)$ if and only if

$$H^{p,1}(\mathfrak{G}) = 0 , \qquad p \geqq 1 .$$

Finally we remark that $\mathfrak{g}_j \otimes \wedge^q{}_i(\mathfrak{m}^*)$ and hence $C^{p,q}(\mathfrak{G})$ are G_0-invariant subspaces of $C^q(\mathfrak{G})$, and we add the next

LEMMA 1.15. *If* $X \in \mathfrak{g}_r$ $(r \geqq 0)$ *and if* $c \in C^{p,q}(\mathfrak{G})$, *then* $c^X \in C^{p+r,q}(\mathfrak{G})$.

§ 2. $G_0^\sharp$-structures of type $\mathfrak{M}$, normal connections of type $\mathfrak{G}$, and the main theorem

In this section except the last paragraph, $\mathfrak{G}$ will be a simple graded Lie algebra of the μ-th kind over $\boldsymbol{R}$, and the differentiability will always mean that of class C^∞.

2.1. Notations. Let B be a manifold.

(1) For example consider a function $F: B \to C^2(\mathfrak{G}) = \mathfrak{g} \otimes \wedge^2(\mathfrak{m}^*)$. For any $X, Y \in \mathfrak{m}$, $F(X \wedge Y)$ denotes the $\mathfrak{g}$-valued function on B given by

$$F(X \wedge Y)(x) = F(x)(X \wedge Y), \qquad x \in B.$$

Let α and β be $\mathfrak{m}$-valued differential forms on B. Take a basis $\{e_i\}_{1 \leqq i \leqq m}$ of $\mathfrak{m}$, and express the forms α an β as follows: $\alpha = \sum_i \alpha_i e_i$ and $\beta = \sum_i \beta_i e_i$. Then $F(\alpha \wedge \beta)$ denotes the $\mathfrak{g}$-valued differential form on B given by

$$F(\alpha \wedge \beta) = \sum_{i,j} \alpha_i \wedge \beta_j \cdot F(e_i \wedge e_j).$$

(2) Let α be a $\mathfrak{g}$-valued differential form on B. α_j denotes the $\mathfrak{g}_j$-component of α with respect to the decomposition $\mathfrak{g} = \sum_j \mathfrak{g}_j$:

$$\alpha = \sum_j \alpha_j = \sum_{j=-\mu}^{\mu} \alpha_j.$$

Let F be a function $B \to C^q(\mathfrak{G}) = \mathfrak{g} \otimes \wedge^q(\mathfrak{m}^*)$. F_j denotes the $\mathfrak{g}_j$-component of F with respect to the decomposition $C^q(\mathfrak{G}) = \sum_j \mathfrak{g}_j \otimes \wedge^q(\mathfrak{m}^*)$, and F^p the $C^{p,q}(\mathfrak{G})$-component of F with respect to the decomposition $C^q(\mathfrak{G}) = \sum_p C^{p,q}(\mathfrak{G})$. Furthermore $F^p{}_j$ denotes the $C^{p,q}{}_j$-component of F with respect to the decomposition $C^q(\mathfrak{G}) = \sum_{p,j} C^{p,q}{}_j$, where

$$C^{p,q}{}_j = \mathfrak{g}_j \otimes \wedge^q{}_{j-p-q+1}(\mathfrak{m}^*)$$

Then we have

$$F = \sum_j F_j = \sum_p F^p,$$

$$F_j = \sum_p F^p{}_j, \qquad F^p = \sum_j F^p{}_j.$$

(3) For a moment let us denote by $\mathscr{C}^q$ the space of all functions F: $B \to C^q(\mathfrak{G})$. The operator $\partial: C^q(\mathfrak{G}) \to C^{q+1}(\mathfrak{G})$ naturally induces an operator $\partial: \mathscr{C}^q \to \mathscr{C}^{q+1}$:

$$(\partial F)(x) = \partial F(x), \qquad F \in \mathscr{C}^q, \qquad x \in B.$$

Similarly we have the operator $\partial^*: \mathscr{C}^{q+1} \to \mathscr{C}^q$. The action of the group G_0 on $C^q(\mathfrak{G})$ naturally induces an action of G_0 on $\mathscr{C}^q$:

$$(F^a)(x) = F(x)^a, \qquad F \in \mathscr{C}^q, \qquad x \in B.$$

Similarly the group G' as well as the Lie algebra $\mathfrak{g}'$ act on $\mathscr{C}^q$.

(4) Let $\{\theta_j\}_{j<0}$ be a system of $\mathfrak{g}_j$-valued 1-forms θ_j, $j<0$, on B. Assume that the system $\{\theta_j\}_{j<0}$ is independent in the sense that $\dim B = m + \dim N_x$, $x \in B$, where $m = \dim \mathfrak{m} = \sum_{j<0} \dim \mathfrak{g}_j$, and N_x denotes the subspace of $T(B)_x$ consisting of all $X \in T(B)_x$ such that $\theta_j(X) = 0$ for all $j<0$. Now let V be a finite dimensional vector space over $\boldsymbol{R}$, and let α and β be V-valued differential forms on B. Let p be any integer $\leqq -1$, and define a subset $I(p)$ of $\boldsymbol{Z} \times \boldsymbol{Z}$ by

$$I(p) = \left\{(r, s) \in \boldsymbol{Z} \times \boldsymbol{Z} \,\middle|\, p < r, s < 0, \text{ and } r+s < p\right\}.$$

Then by $\alpha \underset{p}{\equiv} \beta$ we mean that

$$\alpha \equiv \beta \left\{\bmod \theta_r (r \leqq p)\,;\ \theta_r \wedge \theta_s \big((r, s) \in I(p)\big)\right\}$$

(see [7]). Namely let $\mathscr{D}^*(B)$ denote the exterior algebra of all differential forms on B. Take a basis $\{e_{j\lambda}\}_{1 \leqq \lambda \leqq n_j}$ of $\mathfrak{g}_j$ for each $j<0$, and a basis $\{u_\nu\}_{1 \leqq \nu \leqq k}$ of V. And express the forms θ_j, α and β as follows: $\theta_j = \sum \theta_{j\lambda} e_{j\lambda}$, $\alpha = \sum_\nu \alpha_\nu u_\nu$ and $\beta = \sum_\nu \beta_\nu u_\nu$. Then $\alpha \underset{p}{\equiv} \beta$ means that the forms $\alpha_\nu - \beta_\nu$ are in the ideal of $\mathscr{D}^*(B)$ generated by the following forms:

$$\theta_{r\lambda} (1 \leqq \lambda \leqq n_r,\ r \leqq p)\,;$$

$$\theta_{r\lambda} \wedge \theta_{s\kappa} \big(1 \leqq \lambda \leqq n_r,\ 1 \leqq \kappa \leqq n_s,\ (r, s) \in I(p)\big).$$

2.2. $\mathbf{G}^\#_0$-structures of type $\mathfrak{M}$. In this paragraph we introduce the notion of a $G^\#_0$-structure of type $\mathfrak{M}$, which is one of the subjects in the present paper.

By Lemma 1.8 the natural representation of the group G_0 on the vector space $\mathfrak{m}$, *i.e.*, the representation $\rho: G' \to GL(\mathfrak{m})$, restricted to G_o, is faithful. Since $G_0 = \operatorname{Aut}(\mathfrak{G})$, the image $\rho(G_0)$ of G_0 by ρ is contained in the automorphism group $\operatorname{Aut}(\mathfrak{M})$ of the FGLA, $\mathfrak{M}$. It is not difficult to see that $\rho(G_0)$ is closed in $\operatorname{Aut}(\mathfrak{M})$. Hereafter we shall identify the two groups G_0 and $\rho(G_0)$ through the isomorphism $\rho: G_0 \to \rho(G_0)$.

By using the subgroup G_0 of $\operatorname{Aut}(\mathfrak{M})$, we now define a subgroup $G^\#_0$ of $GL(\mathfrak{m})$ as follows: We first denote by N^0 the subgroup of $GL(\mathfrak{m})$ consisting of all $a \in GL(\mathfrak{m})$ such that

$$aY_p \equiv Y_p \pmod{\mathfrak{d}_{p+1}} \qquad \text{for all } Y_p \in \mathfrak{g}_p \text{ and } p<0\,,$$

were $\mathfrak{d}_{p+1} = \sum_{j=p+1}^{-1} \mathfrak{g}_j$. Then we define $G_0^\sharp$ to be the closed subgroup $G_0 \cdot N^0$ of $GL(\mathfrak{m})$:

$$G^\sharp{}_0 = G_0 \cdot N^0\,.$$

We denote by $\tilde{G}$ the image $\rho(G')$ of G' by the homomorphism $\rho: G' \to GL(\mathfrak{m})$, which is nothing but the linear isotropy group of the homogeneous space G/G' at the origin 0. By Lemmas 1.7 and 1.9 the group $\tilde{G}$ may be expressed as follows:

$$\tilde{G} = G_0 \cdot \rho(\exp \mathfrak{g}_1) \cdots \rho(\exp \mathfrak{g}_{\mu-1})\,.$$

It is easy to see that $\rho(\exp \mathfrak{g}_1) \cdots \rho(\exp \mathfrak{g}_{\mu-1})$ is a closed subgroup of N^0 and hence $\tilde{G}$ is a closed subgroup of $G^\sharp{}_0$.

Remark. By definition the simple graded Lie algebra $\mathfrak{G}$ is of contact type if it is of the second kind and if $\dim \mathfrak{g}_{-2} = 1$. This being said, we remark that the two groups $G^\sharp{}_0$ and $\tilde{G}$ coincide if and only if $\mathfrak{G}$ is of the first kind or of contact type. The equivalence problems associated with these simple graded Lie algebras have been already discussed in our papers [6] and [8].

Now $G^\sharp{}_0$ being a Lie subgroup of $GL(\mathfrak{m})$, we may speak of a $G^\sharp{}_0$-structure. Let $(P^\sharp, \xi)$ be a $G^\sharp{}_0$-structure on a manifold M. Taking values in $\mathfrak{m}$, the basic form ξ may be expressed as follows:

$$\xi = \sum_{j<0} \xi_j\,.$$

Following the paper [7], we say that the $G^\sharp{}_0$-structure $(P^\sharp, \xi)$ is of type $\mathfrak{M}$ if the basic form ξ or the system $\{\xi_j\}_{j<0}$ satisfies the equations

$$d\xi_j + \frac{1}{2} \sum_{r+s=j} [\xi_r, \xi_s] \underset{j}{\equiv} 0\,, \qquad j \leqq -2\,,$$

where the symbols $\underset{j}{\equiv}$ are considered with respect to the system $\{\xi_j\}_{j<0}$. It should be noted that a $G^\sharp{}_0$-structure of type $\mathfrak{M}$ admits a (strongly) regular differential system of type $\mathfrak{M}$ as its underlying structure (see [7]).

2.3. Connections of type $\mathfrak{G}$. Let us consider the homogeneous space G/G' associated with the simple graded Lie algebra $\mathfrak{G}$. According to the previous paper [8], a Cartan connection of type G/G' will be called a connection of type $\mathfrak{G}$.

As usual the group G may be regarded as a principal fibre bundle over the base space G/G' with structure group G'. Let ω be the Maurer-Cartan form of G which is the $\mathfrak{g}$-valued 1-form on G defined by $\omega(X)=X$ for all $X\in\mathfrak{g}$, where $\mathfrak{g}$ should be regarded as the Lie algebra of all left invariant vector fields on G. Then the pair (G, ω) gives a connection of type $\mathfrak{G}$ on G/G', which is called the standard connection of type $\mathfrak{G}$.

Let (P, ω) be a connection of type $\mathfrak{G}$ on a manifold M. We denote by ω_- the $\mathfrak{m}$-component of ω with respect to the decomposition: $\mathfrak{g}=\mathfrak{m}+\mathfrak{g}'$.

LEMMA 2.1. (1) *Let* $z\in P$, *and* $X\in T(P)_z$. *Then* $\omega_-(X)=0$ *if and only if* X *is a vertical vector, i.e.,* X *is of the form* A^*_z *with some* $A\in\mathfrak{g}'$.

(2) $R^*_a\omega_-=\rho(a)^{-1}\omega_-$, $a\in G'$.

PROOF. We first recall conditions (C. 1)$\sim$(C. 3) for the Cartan connection ω. (1) Let A denote the $\mathfrak{g}'$-component of $\omega(X)$ with respect to the decomposition: $\mathfrak{g}=\mathfrak{m}+\mathfrak{g}'$. Using (C. 2), we have

$$\omega(X-A^*_z)=\omega_-(X).$$

Therefore we see from (C. 1) that $\omega_-(X)=0$ if and only if $X=A^*_z$, which clearly proves (1). (2) is clear from (C. 3).

We define a $\mathfrak{g}$-valued 2-form Ω on P by

$$\Omega=d\omega+\frac{1}{2}[\omega, \omega],$$

which is usually called the curvature form.

LEMMA 2.2. *There is a unique function* K: $P\to C^2(\mathfrak{G})=\mathfrak{g}\otimes\Lambda^2(\mathfrak{m}^*)$ *such that*

$$\Omega=\frac{1}{2}K(\omega_-\wedge\omega_-).$$

PROOF. Let $A\in\mathfrak{g}'$. Using (C. 2) and (C. 3), we have

$$A^*\lrcorner\, d\omega=L_{A^*}\omega-d\omega(A^*)=-[A, \omega],$$

whence

$$A^*\lrcorner\,\Omega=A^*\lrcorner\, d\omega+[A, \omega]=0.$$

Now the lemma follows from this fact and (1) of Lemma 2.1.

The equation in Lemma 2.2 will be called the structure equation, and the function K will be called the curvature function or simply the curvature.

For any $X\in\mathfrak{g}$ we define a vector field $\check{\omega}(X)$ on P by

$$\omega\big(\check{\omega}(\mathrm{X})\big)=\mathrm{X}.$$

Then we have the following:

(1) For any $z\in P$ the assignment $X\to\check{\omega}(X)_z$ gives a linear isomorphism of $\mathfrak{g}$ onto $T(P)_z$.

(2) $\check{\omega}(X)=X^*$, $X\in\mathfrak{g}'$.

(3) $(R_a)_*\check{\omega}(X)=\check{\omega}(\mathrm{Ad}(a^{-1})X)$, $X\in\mathfrak{g}$, $a\in G'$.

Note that the curvature function K may be explicitly given by

$$K(X\wedge Y)=[X, Y]-\omega\left(\left[\check{\omega}(X), \check{\omega}(Y)\right]\right), \qquad X, Y\in\mathfrak{m}.$$

LEMMA 2.3. $R^*_aK=K^a$, $a\in G'$.

PROOF. Using (C. 3), we have

$$R^*_a\Omega=\mathrm{Ad}(a^{-1})\,\Omega=\frac{1}{2}\mathrm{Ad}(a^{-1})\,K(\omega_-\wedge\omega_-).$$

On the other hand using (2) of Lemma 2.1, we have

$$R^*_a\Omega=\frac{1}{2}(R^*_aK)\left(\rho(a)^{-1}\omega_-\wedge\rho(a)^{-1}\omega_-\right).$$

It follows that

$$\mathrm{Ad}(a^{-1})\,K(X\wedge Y)=(R^*_aK)\left(\rho(a)^{-1}X\wedge\rho(a)^{-1}Y\right), \qquad X, Y\in\mathfrak{m},$$

proving the lemma.

Following the general notations given in 2.1, the connection form ω is decomposed as follows:

$$\omega=\sum_j\omega_j,$$

the curvature form Ω as follows:

$$\Omega=\sum_j\Omega_j,$$

and the curvature function K as follows:

$$K=\sum_j K_j=\sum_p K^p,$$

$$K_j=\sum_p K^p{}_j, \qquad K^p=\sum_j K^p{}_j.$$

We note that (C. 2) may be described as follows:

$$\omega_j(X^*_r)=\delta_{jr}X_r, \qquad X_r\in\mathfrak{g}_r, \qquad 0\leqq r\leqq\mu,$$

and (C. 3) as follows

(i) $\qquad R^*_a\omega_j=\mathrm{Ad}(a^{-1})\,\omega_j, \qquad a\in G_0,$

(ii) $\qquad \mathscr{L}_{X^*_r}\omega_j=-[X_r, \omega_{j-r}], \qquad X_r\in\mathfrak{g}_r.$

(In the proof of this last fact we use Lemma 1.7.) We also note that Lemma 2.3 is equivalent to the following two assertions:

(i) $\quad R^*_a K^p = (K^p)^a\,, \qquad a \in G_0\,,$

(ii) $\quad \mathscr{L}_{X^*_r} K^p = (K^{p-r})^{X_r}\,, \qquad X_r \in \mathfrak{g}_r\,, \qquad 1 \leqq r \leqq \mu\,,$

from which follows the following

LEMMA 2. 4. *If for some p the functions K^q vanish for all $q<p$, then we have*

$$R^*_a K^p = (K^p)^b\,, \qquad a \in G'\,,$$

where b denotes the G_0-component of a with respect to the decomposition given in Lemma 1. 7.

The structure equation is decomposed into the equations

$$(E_j) \qquad \Omega_j = \frac{1}{2} K_j(\omega_- \wedge \omega_-) = \frac{1}{2} \sum_p K^p{}_j(\omega_- \wedge \omega_-)\,,$$

and the 2-forms Ω_j and $K^p{}_j(\omega_- \wedge \omega_-)$ can be expressed respectively as follows:

$$\Omega_j = d\omega_j + \frac{1}{2} \sum_{u+v=j} [\omega_u, \omega_v]\,,$$

$$K^p{}_j(\omega_- \wedge \omega_-) = \sum_{\substack{r+s=j-p-1,\\ r,s<0}} K^p(\omega_r \wedge \omega_s)\,.$$

For any p and $j \leqq p-2$ we define a $\mathfrak{g}_j$-valued 2-form $\Omega^p{}_j$ on P by

$$\Omega^p{}_j = d\omega_j + \frac{1}{2} \sum_{\substack{u+v=j,\\ u,v \leqq p-1}} [\omega_u, \omega_v]\,.$$

The next lemma can be easily derived from equations (E_j).

LEMMA 2. 5. *For every p we have the equations*

$$(E^p{}_j) \qquad \Omega^p{}_j \underset{j-p}{\equiv} \frac{1}{2} \sum_{l \leqq p-1} K^l{}_j(\omega_- \wedge \omega_-)\,, \qquad j \leqq p-2\,,$$

where the symbols $\underset{j-p}{\equiv}$ are considered with respect to the system $\{\omega_j\}_{j<0}$.

The system of equations $(E^p{}_j)$, $j \leqq p-2$, is, so to speak, the structure equation which is satisfied by the system $\omega^{(p)} = \{\omega_j\}_{j \leqq p-1}$. Fix any integer j. Then equation $(E^p{}_j)$ is nothing but equation (E_j) for sufficiently large p or symbolically

$$(E_j) = \lim_{p \to \infty} (E^p{}_j)\,.$$

Let us now show that to every connection of type $\mathfrak{G}$, (P, ω), on a manifold M there is associated a $\tilde{G}$-structure $(\tilde{P}, \xi)$ on M in a natural manner: Let G'' denote the kernel of the homomorphism $\rho\colon G' \to \tilde{G}$. Then we define

$\tilde{P}$ to be the factor bundle P/G'', which is a principal fibre bundle over the base space M with structure group $\tilde{G}=G'/G''$. Let ρ denote the projection $P\to\tilde{P}$. Then we see from Lemma 2. 1 that there is a unique $\mathfrak{m}$-valued 1-form $\tilde{\xi}$ on $\tilde{P}$ such that $\rho^*\tilde{\xi}=\omega_-$, and that the pair $(\tilde{P}, \tilde{\xi})$ gives a $\tilde{G}$-structure on M.

Now recall that the group $\tilde{G}$ was extended to the group $G^{\sharp}{}_0\subset GL(\mathfrak{m})$. Correspondingly the $\tilde{G}$-structure $(\tilde{P}, \tilde{\xi})$ is extensible to the group $G^{\sharp}{}_0$ (see Preliminary remarks). We denote by $(P^{\sharp}, \xi)$ the extended $G^{\sharp}{}_0$-structure on M.

In this way we have seen that every connection of type $\mathfrak{G}$, (P, ω), induces a $G^{\sharp}{}_0$-structure $(P^{\sharp}, \xi)$ in a natural manner. Let (P, ω) (resp. (P', ω')) be a connection of type $\mathfrak{G}$ on a manifold M (resp. on M'), and $(P^{\sharp}, \xi)$ (resp. $(P'^{\sharp}, \xi')$) the corresponding $G^{\sharp}{}_0$-structure on M (resp. on M'). Then we remark that every isomorphism $\varphi:(P, \omega)\to(P', \omega')$ induces an isomorphism $\varphi^{\sharp}:(P^{\sharp}, \xi)\to(P'^{\sharp}, \xi')$ in a natural manner. More precisely let ρ (resp. ρ') be the natural homomorphism $P\to P^{\sharp}$ (resp. $P'\to P'^{\sharp}$). Then there is a unique isomorphism $\varphi^{\sharp}:(P^{\sharp}, \xi)\to(P'^{\sharp}, \xi')$ with $\rho'\circ\varphi=\varphi^{\sharp}\circ\rho$.

2. 4. Normal connections of type $\mathfrak{G}$, and the main theorem. We say that a connection of type $\mathfrak{G}$, (P, ω), on a manifold M is normal if the curvature K satisfies the following conditions:

(NC. 1) $\qquad K^p=0 \quad$ for $\quad p<0$;

(NC. 2) $\qquad \partial^* K^p=0 \quad$ for $\quad p\geqq 0$.

It is clear that the standard connection of type $\mathfrak{G}$, (G, ω), on G/G' is normal, because the curvature K vanishes.

LEMMA 2. 6. *Let (P, ω) be a connection of type $\mathfrak{G}$ on a manifold M, and $(P^{\sharp}, \xi)$ the corresponding $G^{\sharp}{}_0$-structure on M. If (P, ω) is normal, then $(P^{\sharp}, \xi)$ is of type $\mathfrak{M}$.*

PROOF. Let $(\tilde{P}, \tilde{\xi})$ be the $\tilde{G}$-structure on M corresponding to (P, ω). We have $\rho^*\tilde{\xi}=\omega_-$, and from (NC. 1) and Lemma 2. 5 we obtain

$$\Omega^0{}_j \underset{j}{\equiv} 0, \qquad j\leqq -2.$$

Hence we see that the basic form $\tilde{\xi}$ of $\tilde{P}$ satisfies the equations

$$d\tilde{\xi}_j+\frac{1}{2}\sum_{r+s=j}[\tilde{\xi}_r, \tilde{\xi}_s] \underset{j}{\equiv} 0, \qquad j\leqq -2,$$

where the symbols $\underset{j}{\equiv}$ are, of course, considered with respect to the system $\{\tilde{\xi}_j\}_{j<0}$. Using this fact, we show that $(P^{\sharp}, \xi)$ is of type $\mathfrak{M}$, *i. e.*,

$$d\xi_j+\frac{1}{2}\sum_{r+s=j}[\xi_r, \xi_s] \underset{j}{\equiv} 0, \qquad j\leqq -2.$$

Indeed since the matter is of local character, we may assume that $\tilde{P}$ admits a global cross section, say g. Then there is a unique map $a: P^{\#}\to G^{\#}_0$ such that $z=g(\pi(z))\cdot a(z)$, $z\in P^{\#}$, π being the projection $P^{\#}\to M$. If we put $\eta=g^*\xi=g^*\tilde{\xi}$, we have $\xi=a^{-1}\cdot\pi^*\eta$. Let b be the G_0-component of a with respect to the decomposition $G^{\#}_0=G_0\cdot N^0$. Then we easily see that

$$\xi_j\equiv b^{-1}\cdot\pi^*\eta_j \qquad \Big(\mathrm{mod}\ \pi^*\eta_r(r<j)\Big),$$

$$\pi^*\eta_j\equiv b\cdot\xi_j \qquad \Big(\mathrm{mod}\ \xi_r(r<j)\Big).$$

Furthermore we see from the equations above for $\tilde{\xi}$ that

$$d\eta_j+\frac{1}{2}\sum_{r+s=j}[\eta_r,\eta_s]\equiv 0 \qquad \Big(\mathrm{mod}\ \eta_r\wedge\eta_s(r+s<j)\Big).$$

Now our assertion follows easily from these facts. We have thus proved the lemma.

We are now in a position to state the main theorem in the present paper:

THEOREM 2.7. *Let $\mathfrak{G}$ be a simple graded Lie algebra over $\boldsymbol{R}$. Assume that $\mathfrak{G}$ is the prolongation of* $(\mathfrak{M},\mathfrak{g}_0)$.

(1) *Every normal connection of type $\mathfrak{G}$, (P,ω), on a manifold M induces a $G^{\#}_0$-structure of type $\mathfrak{M}$, $(P^{\#},\xi)$, on M in a natural manner. Conversely if $(P^{\#},\xi)$ is a $G^{\#}_0$-structure of type $\mathfrak{M}$ on M, there is a normal connection of type $\mathfrak{G}$, (P,ω), on M which induces the given $(P^{\#},\xi)$.*

(2) *Let (P,ω) (resp. (P',ω')) be a normal connection of type $\mathfrak{G}$ on a manifold M (resp. on M'), and $(P^{\#},\xi)$ (resp. $(P'^{\#},\xi')$) the corresponding $G^{\#}_0$-structure of type $\mathfrak{M}$ on M (resp. on M'). Then every isomorphism $\varphi: (P,\omega)\to(P',\omega')$ induces an isomorphism $\varphi^{\#}: (P^{\#},\xi)\to(P'^{\#},\xi')$ in a natural manner. Conversely if $\varphi^{\#}: (P^{\#},\xi)\to(P'^{\#},\xi')$ is an isomorphism, there is a unique isomorphism $\varphi: (P,\omega)\to(P',\omega')$ which induces the given $\varphi^{\#}$.*

This theorem will be proved in § 3～§ 5.

REMARK. Let (P,ω) be a connection of type $\mathfrak{G}$ on a manifold M. By the very definition of the operator ∂^*, the function ∂^*K can be described as follows:

$$\partial^*K=K^*+R,$$

where the functions K^* and R are respectively given by

$$K^*(X)=\sum_i\Big[e^*_i,K(e_i\wedge X)\Big],$$

$$R(X)=\frac{1}{2}\sum_i K\Big([e^*_i,X]_-\wedge e_i\Big),\qquad X\in\mathfrak{m}.$$

The function K^* was already introduced in the previous paper [8] as the $*$-curvature. Furthermore in the special case where $\mathfrak{G}$ is of contact type, the connection (P, ω) was defined to be normal if the $*$-curvature K^* vanishes: However we have the following.

PROPOSITION 2.8. *Assume that $\mathfrak{G}$ is of contact type and that it is the prolongation of $(\mathfrak{M}, \mathfrak{g}_0)$. Assume further that the torsion part of K vanishes, i.e., $K_{-2}=K_{-1}=0$. Then the connection (P, ω) is normal in our present sense if and only if it is normal in our previous sense.*

The proof of this fact uses the results in [8] (especially § 8 there) together with Theorem 2.7. For example the assumptions in the proposition are satisfied by the connection (see [8]) which is naturally attached to a non-degenerate real hypersurface of a complex manifold.

2.5. Fundamental systems of invariants. Let (P, ω) be a normal connection of type $\mathfrak{G}$ on a manifold M. (We do not assume that $\mathfrak{G}$ is the prolongation of $(\mathfrak{M}, \mathfrak{g}_0)$.) The main aim of this paragraph is to show that the harmonic part $H(K)$ of the curvature K gives a fundamental system of invariants of the connection (P, ω) (Theorem 2.9).

We first explain the notations, which will be necessary for the statement of the theorem.

We denote by $\mathscr{F}(P)$ the algebra of all differentiable functions on P. In general let V be a finite dimensional vector space over $\boldsymbol{R}$, and L a differentiable function $P \to V$. We denote by $\mathscr{I}_0(L)$ the subspace of $\mathscr{F}(P)$ consisting of all the functions of the form

$$\langle L, v^* \rangle ,$$

where $v^* \in V^*$. And we denote by $\hat{\mathscr{I}}_0(L)$ the subspace of $\mathscr{F}(P)$ spanned by the functions in $\mathscr{I}_0(L)$ and their successive covariant derivatives, *i.e.*, all the functions of the form:

$$\check{\omega}(X_1) \cdots \check{\omega}(X_l) f ,$$

where $f \in \mathscr{I}_0(L)$, and $X_1, \cdots, X_l \in \mathfrak{m}$ $(l=0, 1, 2, \cdots)$. Furthermore we denote by $\hat{\mathscr{I}}(L)$ the subalgebra of $\mathscr{F}(P)$ generated by $\hat{\mathscr{I}}_0(L)$.

Applying the notations above to the curvature $K: P \to C^2(\mathfrak{G})$, we have the spaces $\hat{\mathscr{I}}_0(K)$ and $\hat{\mathscr{I}}(K)$. Let us now recall that the sapce $C^2(\mathfrak{G})$ is orthogonally decomposed as follows:

$$C^2(\mathfrak{G}) = H^2(\mathfrak{G}) + \varDelta C^2(\mathfrak{G}) .$$

We denote by H the orthogonal projection $C^2(\mathfrak{G}) \to H^2(\mathfrak{G})$. Applying the operator H to the curvature K, we have the function $H(K): P \to H^2(\mathfrak{G})$, the harmonic part of K. Then the function $H(K)$ gives rise to the spaces $\hat{\mathscr{I}}_0(H(K))$ and $\hat{\mathscr{I}}(H(K))$. Clearly we have $\hat{\mathscr{I}}_0(H(K))\subset\hat{\mathscr{I}}_0(K)$ and $\hat{\mathscr{I}}(H(K))\subset\hat{\mathscr{I}}(K)$. Finally we denote by K_- the torsion part of the curvature K, *i. e.*,

$$K_-=\sum_{j<0} K_j\,.$$

These being prepared, we state the following

THEOREM 2.9. (1) $\hat{\mathscr{I}}(K)=\hat{\mathscr{I}}(H(K))$.

(2) *If* $K_-=0$, *then* $\hat{\mathscr{I}}_0(K)=\hat{\mathscr{I}}_0(H(K))$.

COROLLARY. *The curvature K vanishes if and only if the harmonic part $H(K)$ of K vanishes.*

By E. Cartan [1], the space $\hat{\mathscr{I}}_0(K)$ or $\hat{\mathscr{I}}(K)$ gives a complete system of invariants of the connection (P,ω). Theorem 2.9 indicates that the harmonic part $H(K)$ of the curvature K gives a fundamental system of invariants. The study of the fundamental system of invariants is preceded by the calculation of the spaces of harmonic forms, $H^{p,2}(\mathfrak{G})$. In the forthcoming papers we shall calculate the spaces $H^{p,2}(\mathfrak{G})$ for various simple graded Lie algebras $\mathfrak{G}$.

We shall now prove Theorem 2.9. The proof is based on the Bianchi identity for the connection (P,ω), which may be stated as follows:

LEMMA 2.10. (cf. [8], Lemma 8.2).

$$(\partial K)(X_1\wedge X_2\wedge X_3)=-\mathfrak{S}\check{\omega}(X_1)\,K(X_2\wedge X_3)-\mathfrak{S}K\Big(K_-(X_1\wedge X_2)\wedge X_3\Big)\,,$$

where $X_1, X_2, X_3\in\mathfrak{m}$ *and* $\mathfrak{S}$ *stands for the cyclic sum with respect to* (X_1, X_2, X_3).

The curvature K is decomposed as follows:

$$K=\sum_p K^p\,.$$

For any $p\geqq 0$, we define a function $\Psi^{p-1}\colon P\rightarrow C^{p-1,3}(\mathfrak{G})$ in the following manner: Take any negative integers r_1, r_2, and r_3. For each $1\leqq i\leqq 3$, take any vector $X_i\in\mathfrak{g}_{r_i}$. Then the function $\Psi^{p-1}(X_1\wedge X_2\wedge X_3)$ is defined by the following formula:

$$\begin{aligned}\Psi^{p-1}(X_1\wedge X_2\wedge X_3)=&-\check{\omega}(X_1)\,K^{p+r_1}(X_2\wedge X_3)-\check{\omega}(X_2)\,K^{p+r_2}(X_3\wedge X_1)\\&-\check{\omega}(X_3)\,K^{p+r_3}(X_1\wedge X_2)-\sum_{l=0}^{-r_1-r_2-2}K^{p-l-1}\Big(K^l(X_1\wedge X_2)\wedge X_3\Big)\\&-\sum_{l=0}^{-r_2-r_3-2}K^{p-l-1}\Big(K^l(X_2\wedge X_3)\wedge X_1\Big)-\sum_{l=0}^{-r_3-r_1-2}K^{p-l-1}\Big(K^l(X_3\wedge X_1)\wedge X_2\Big)\,.\end{aligned}$$

(Putting $i=r_1+r_2+r_3$, we see that $\Psi^{p-1}(X_1\wedge X_2\wedge X_3)$ takes values in $\mathfrak{g}_{i+p+1}$. Hence the formula above really defines a function $\Psi^{p-1}\colon P\rightarrow C^{p-1,3}(\mathfrak{G})$.)

LEMMA 2.11. *The Bianchi identity may be described as follows*:

$$\partial K^p = \Psi^{p-1}, \qquad p \geqq 0.$$

PROOF. The notations being as above, we first remark that $(\partial K^p)(X_1 \wedge X_2 \wedge X_3)$ is the $\mathfrak{g}_{i+p+1}$-component of $(\partial K)(X_1 \wedge X_2 \wedge X_3)$. Clearly the $\mathfrak{g}_{i+p+1}$-component of $\check{\omega}(X_1) K(X_2 \wedge X_3)$ is $\check{\omega}(X_1) K^{p+r_1}(X_2 \wedge X_3)$. By using (NC. 1), we see that

$$K_-(X_1 \wedge X_2) = \sum_{l=0}^{-r_1-r_2-2} K^l(X_1 \wedge X_2),$$

and hence that the $\mathfrak{g}_{i+p+1}$-component of $K(K_-(X_1 \wedge X_2) \wedge X_3)$ is

$$\sum_{l=0}^{-r_1-r_2-2} K^{p-l-1}\left(K^l(X_1 \wedge X_2) \wedge X_3\right).$$

It is now clear that the Bianchi identity may be described as in Lemma 2.11.

LEMMA 2.12. (1) $\mathscr{I}_0(\Psi^{p-1}) \subset \hat{\mathscr{I}}\left(\sum_{l=0}^{p-1} K^l\right)$.

(2) *If* $K_-=0$, *then* $\mathscr{I}_0(\Psi^{p-1}) \subset \hat{\mathscr{I}}_0\left(\sum_{l=0}^{p-1} K^l\right)$.

These facts are clear from the definition of Ψ^{p-1} and the proof of Lemma 2.11.

We put

$$L^p = K^p - H(K^p),$$

which takes values in $\varDelta C^{p,2}(\mathfrak{G})$.

LEMMA 2.13. $L^p = \varDelta^{-1}\partial^* \Psi^{p-1}$, $p \geqq 0$.

PROOF. By Lemma 2.11 we have $\partial L^p = \partial(K^p - H(K^p)) = \partial K^p = \Psi^{p-1}$. By (NC. 2) we have $\partial^* L^p = \partial^*(K^p - H(K^p)) = \partial^* K^p = 0$. It follows that $\varDelta L^p = \partial^* \Psi^{p-1}$ and hence $L^p = \varDelta^{-1}\partial^* \Psi^{p-1}$.

We are now in a position to prove Theorem 2.9. By (1) of Lemma 2.12. and Lemma 2.13 we have

$$\mathscr{I}_0(L^p) \subset \hat{\mathscr{I}}\left(\sum_{l=0}^{p-1} K^l\right) = \hat{\mathscr{I}}\left(\sum_{l=0}^{p-1} H(K^l) + \sum_{l=0}^{p-1} L^l\right).$$

Since $\partial K^0 = \Psi^{-1} = 0$, we have $K^0 = H(K^0)$. Therefore it follows that

$$\mathscr{I}_0(L^p) \subset \hat{\mathscr{I}}\left(\sum_{l=0}^{p-1} H(K^l)\right), \qquad p \geqq 0.$$

This fact clearly implies that $\mathscr{I}_0(K) \subset \hat{\mathscr{I}}(H(K))$ and hence $\hat{\mathscr{I}}(K) \subset \hat{\mathscr{I}}(H(K))$. Since $\hat{\mathscr{I}}(H(K)) \subset \hat{\mathscr{I}}(K)$, we obtain $\hat{\mathscr{I}}(K) = \hat{\mathscr{I}}(H(K))$, proving (1). Now assume that $K_-=0$. By (2) of Lemma 2.12, and Lemma 2.13 we see as above that

$$\mathscr{I}_0(L^p)\subset\hat{\mathscr{I}}_0\left(\sum_{l=0}^{p-1}H(K^l)\right), \qquad p\geqq 0\,.$$

Hence we obtain $\hat{\mathscr{I}}_0(K)=\hat{\mathscr{I}}_0(H(K))$, proving (2).

We have thus completed the proof of Theorem2.9.

2.6. Normal connections of type $\mathfrak{G}$ in the complex analytic category. Let $\mathfrak{G}$ be a simple graded Lie algebra over the field $\boldsymbol{C}$ of complex numbers. There correspond to $\mathfrak{G}$ the FGLA over $\boldsymbol{C}$, $\mathfrak{M}$, and the complex Lie groups G, G' and G_0 (see §1). Thus we have the notion of a $G^{\#}{}_0$-structure of type $\mathfrak{M}$ in the complex analytic category as well as the notion of a normal connection of type $\mathfrak{G}$ in the complex analytic categroy. (These notions can be defined in the same manner as in the real case by considering everything in the complex analtyic category and by localizing the definition of $\underset{p}{\equiv}$.)

Here we remark that Theorem 2.7 holds in the complex analytic category, which can be deduced from the proof of the theorem given in §3~§5. Moreover we remark that Theorem 2.9 also holds in the complex analytic category.

§3. A reduction theorem for $G^{\#}{}_0$-structures of type $\mathfrak{M}$

This and the subsequent two sections will be devoted to the proof of Theorem 2.7, as we promised. In the following $\mathfrak{G}$ will be a simple graded Lie algebra of the μ-th kind over $\boldsymbol{R}$, and we shall assume that $\mathfrak{G}$ is the prolongation of $(\mathfrak{M}, \mathfrak{g}_0)$. The spaces $C^{p,q}(\mathfrak{G})$ will be simply written as $C^{p,q}$.

In this section we shall prove the important fact that every $G^{\#}{}_0$-structure of type $\mathfrak{M}$ is naturally reduced to a $\tilde{G}$-structure (Theorem 3.7).

3.1. Algebraic preliminaries. For any $p\geqq 0$, we define subspaces $C^{p,1}{}_-$ and $C^{p,1}{}_+$ of $C^{p,1}$ respectively as follows:

$$C^{p,1}{}_-=\sum_{j<0}\mathfrak{g}_j\otimes\mathfrak{g}^*{}_{j-p}\,,$$

$$C^{p,1}{}_+=\sum_{0\leqq j<p}\mathfrak{g}_j\otimes\mathfrak{g}^*{}_{j-p}\,.$$

Clearly we hvae

$$C^{p,1}=C^{p,1}{}_-+C^{p,1}{}_+ \qquad \text{(direct sum)}\,.$$

Note that $C^{0,1}{}_+=0$ and $C^{p,1}{}_-=0$, $p\geqq\mu$. We denote by π_- (resp. by π_+) the projection of $C^{p,1}$ onto $C^{p,1}{}_-$ (resp. onto $C^{p,1}{}_+$).

By Lemma 1.14 we easily have the following

LEMMA 3.1. $\pi_-(\partial\mathfrak{g}_p)=\{c\in C^{p,1}{}_-|\partial c\in\partial C^{p,1}{}_+\}$, $p\geqq 1$.

LEMMA 3.2. (1) *If* $p\geqq 1$, *then the map* $\pi_+\circ\partial: \mathfrak{g}_p\to C^{p,1}{}_+$ is injective.
(2) *If* $1\leqq p\leqq\mu-1$, *then the map* $\pi_-\circ\partial: \mathfrak{g}_p\to C^{p,1}{}_-$ *is injective.*

The first assertion of this lemma follows from Lemma 1.3, and the second from Lemma 1.6. The next lemma is clear from Lemmas 1.14 and 3.2.

LEMMA 3.3. (1) *If* $p\geqq 1$, *then the map* $\partial: C^{p,1}{}_-\to C^{p-1,2}$ *is injective.*
(2) *If* $1\leqq p\leqq\mu-1$, *then the map* $\partial: C^{p,1}{}_+\to C^{p-1,2}$ *is injective.*

For any p and q we define a subspace $Z^{p,q}{}_*$ of $C^{p,q}$ by

$$Z^{p,q}{}_*=\left\{c\in C^{p,q}\middle|\partial^* c=0\right\}.$$

Clearly we have

$$C^{p,q}=Z^{p,q}{}_*+\partial C^{p+1,q-1}\qquad\text{(direct sum)}.$$

For any $p\geqq 1$ we now define a subspace $W^{p,1}$ of $C^{p,1}{}_-$ by

$$W^{p,1}=Z^{p,1}{}_*\cap C^{p,1}{}_-.$$

LEMMA 3.4. *Let* $p\geqq 1$.
(1) $C^{p,1}{}_-=W^{p,1}+\pi_-(\partial\mathfrak{g}_p)$ *(direct sum).*
(2) $C^{p-1,2}=Z^{p-1,2}{}_*+\partial W^{p,1}+\partial C^{p,1}{}_+$ *(direct sum).*

PROOF. Let $c\in C^{p,1}{}_-$. Since $C^{p,1}{}_-\perp C^{p,1}{}_+$ with respect to the inner product (,), we have $(c,\pi_-(\partial\mathfrak{g}_p))=(c,\partial\mathfrak{g}_p)=(\partial^* c,\mathfrak{g}_p)$. Hence it follows that $W^{p,1}$ is the orthogonal complement of $\pi_-(\partial\mathfrak{g}_p)$ in $C^{p,1}{}_-$, proving (1). Let us prove (2). Using (1) we have

$$\begin{aligned}\partial C^{p,1}&=\partial C^{p,1}{}_-+\partial C^{p,1}{}_+\\&=\partial W^{p,1}+\partial\pi_-(\partial\mathfrak{g}_p)+\partial C^{p,1}{}_+.\end{aligned}$$

By Lemma 3.1 we have $\partial\pi_-(\partial\mathfrak{g}_p)\subset\partial C^{p,1}{}_+$, and by Lemma 3.1 and (1) we have $\partial W^{p,1}\cap\partial C^{p,1}{}_+=\partial(W^{p,1}\cap\pi_-(\partial\mathfrak{g}_p))=0$. Hence we obtain

$$\partial C^{p,1}=\partial W^{p,1}+\partial C^{p,1}{}_+\qquad\text{(direct sum)}.$$

Since $C^{p-1,2}=Z^{p-1,2}{}_*+\partial C^{p,1}$ (direct sum), we have proved Lemma 3.4.

The Lie algebras $\tilde{\mathfrak{g}}$ and $\mathfrak{g}^\#{}_0$. We denote by $\tilde{\mathfrak{g}}$ and $\mathfrak{g}^\#{}_0$ the Lie algebras of $\tilde{G}$ and $G^\#{}_0$ respectively, which are subalgebras of $\mathfrak{gl}(\mathfrak{m})=\mathfrak{m}\otimes\mathfrak{m}^*$. Then we have

$$\tilde{\mathfrak{g}}=\sum_{p=0}^{\mu-1}\rho(\mathfrak{g}_p),$$

$$\mathfrak{g}^\#{}_0=\rho(\mathfrak{g}_p)+\sum_{p=0}^{\mu-1}C^{p,1}{}_-.$$

Here we notice that $\rho(\mathfrak{g}_p)\subset C^{p,1}{}_-$ or more precisely

$$\rho(X_p) = -\pi_-(\partial X_p)\,, \qquad X_p\in\mathfrak{g}_p\,.$$

The subalgebras $\mathfrak{u}^p$ of $\mathfrak{g}^\sharp{}_0$. For each $0\leqq p\leqq\mu-1$ we define a subalgebra $\mathfrak{u}^p$ of $\mathfrak{g}^\sharp{}_0$ by

$$\mathfrak{u}^p = \sum_{j=0}^{p}\rho(\mathfrak{g}_j)+\mathfrak{q}^p\,,$$

where

$$\mathfrak{q}^p = \sum_{j=p+1}^{\mu-1} C^{j,1}{}_-\,.$$

Clearly we have

$$\mathfrak{g}^\sharp{}_0=\mathfrak{u}^0\supset\mathfrak{u}^1\supset\cdots\supset\mathfrak{u}^{\mu-1}=\tilde{\mathfrak{g}}\,.$$

The subgroups U^p of $G^\sharp{}_0$. For each $0\leqq p\leqq\mu-1$ we define a subalgebra $\mathfrak{n}^p$ of $\mathfrak{u}^p$ by

$$\mathfrak{n}^p = \sum_{j=1}^{p}\rho(\mathfrak{g}_j)+\mathfrak{q}^p\,,$$

and denote by N^p the (closed) Lie subgroup of $G^\sharp{}_0$ generated by $\mathfrak{n}^p$. (The group N^0 defined here coincides with the group N^0 defined in 2. 2.) We have

$$N^0\supset N^1\supset\cdots\supset N^{\mu-1}\,.$$

We then define a closed subgroup U^p of $G^\sharp{}_0$ by

$$U^p = G_0\cdot N^p\,.$$

It is clear that the Lie algebra of U^p is $\mathfrak{u}^p$, and that

$$G^\sharp{}_0=U^0\supset U^1\supset\cdots\supset U^{\mu-1}=\tilde{G}\,.$$

3. 2. Normal p-systems in U^p-structures. U^p being a Lie subgroup of $GL(\mathfrak{m})$, we have the notion of a U^p-structure.

Let (B^p,ξ) be a U^p-structure on a manifold M. Taking values in $\mathfrak{m}$, the baisc form ξ may be expressed as follows:

$$\xi=\sum_{j<0}\xi_j\,.$$

LEMMA 3. 5. (1) $\xi_j(A^*)=0,\ A\in\mathfrak{u}^p$.

(2) $R^*{}_a\xi_j=\mathrm{Ad}(a^{-1})\,\xi_j,\ a\in G_0$.

(3) $\mathscr{L}_{\rho(X_r)^*}\xi_j=-[X_r,\xi_{j-r}],\ X_r\in\mathfrak{g}_r,\ 0\leqq r\leqq p$.

(4) $\mathscr{L}_{A^*}\xi_j=-A(\xi_{j-i}),\ A\in C^{i,1}{}_-,\ p+1\leqq i\leqq\mu-1$.

This lemma is clear from the following conditions for the U^p-structure:

i) $\xi(A^*)=0,\ A\in\mathfrak{u}^p$; ii) $R^*_a\xi=a^{-1}\xi,\ a\in U^p$.

We shall now introduce the notion of a p-system in (B^p, ξ). Let $\theta^{(p)}=\{\theta_j\}_{j\leqq p-1}$ be a system of $\mathfrak{g}_j$-valued 1-forms θ_j, $j\leqq p-1$, on B^p. Assume that $\theta^{(p)}$ is compatible with the basic form ξ, *i.e.*, $\theta_i=\xi_i$, $i<0$. Then we say that $\theta^{(p)}$ is a p-system in (B^p, ξ) if it satisfies the following conditions:

$(p.\ 1)$ i) $\theta_j(\rho(X_r)^*)=\delta_{jr}X_r,\ X_r\in\mathfrak{g}_r,\ 0\leqq r\leqq p,$

ii) $\theta_j(A^*)=0,\ A\in\mathfrak{q}^p.$

$(p.\ 2)$ i) $R^*_a\theta_j=\mathrm{Ad}(a^{-1})\,\theta_j,\ a\in G_0,$

ii) $\mathscr{L}_{\rho(X_r)^*}\theta_j \underset{j-p-1}{\equiv} -[X_r, \theta_{j-r}],\ X_r\in\mathfrak{g}_r,\ 0\leqq r\leqq p,$

iii) $\mathscr{L}_{A^*}\theta_j \underset{j-p-1}{\equiv} 0,\ A\in\mathfrak{q}^p,$

where the symbols $\underset{j-p-1}{\equiv}$ are considered with respect to the system $\{\theta_j\}_{j<0}$.

It should be noted that these conditions are compatible with Lemma 3.5.

Let $\theta^{(p)}$ be a p-system in (B^p, ξ). Let $q\leqq p$ and $j\leqq q-2$. Using the 1-forms θ_i, $i\leqq q-1$, we define a $\mathfrak{g}_j$-valued 2-form Θ^q_j on B^p by

$$\Theta^q_j=d\theta_j+\frac{1}{2}\sum_{\substack{u+v=j,\\ u,v\leqq q-1}}[\theta_u, \theta_v]\,.$$

Hereafter the symbols $\underset{k}{\equiv}$ will always be considered with respect to the system $\{\theta_j\}_{j<0}$.

LEMMA 3.6. *There are unique functions R^l: $B^p\rightarrow C^{l,2}$, $l\leqq p-1$, such that*

$$\Theta^p_j\underset{j-p}{\equiv}\frac{1}{2}\sum_{l\leqq p-1}R^l_j(\theta_-\wedge\theta_-)\,,\qquad j\leqq p-2\,,$$

where $\theta_-=\sum_{j<0}\theta_j=\xi$.

PROOF. This lemma can be easily derived from the equalities

$$A^*\rfloor\,\Theta^p_j\underset{j-p}{\equiv}0\,,\qquad A\in\mathfrak{u}^p\,,\qquad j\leqq p-2\,,$$

which we shall prove from now on. By $(p.\ 1)$ we have

$$\mathscr{L}_{A^*}\theta_j=A^*\rfloor\,d\theta_j+d\theta_j(A^*)=A^*\rfloor\,d\theta_j\,,$$

whence

$$A^*\rfloor\,\Theta^p_j=\mathscr{L}_{A^*}\theta_j+\sum_{u+v=j}\Big[\theta_u(A^*), \theta_v\Big]$$

If A is of the form $\rho(X_r)$ with some $0\leqq r\leqq p-1$, it follows from $(p.\ 1)$ and $(p.\ 2)$ that

$$\rho(X_r)^* \lrcorner\, \Theta_j^p \underset{j-p-1}{\equiv} -[X_r, \theta_{j-r}] + [X_r, \theta_{j-r}]$$
$$\underset{j-p-1}{\equiv} 0 .$$

Similarly if $A = \rho(X_p)$, we have

$$\rho(X_p)^* \lrcorner\, \Theta_j^p \underset{j-p-1}{\equiv} -[X_p, \theta_{j-p}] \underset{j-p}{\equiv} 0 ,$$

and if $A \in \mathfrak{q}^p$, we have

$$A^* \lrcorner\, \Theta_j^p \underset{j-p-1}{\equiv} 0 .$$

We have thus shown that $A^* \lrcorner\, \Theta_j^p \underset{j-p}{\equiv} 0$ for all $A \in \mathfrak{u}^p$, proving Lemma 3. 6.

The system of the equations in Lemma 3. 6 will be called the structure equation, and the system of functions, $\{R^l\}_{l \leqq p-1}$, will be called the curvature. Clearly the structure equation induces the equations

$$\Theta_j^q \underset{j-q}{\equiv} \frac{1}{2} \sum_{l \leqq q-1} R_j^l(\theta_- \wedge \theta_-) , \qquad j \leqq q-2 , \qquad q \leqq p .$$

Finally we say that the p-system $\theta^{(p)}$ is normal if its curvature satisfies the following conditions:

i) $R^l = 0$ for $l < 0$,

ii) $\partial^* R^l = 0$ for $0 \leqq l \leqq p-1$.

3. 3. Reduction theorems. We say that a U^p-structure is of type $(\mathfrak{M}, p)$ if it admits a normal p-system. We have $U^0 = G^\sharp_0$. Clearly a U^0-structure of type $(\mathfrak{M}, 0)$ means a $G^\sharp_0$-structure of type $\mathfrak{M}$. We have $U^{\mu-1} = \tilde{G}$. Thus we have the notion of a $\tilde{G}$-structure of type $(\mathfrak{M}, \mu-1)$.

The main aim of this section is to prove the following

THEOREM 3. 7. *Every $G^\sharp_0$-structure of type $\mathfrak{M}$ is reduced to a unique $\tilde{G}$-structure of type $(\mathfrak{M}, \mu-1)$.*

The existence part of this theorem is derived from the following

THEOREM 3. 8. *If $0 \leqq p \leqq \mu-2$, then every U^p-structure of type $(\mathfrak{M}, p)$ is naturally reduced to a U^{p+1}-structure of type $(\mathfrak{M}, p+1)$.*

In the subsequent paragraphs we shall prove these theorems together with some related facts.

3. 4. The invariance of normal p-systems in U^p-structures of type $(\mathfrak{M}, p)$.

LEMMA 3. 9. *Let (B^p, ξ) be a U^p-structure of type $(\mathfrak{M}, p)$ on a manifold M. Let $\theta^{(p)} = \{\theta_j\}_{j \leqq p-1}$ and $\theta'^{(p)} = \{\theta'_j\}_{j \leqq p-1}$ be two normal p-systems in (B^p, ξ). Then we have*

$$\theta'_j \underset{j-p-1}{\equiv} \theta_j , \qquad j \leqq p-1 .$$

Let $\{R^l\}_{l \leqq p-1}$ and $\{R'^l\}_{l \leqq p-1}$ be the curvatures of $\theta^{(p)}$ and $\theta'^{(p)}$ respectively. The proof below will indicate that

$$R'^l = R^l, \qquad l \leqq p-1.$$

Accordingly we know that the curvature of a normal p-system is an invariant, while the p-system itself is only an invariant in an equivalence relation.

PROOF of LEMMA 3.9. By induction we shall prove the following statements:

$$(S_q) \qquad \theta'_j \underset{j-q-1}{\equiv} \theta_j. \qquad j \leqq q-1,$$

where $0 \leqq q \leqq p$. Clearly (S_0) is true, because $\theta'_j = \theta_j \ (=\xi_j)$, $j<0$. Assuming (S_q) for some $0 \leqq q < p$, we shall prove (S_{q+1}).

By $(p.\ 1)$ we have $\theta'_q(A^*) = \theta_q(A^*)$ for every $A \in \mathfrak{u}^p$, meaning that $\theta'_q \underset{-1}{\equiv} \theta_q$. As we have just remarked, $\theta'_j = \theta_j$, $j<0$. From these facts together with (S_q) we see that there is a unique function $f^{q+1}: B^p \to C_+^{q+1,1}$ such that

$$\theta'_j \underset{j-q-2}{\equiv} \theta_j + f^{q+1}(\theta_{j-q-1}), \qquad j \leqq q. \tag{3.1}$$

The structure equations for $\theta^{(p+1)}$ and $\theta'^{(p+1)}$ yield the equalities

$$\Theta'^{q+1}_j - \Theta^{q+1}_j \underset{j-q-1}{\equiv} \frac{1}{2} \sum_{l \leqq q} (R'^l_j - R^l_j)(\theta_- \wedge \theta_-), \qquad j \leqq q-1, \tag{3.2}$$

where

$$\Theta^{q+1}_j = d\theta_j + \frac{1}{2} \sum_{u+v=j} [\theta_u, \theta_v], \tag{3.3}$$

$$\Theta'^{q+1}_j = d\theta'_j + \frac{1}{2} \sum_{u+v=j} [\theta'_u, \theta'_v]. \tag{3.3'}$$

(In the following, the letters u and v mean integers $\leqq q$, while the letters r and s negative integers.) We shall now calculate the left side of (3.2).

Since $\theta^{(p)}$ is normal, we have $R^l = 0$, $l<0$. Hence we obtain $\Theta^0_i \underset{i}{\equiv} 0$, $i \leqq -2$, whence

$$d\theta_i \underset{j-q-1}{\equiv} 0, \qquad i \leqq j-q-2,$$

$$d\theta_{j-q-1} \underset{j-q-1}{\equiv} -\frac{1}{2} \sum_{r+s=j-q-1} [\theta_r, \theta_s].$$

Therefore we see from (3.1) that

$$\begin{aligned} d\theta'_j - d\theta_j &\underset{j-q-1}{\equiv} f^{q+1}(d\theta_{j-q-1}) \\ &\underset{j-q-1}{\equiv} -\frac{1}{2} \sum_{r+s=j-q-1} f^{q+1}\big([\theta_r, \theta_s]\big). \end{aligned} \tag{3.4}$$

Moreover we see form (3. 1) that

$$(3.5)\qquad \frac{1}{2}\sum_{u+v=j}[\theta'_u, \theta'_v]-\frac{1}{2}\sum_{u+v=j}[\theta_u, \theta_v]$$
$$\underset{j-q-1}{\equiv}\sum_{u+v=j}\left[f^{q+1}(\theta_{u-q-1}), \theta_v\right]$$
$$\underset{j-q-1}{\equiv}\sum_{r+s=j-q-1}\left[f^{q+1}(\theta_r), \theta_s\right].$$

Now recalling the definition of the operator ∂, we have

$$(3.6)\qquad \frac{1}{2}(\partial f^{q+1})_j(\theta_-\wedge\theta_-)=\frac{1}{2}\sum_{r+s=j-q-1}(\partial f^{q+1})(\theta_r\wedge\theta_s)$$
$$=\sum_{r+s=j-q-1}\left[f^{q+1}(\theta_r), \theta_s\right]-\frac{1}{2}\sum_{r+s=j-q-1}f^{q+1}\left([\theta_r, \theta_s]\right).$$

From (3. 3)$\sim$(3. 6) it follows immediately that

$$(3.7)\qquad \Theta'^{q+1}_j-\Theta^{q+1}_j\underset{j-q-1}{\equiv}\frac{1}{2}(\partial f^{q+1})_j(\theta_-\wedge\theta_-).$$

From (3. 2) and (3. 7) we obtain

$$\sum_{l\leqq q}(R'^l_j-R^l_j)(\theta_-\wedge\theta_-)\underset{j-q-1}{\equiv}(\partial f^{q+1})_j(\theta_-\wedge\theta_-), \qquad j\leqq q-1,$$

which clearly mean that $R'^l=R^l$, $l<q$, and

$$R'^q-R^q=\partial f^{q+1}.$$

Since both R^q and R'^q take values in $Z^{q,2}_*$, it follows that $\partial f^{q+1}=0$ (and hence $R'^q=R^q$). Therefore we have $f^{q+1}=0$ by Lemma 3. 3, which proves (S_{q+1}). We have thus found (S_p) to be true, completing the proof of Lemma 3. 9. Incidentally the proof above indicates that $R'^l=R^l$, $l\leqq p-1$.

REMARK. Hereafter we shall frequently use the reasoning in the proof of Lemma 3. 9. See the proofs of the following: Lemma 3. 11, Lemma 3. 12, Lemma 3. 17, Lemma 4. 16, Lemma 4. 17, Lemma 5. 3, and Lemma 5. 4.

3. 5. Normal pre-$(p+1)$-systems in U^p-structures. Let (B^p, ξ) be a U^p-structure on a manifold M. Let $\theta^{(p+1)}=\{\theta_j\}_{j\leqq p}$ be a system of $\mathfrak{g}_j$-valued 1-forms θ_j, $j\leqq p$, on B^p. Let $\theta^{(p)}$ denote the system $\{\theta_j\}_{j\leqq p-1}$. Then we say that $\theta^{(p+1)}$ is a pre-$(p+1)$-system in (B^p, ξ) if it satisfies the following conditions:

$(p+1.$ a) $\theta^{(p)}$ is a p-system in (B^p, ξ).

$(p+1.$ b) i) $\theta_p(\rho(X_r)^*)=\delta_{pr}X_r$, $X_r\in\mathfrak{g}_r$, $0\leqq r\leqq p$,

ii) $\theta_p(A^*)=0$, $A\in\mathfrak{q}^p$.

$(p+1.\ \mathrm{c})\quad R^*{}_a\theta_p=\mathrm{Ad}(a^{-1})\theta_p,\ a\in G_0.$

Let $\theta^{(p+1)}$ be a pre-$(p+1)$-system in (B^p, ξ). For any $j\leqq p-1$ we define a $\mathfrak{g}_j$-valued 2-form Θ_j^{p+1} on B^p in the same manner as before, that is,

$$\Theta_j^{p+1}=d\theta_j+\frac{1}{2}\sum_{\substack{u+v=j,\\ u,v\leqq p}}[\theta_u, \theta_v]\,.$$

Then we deduce from the proof of Lemma 3.6 that

$$A^*\rfloor\,\Theta_j^{p+1}\underset{j-p-1}{\equiv}0\,,\qquad A\in\mathfrak{u}^p\,,\qquad j\leqq p-1\,.$$

Hence there are unique functions $R^l: B^p\rightarrow C^{l,2}$, $l\leqq p$, such that

$$\Theta_j^{p+1}\underset{j-p-1}{\equiv}\frac{1}{2}\sum_{l\leqq p}R_j^l(\theta_-\wedge\theta_-)\,,\qquad j\leqq p-1\,.$$

As before the system of the equations above will be called the structure equation (for $\theta^{(p+1)}$), and the system of functions, $\{R^l\}_{l\leqq p}$, will be called the curvature (of $\theta^{(p+1)}$). The structure equation for $\theta^{(p+1)}$ induces the equations

$$\Theta_j^p\underset{j-p}{\equiv}\frac{1}{2}\sum_{l\leqq p-1}R_j^l(\theta_-\wedge\theta_-)\,,\qquad j\leqq p-2\,,$$

which together form the structure equation for the p-system $\theta^{(p)}$.

LEMMA 3.10. $R_a{}^*R^p=(R^p)^a$, $a\in G_0$.

PROOF. By $(p.\ 2)$ $(p+1.\ \mathrm{c})$ we have $R_a{}^*\theta_j=\mathrm{Ad}(a^{-1})\,\theta_j$ for all $a\in G_0$ and $j\leqq p$. Therefore it follows from the structure equation for $\theta^{(p+1)}$ that

$$\mathrm{Ad}(a^{-1})\,\Theta_j^{p+1}\underset{j-p-1}{\equiv}\frac{1}{2}\sum_{l\leqq p}(R_a{}^*R_j^l)\left(\mathrm{Ad}(a^{-1})\,\theta_-\wedge\mathrm{Ad}(a^{-1})\,\theta_-\right).$$

These equations together with the structure equation yield the equalities

$$\mathrm{Ad}(a)\,(R_a{}^*R_j^l)\left(\mathrm{Ad}(a^{-1})\,\theta_-\wedge\mathrm{Ad}(a^{-1})\,\theta_-\right)\underset{j-p-1}{\equiv}R_j^l(\theta_-\wedge\theta_-)\,,$$

which clearly mean that $R_a{}^*R^l=(R^l)^a$, $l\leqq p$.

Finally we say that the pre-$(p+1)$-system $\theta^{(p+1)}$ in (B^p, ξ) is normal if it satisfies the following conditions:

i) The p-system $\theta^{(p)}$ in (B^p, ξ) is normal,

ii) The function R^p takes values in $Z_*^{p,2}+\partial W^{p+1,1}$.

3. 6. The existence and invariance of normal pre-$(p+1)$-systems in U^p-structures of type $(\mathfrak{M}, p)$.

LEMMA 3.11. *Every U^p-structure of type $(\mathfrak{M}, p)$ admits a normal pre-$(p+1)$-system.*

PROOF. Let (B^p, ξ) be a U^p-structure of type $(\mathfrak{M}, p)$ on a manifold M. As is well known, the principal fibre bundle B^p admits a connection (in the usual sense) (cf. [3]), that is, there is a $\mathfrak{u}^p$-valued 1-form α on B^p such that i) $\alpha(A^*)=A$, $A\in\mathfrak{u}^p$, and ii) $R_a{}^*\alpha=a^{-1}\alpha a$, $a\in U^p$. Let α_p denote the $\rho(\mathfrak{g}_p)$-component of α with respect to the decomposition: $\mathfrak{u}^p=\sum_{j=0}^{p}\rho(\mathfrak{g}_j)+\mathfrak{q}^p$. Clearly the 1-form α_p satisfies the following:

$$\alpha_p\big(\rho(X_r)^*\big)=\delta_{pr}\,\rho(X_r)\,,\qquad X_r\in\mathfrak{g}_r\,,\qquad 0\leqq r\leqq p\,;$$

$$\alpha_p(A^*)=0\,,\qquad A\in\mathfrak{q}^p\,;$$

$$R_a{}^*\alpha_p=a^{-1}\alpha_p a\,,\qquad a\in G_0\,.$$

By Lemma 1.9 or Lemma 3. 2 there is a unique $\mathfrak{g}_p$-valued 1-form θ_p on B^p such that $\alpha_p=\rho(\theta_p)$. Then the formulas above for α_p mean that θ_p satisfies $(p+1.$ b) and $(p+1.$ c).

Now take a normal p-system $\theta^{(p)}=\{\theta_j\}_{j\leqq p-1}$ in (B^p, ξ), and consider the system $\theta^{(p+1)}=\{\theta_j\}_{j\leqq p}$ formed by $\{\theta_j\}_{j\leqq p-1}$ and θ_p. Then it is clear from the remark above that $\theta^{(p+1)}$ gives a pre-$(p+1)$-system in (B^p, ξ). We shall modify $\theta^{(p+1)}$ to obtain a normal pre-$(p+1)$-system.

By Lemma 3. 4 the space $C^{p,2}$ is decomposed as follows:

$$C^{p,2}=Z_*{}^{p,2}+\partial W^{p+1,1}+\partial C_+^{p+1,1}\,.$$

Let $\{R^l\}_{l\leqq p}$ be the curvature of $\theta^{(p+1)}$. R^p taking values in $C^{p,2}$, we denote by L^p the $\partial C_+^{p+1,1}$-component of R^p. By Lemma 3. 10 we have $R_a{}^*R^p=(R^p)^a$, $a\in G_0$, and by Lemma 1. 11 each subspaces in the decomposition above is G_0-invariant. Hence it follows that

$$(3.8)\qquad R_a{}^*L^p=(L^p)^a\,,\qquad a\in G_0\,.$$

We now show that there is a function $f^{p+1}: B^p\rightarrow C_+^{p+1,1}$ such that $L^p=-\partial f^{p+1}$ and

$$(3.9)\qquad R_a{}^*f^{p+1}=(f^{p+1})^a\,,\qquad a\in G_0\,.$$

First consider the case where $0\leqq p\leqq\mu-2$. By Lemma 3. 3 there is a unique function $f^{p+1}: B^p\rightarrow C_+^{p+1,1}$ such that $L^p=-\partial f^{p+1}$. By (3. 8) and Lemma 1. 11 we see that f^{p+1} satisfies (3. 9). Next consider the case where $p=\mu-1$. We have $C_+^{\mu,1}=C^{\mu,1}=Z_*^{\mu,1}+\partial\mathfrak{g}_\mu$, and we know that $H^{\mu,1}(\mathfrak{G})=0$ (see 1. 4). Hence there is a unique function $f^\mu: B^{\mu-1}\rightarrow Z_*^{\mu,1}$ such that $L^{\mu-1}=-\partial f^\mu$, and as above we see that f^μ satisfies (3. 9).

Using the function f^{p+1} thus obtained, we now modify $\theta^{(p+1)}$ as follows:

$$\theta'_j=\theta_j+f^{p+1}(\theta_{j-p-1})\,,\qquad j\leqq p\,.$$

Let $\theta'^{(p+1)}=\{\theta'_j\}_{j\leqq p}$. Then it is easy to see that $\theta'^{(p+1)}$ gives a pre-$(p+1)$-system in (B^p,ξ). (In the proof of this fact we use (3. 9).) We assert that $\theta'^{(p+1)}$ is normal. Indeed let $\{R'^l\}_{l\leqq p}$ denote the curvature of $\theta'^{(p+1)}$. Then we deduce from the proof of Lemma 3. 9 that $R'^l=R^l$, $l<p$, and $R'^p=R^p+\partial f^{p+1}$. (Since $\theta^{(p)}$ is normal, we have $\Theta_i^0 \underset{i}{\equiv} 0$, $i<0$. Hence it follows that

$$\Theta'^{p+1}_j-\Theta^{p+1}_j \underset{j-p-1}{\equiv} \frac{1}{2}(\partial f^{p+1})_j(\theta_-\wedge\theta_-)\,,\qquad j\leqq p-1\,,$$

which together with the structure equations for $\theta^{(p+1)}$ and $\theta'^{(p+1)}$ yield the desired equalities.) Since $\theta^{(p)}$ is normal and since $R'^p=R^p-L^p$ takes values in $Z^{p,2}_*+\partial W^{p+1,1}$, we therefore see that $\theta'^{(p+1)}$ is normal, proving our assertion. We have thus completed the proof of Lemma 3. 11.

LEMMA 3. 12. *Let (B^p,ξ) be a U^p-structure of type $(\mathfrak{M},p)$ on a manifold M, and let $\theta^{(p+1)}$ and $\theta'^{(p+1)}$ be two normal pre-$(p+1)$-systems in (B^p,ξ).*

(1) *If $0\leqq p\leqq\mu-2$, then*

$$\theta'_j \underset{j-p-2}{\equiv} \theta_j\,,\qquad j\leqq p\,.$$

(2) *If $p=\mu-1$, then there is a unique function $g_\mu\colon B^p\to\mathfrak{g}_\mu$ such that*

$$\theta'_j \underset{j-\mu-1}{\equiv} \theta_j+[g_\mu,\theta_{j-\mu}]\,,\qquad j\leqq\mu-1\,.$$

PROOF. By $(p+1.\ \mathrm{b})$ we have $\theta'_p(A^*)=\theta_p(A^*)$ for every $A\in\mathfrak{u}^p$, whence $\theta'_p \underset{-1}{\equiv} \theta_p$. We have $\theta'_j=\theta_j$, $j<0$, and by Lemma 3. 9 we have $\theta'_j \underset{j-p-1}{\equiv} \theta_j$, $j<p$. From these facts we see that there is a unique function $f^{p+1}\colon B^p\to C^{p+1,1}_+$ such that

$$\theta'_j \underset{j-p-2}{\equiv} \theta_j+f^{p+1}(\theta_{j-p-1})\,,\qquad j\leqq p\,.$$

Let $\{R^l\}_{l\leqq p}$ and $\{R'^l\}_{l\leqq p}$ be the curvatures of $\theta^{(p+1)}$ and $\theta'^{(p+1)}$ respectively. Then it follows that

$$R'^p=R^p+\partial f^{p+1}$$

(cf. the proof of Lemma 3. 9). Here we notice that both R^p *and* R'^p take values in $Z^{p,2}_*+\partial W^{p+1,1}$. Therefore we see from Lemma 3. 4 that $\partial f^{p+1}=0$ (and hence $R'^p=R^p$). If $0\leqq p\leqq\mu-2$, we have $f^{p+1}=0$ by Lemma 3. 3. Now suppose that $p=\mu-1$. We have $C^{\mu,1}_+=C^{\mu,1}$, and we see from Lemma 1. 14 that there is a unique function $g_\mu\colon B^{\mu-1}\to\mathfrak{g}_\mu$ such that $f^\mu=-\partial g_\mu$; We have $f^\mu(\theta_{j-\mu})=[g_\mu,\theta_{j-\mu}]$. We have thereby proved Lemma 3. 12.

The notations being as above, we already know that $R'^l=R^l$, $l<p$ (see 3. 4). The proof above also indicates that

$$R'^p = R^p .$$

Accordingly we know that the curvature of a normal pre-$(p+1)$-system in a U^p-structure of type $(\mathfrak{M}, p)$ is an invariant.

REMARK. We have $U^{\mu-1}=\tilde{G}$. Thus we may speak of a (normal) pre-μ-system in a $\tilde{G}$-structure. We have $W^{\mu,1}\subset C_-^{\mu,1}=0$. Hence a pre-$\mu$-system $\theta^{(\mu)}$ in a $\tilde{G}$-structure is normal if and only if its curvature $\{R^l\}_{l\leqq\mu-1}$ satisfies the following conditions: i) $R^l=0$ for $l<0$; ii) $\partial^* R^l=0$ for $0\leqq l\leqq\mu-1$. By Lemma 3.11 every $\tilde{G}$-structure of type $(\mathfrak{M}, \mu-1)$ admits a normal pre-μ-system. Let us consider the special case where $\mathfrak{G}$ is of the first kind. Clearly every $\tilde{G}$-structure is of type $(\mathfrak{M}, 0)$. (Note that $G_0^{\#}=\tilde{G}=G_0$). Let $\theta^{(1)}$ be a normal pre-1-system in a $\tilde{G}$-structure, and $\{R^l\}_{l\leqq 0}$ its curvature. Then the function R^0 is nothing but the torsion or the structure function of the $\tilde{G}$-structure (cf. [6]).

3.7. Proof of Theorem 3.8. Let $0\leqq p\leqq\mu-2$, and let (B^p, ξ) be a U^p-structure of type $(\mathfrak{M}, p)$ on a manifold M. By Lemma 3.11 (B^p, ξ) admits a normal pre-$(p+1)$-system. We take any normal pre-$(p+1)$-system $\theta^{(p+1)}=\{\theta_j\}_{j\leqq p}$ in (B^p, ξ), and denote by $\{R^l\}_{l\leqq p}$ its curvature.

First of all we recall that the Lie algebra $\mathfrak{u}^p$ is decomposed as follows:

$$\mathfrak{u}^p = \sum_{j=0}^{p} \rho(\mathfrak{g}_j) + \mathfrak{q}^p ,$$

$$\mathfrak{q}^p = C_-^{p+1,1} + \mathfrak{q}^{p+1} ,$$

$$C_-^{p+1,1} = \rho(\mathfrak{g}_{p+1}) + W^{p+1,1} .$$

Especially every element Y of $C_-^{p+1,1}$ can be written (uniquely) in the form:

$$Y = \rho(X_{p+1}) + Y' ,$$

where $X_{p+1}\in\mathfrak{g}_{p+1}$ and $Y'\in W^{p+1,1}$. This being said, we state the next three lemmas, which will be proved in 3.9~3.11.

LEMMA 3.13. *Let* $1\leqq k\leqq p$ *and* $X_k\in\mathfrak{g}_k$.

(1) $\mathscr{L}_{\rho(X_k)^*}\theta_j \underset{j-p-2}{\equiv} -[X_k, \theta_{j-k}],\ j\leqq p$.

(2) $\mathscr{L}_{\rho(X_k)^*} R^p = (R^{p-k})^{X_k}$.

LEMMA 3.14. *Let* $Y\in C_-^{p+1,1}$.

(1) $\mathscr{L}_{Y^*}\theta_j \underset{j-p-2}{\equiv} -[X_{p+1}, \theta_{j-p-1}] - Y'(\theta_{j-p-1}),\ j\leqq p$.

(2) $\mathscr{L}_{Y^*} R^p = -\partial Y'$.

LEMMA 3.15. *Let* $Z\in\mathfrak{q}^{p+1}$.

(1) $\mathscr{L}_{Z^*}\theta_j \underset{j-p-2}{\equiv} 0,\ j\leqq p$.

(2) $\mathscr{L}_{Z^*} R^p = 0$.

Every element a of U^p can be written uniquely in the form:

$$a=b\cdot c_1\cdots c_{p+1}\cdot d\,,$$

where $b\in G_0$, $c_k\in\exp\rho(\mathfrak{g}_k)$, $1\leqq k\leqq p$, $c_{p+1}\in\exp C_{-}^{p+1,1}$, and $d\in\exp\mathfrak{q}^{p+1}$. Furthermore the elements c_k, c_{p+1} and d can be expressed uniquely as follows: $c_k=\exp\rho(X_k)$ with $X_k\in\mathfrak{g}_k$, $c_{p+1}=\exp Y$ with $Y\in C_{-}^{p+1,1}$, and $d=\exp Z$ with $Z\in\mathfrak{q}^{p+1}$. We also note that as before Y can be expressed as $Y=\rho(X_{p+1})+Y'$. Clearly a is in the subgroup U^{p+1} of U^p if and only if $Y'=0$.

These being prepared, we shall prove the following

LEMMA 3.16. *Let $z\in B^p$ and $a\in U^p$.*

(1) $R^p(z\cdot b)=R^p(z)^b$.

(2) $R^p(z\cdot c_k)-R^p(z)\in Z_*^{p,2}$, $1\leqq k\leqq p$.

(3) $R^p(z\cdot c_{p+1})=R^p(z)-\partial Y'$.

(4) $R^p(z\cdot d)=R^p(z)$.

(5) *Assume that $R^p(z)\in Z_*^{p,2}$. Then $R^p(z\cdot a)\in Z_*^{p,2}$ if and only if $a\in U^{p+1}$.*

First of all (1) is nothing but Lemma 3.10.

(2) By Lemma 3.13 we have $\mathscr{L}_{\rho(X_k)*}R^p=(R^{p-k})^{X_k}$. Since $p-k<p$, R^{p-k} takes values in $Z_*^{p-k,2}$. (Note that $\theta^{(p+1)}$ is normal.) Therefore we see from Lemma 1.12 that $(R^{p-k})^{X_k}=\mathscr{L}_{\rho(X_k)*}R^p$ takes values in $Z_*^{p,2}$, which clearly means (2).

(3) and (4) follow immediately from Lemmas 3.14 and 3.15 respectively.

(5) By (3) and (4) we have

$$R^p(z\cdot a)=R^p(z\cdot b\cdot c_1\cdots c_p)-\partial Y'\,.$$

Since $R^p(z)\in Z_*^{p,2}$, it follows from (1), (2) and Lemma 1.11 that $R^p(z\cdot b\cdot c_1\cdots c_p)\in Z_*^{p,2}$. Therefore we know from Lemma 3.3 that $R^p(z\cdot a)\in Z_*^{p,2}$ if and only if $Y'=0$, *i.e.*, $a\in U^{p+1}$.

We have thus proved Lemma 3.16.

Using the invariant function R^p, we define a subset B^{p+1} of B^p by

$$B^{p+1}=\left\{z\in B^p\,\middle|\,R^p(z)\in Z_*^{p,2}\right\}.$$

We want to show that B^{p+1} defines a reduction of the principal bundle B^p to the group U^{p+1}. For this purpose we first prove that, for each point $x_0\in M$, there is a local cross section s of B^p defined on a neighborhood V of x_0 such that $s(V)\subset B^{p+1}$. Indeed take a local cross section $\bar{s}$ of B^p defined on a neighborhood V of x_0. Fix any point $x\in V$. Since $R^p(\bar{s}(x))$ takes values in $Z_*^{p,2}+\partial W^{p+1,1}$, there is a unique $A(x)\in W^{p+1,1}$ such that $R^p(\bar{s}(x))-\partial A(x)\in Z_*^{p,2}$. Let s denote the local cross section of B^p defined by

$s(x)=\bar{s}(x)\cdot a(x)$ for all $x\in V$, where $a(x)=\exp A(x)$. Then we see from (3) of Lemma 3.16 that $R^p(s(x))=R^p(\bar{s}(x))-\partial A(x)\in Z_*{}^{p,2}$ for all $x\in V$, meaning that $s(V)\subset B^{p+1}$. This proves our assertion. Now let $z\in B^{p+1}$ and $a\in U^p$. Then (5) of Lemma 3.16 means that $z\cdot a\in B^{p+1}$ if and only if $a\in U^{p+1}$. We have thus seen that B^{p+1} is a principal fibre bundle over the base space M with structure group U^{p+1} and it is a reduction of B^p to U^{p+1}.

Let ι be the injection $B^{p+1}\rightarrow B^p$. Then the pair $(B^{p+1}, \iota^*\xi)$ gives a U^{p+1}-structure on M. We assert that system $\theta'^{(p+1)}=\{\iota^*\theta_j\}_{j\leqq p}$ is a normal $(p+1)$-system in $(B^{p+1}, \iota^*\xi)$. Firstly it is clear that $\theta'^{(p+1)}$ is compatible with the basic form $\iota^*\xi$. Secondly $\theta'^{(p+1)}$ is a $(p+1)$-system in $(B^{p+1}, \iota^*\xi)$, which can be easily verified from conditions (p. 1)$\sim$($p+1$. c) for $\theta^{(p+1)}$, and Lemmas 3.13$\sim$3.15. Thirdly $\theta'^{(p+1)}$ is normal. Indeed the structure equation for $\theta^{(p+1)}$ induces the equations

$$\iota^*\Theta_j^{p+1} \underset{j-p-1}{\equiv} \frac{1}{2}\sum_{l\leqq p}(\iota^* R^l)(\iota^*\theta_-\wedge\iota^*\theta_-)\,, \qquad j\leqq p-1\,,$$

which together form the structure equation for $\theta'^{(p+1)}$. Since $\theta^{(p+1)}$ is normal and since ι^*R^p takes values in $Z_*{}^{p,2}$, we see that $\theta'^{(p+1)}$ is normal, proving our assertion.

In this way we have seen that every U^p-structure of type $(\mathfrak{M}, p)$ is naturally reduced to a U^{p+1}-structure of type $(\mathfrak{M}, p+1)$. (The naturality follows from the fact that the function R^p is an invariant.) We have thereby completed the proof of Theorem 3.8.

3.8. Proof of Theorem 3.7. Our task here is to prove the uniqueness part of the theorem.

Let $(P^\#, \xi)$ be a $G_0{}^\#$-structure of type $\mathfrak{M}$ on a manifold M. By Theorem 3.8 there is a sequence of U^p-structures of type $(\mathfrak{M}, \mathrm{p})$, (B^p, ξ^p), on M $(0\leqq p\leqq\mu-1)$ such that $(B^0, \xi^0)=(P^\#, \xi)$ and such that, for every $0\leqq p\leqq\mu-2$, (B^{p+1}, ξ^{p+1}) is the natural reduction of (B^p, ξ^p) to U^{p+1}:

$$P^\# = B^0\supset B^1\supset\cdots\supset B^{\mu-1}\,.$$

Let $(\tilde{P}, \tilde{\xi})$ be any $\tilde{G}$-structure of type $(\mathfrak{M}, \mu-1)$ on M which is a reduction of $(P^\#, \xi)$ to $\tilde{G}$. Then we must show that $(\tilde{P}, \tilde{\xi})=(B^{\mu-1}, \xi^{\mu-1})$ or equivalently $\tilde{P}=B^{\mu-1}$.

For this purpose we shall prove by induction that $\tilde{P}\subset B^p$ for all $0\leqq p\leqq\mu-1$. Assume that $\tilde{P}\subset B^p$ for some $0\leqq p\leqq\mu-2$. Let $\theta^{(\mu-1)}=\{\theta_j\}_{j\leqq\mu-2}$ be a normal $(\mu-1)$-system in $(\tilde{P}, \tilde{\xi})$, and $\{R^l\}_{l\leqq\mu-2}$ its curvature. Similarly let $\theta'^{(p+1)}=\{\theta'_j\}_{j\leqq p}$ be a normal pre-$(p+1)$-system in (B^p, ξ^p), and $\{R'^l\}_{l\leqq p}$ its curvature. Let ι be the injection $\tilde{P}\rightarrow B^{p+1}$. Then by Lemma 3.17 below we have

$$\iota^* R'^p = R^p .$$

Since $B^{p+1}=\{z\in B^p | R'^p(z)\in Z_*{}^{p,2}\}$ and since R^p takes values in $Z_*{}^{p,2}$, it follows that $\tilde{P}\subset B^{p+1}$, completing our induction. Thus we have $\tilde{P}\subset B^{\mu-1}$ and hence $\tilde{P}=B^{\mu-1}$, which proves the uniqueness part of Theorem 3.7.

LEMMA 3.17. $\iota^* R'^p=R^p$.

PROOF. Clearly we have $\iota^* \theta'_-=\theta_-$. The structure equation for $\theta^{(\mu-1)}$ induces the equations

$$\Theta_j^{q+1} \underset{j-q-1}{\equiv} \frac{1}{2}\sum_{l\leqq q} R_j^l(\theta_-\wedge\theta_-)\,, \qquad j\leqq q-1, \qquad 0\leqq q\leqq p,$$

and similarly the structure equation for $\theta'^{(p+1)}$ the equations

$$\iota^*\Theta'^{q+1}_j \underset{j-q-1}{\equiv} \frac{1}{2}\sum_{l\leqq q}(\iota^* R'^l)\,(\theta_-\wedge\theta_-)\,, \qquad j\leqq q-1\,, \qquad 0\leqq q\leqq p\,.$$

By condition ($\mu-1$. 1) for $\theta^{(\mu-1)}$, and conditions (p. 1), ($p+1$. b) for $\theta'^{(p+1)}$ it follows that

$$\iota^*\theta'_q \underset{-1}{\equiv} \theta_q\,, \qquad 0\leqq q\leqq p\,.$$

We further notice that both $\theta^{(\mu-1)}$ and $\theta'^{(p+1)}$ are normal, and especially $\Theta^0{}_j\underset{j}{\equiv}0$, $j\leqq -2$. Therefore we deduce from the proof of Lemma 3.9 that $\iota^*\theta'_j \underset{j-p-2}{\equiv} \theta_j$, $j\leqq p$, and $\iota^* R'^l=R^l$, $l\leqq p$. (See also the proof of Lemma 3.12.) Thus we obtain Lemma 3.17.

3.9. Proof of Lemma 3.13.

LEMMA 3.18. $\mathscr{L}_{\rho(X_k)^*}\theta_p \underset{-1}{\equiv} -[X_k, \theta_{p-k}]$.

PROOF. This fact can be derived from (p. 1) and ($p+1$. b) as follows: Putting

$$\alpha = \mathscr{L}_{\rho(X_k)^*}\theta_p + [X_k, \theta_{p-k}]\,,$$

we must show that $\alpha(A^*)=0$ for every $A\in\mathfrak{u}^p$. Let $A\in\mathfrak{u}^p$. Then we have

$$(\mathscr{L}_{\rho(X_k)^*}\theta_p)\,(A^*) = \rho(X_k)^*\theta_p(A^*) - \theta_p\Big(\big[\rho(X_k)^*, A^*\big]\Big)$$
$$= -\theta_p\Big(\big[\rho(X_k), A\big]^*\Big)\,,$$

and hence

$$\alpha(A^*) = -\theta_p\Big(\big[\rho(X_k), A\big]^*\Big) + \big[X_k, \theta_{p-k}(A^*)\big]\,.$$

If $A\in\mathfrak{q}^p$, we have $[\rho(X_k), A]\in\mathfrak{q}^p$, because $\mathfrak{q}^p$ is an ideal of $\mathfrak{u}^p$, and hence it follows that $\alpha(A^*)=0$. If A is of the form $\rho(Y_r)$ with some $0\leqq r\leqq p$ and some $Y_r\in\mathfrak{g}_r$, it follows that

$$\alpha\left(\rho(Y_r)^*\right) = -\theta_p\left(\rho\left([X_k, Y_r]\right)^*\right) + \left[X_k, \theta_{p-k}\left(\rho(Y_r)^*\right)\right] = 0\,.$$

We have thus shown $\alpha(A^*)=0$ for every $A\in\mathfrak{u}^p$, proving Lemma 3.18.

By (p. 2) we have $\mathscr{L}_{\rho(X_k)^*}\theta_j \underset{j-p-1}{\equiv} -[X_k, \theta_{j-k}]$, $j\leqq p-1$, and by Lemma 3.5 we have $\mathscr{L}_{\rho(X_k)^*}\theta_j = -[X_k, \theta_{j-k}]$, $j<0$. From these facts together with Lemma 3.18 we see that there is a unique function $f^{p+1}: B^p \to C_+^{p+1,1}$ such that

$$(3.10)\qquad \mathscr{L}_{\rho(X_k)^*}\theta_j \underset{j-p-2}{\equiv} -[X_k, \theta_{j-k}] + f^{p+1}(\theta_{j-p-1})\,, \qquad j\leqq p\,.$$

LEMMA 3.19. $\mathscr{L}_{\rho(X_k)^*}R^p = (R^{p-k})^{X_k} + \partial f^{p+1}$.

Now Lemma 3.13 can be obtained from (3.10) and Lemma 3.19 in the following manner. We first remark that R^p takes values in $Z_*{}^{p,2} + \partial W^{p+1,1}$. From Lemmas 1.12 and 1.15 we see that $(R^{p-k})^{X_k}$ takes values in $Z_*{}^{p,2}$ (cf. the proof of Lemma 3.16). Therefore it follows from Lemmas 3.3, 3.4 and 3.19 that $f^{p+1}=0$. Thus Lemma 3.13 follows.

We shall now prove Lemma 3.19. The structure equation for $\theta^{(p+1)}$ gives the equations

$$(3.11)\qquad \Theta_j^{p+1} \underset{j-p-1}{\equiv} \frac{1}{2}\sum_{l\leqq p} R_j^l(\theta_-\wedge\theta_-)\,, \qquad j\leqq p-1\,,$$

where

$$(3.12)\qquad \Theta_j^{p+1} = d\theta_j + \frac{1}{2}\sum_{u+v=j}[\theta_u, \theta_v]\,.$$

Since $\mathscr{L}_{\rho(X_k)^*}\theta_i \underset{i}{\equiv} 0$, $i<0$, it follows from (3.11) and (3.12) that

$$(3.13)\qquad d\mathscr{L}_{\rho(X_k)^*}\theta_j + \sum_{u+v=j}[\mathscr{L}_{\rho(X_k)^*}\theta_u, \theta_v]$$
$$\underset{j-p-1}{\equiv} \frac{1}{2}\sum_{l\leqq p}(\mathscr{L}_{\rho(X_k)^*}R_j^l)(\theta_-\wedge\theta_-) - \sum_{l\leqq p}R_j^l\left(\rho(X_k)\,\theta_-\wedge\theta_-\right).$$

(Note that $\mathscr{L}_{\rho(X_k)^*}\theta_- = -\rho(X_k)\,\theta_-$).

We have $d\theta_i \underset{j-p-1}{\equiv} 0$, $i\leqq j-p-2$, and

$$d\theta_{j-p-1} \underset{j-p-1}{\equiv} -\frac{1}{2}\sum_{r+s=j-p-1}[\theta_r, \theta_s]\,.$$

(In the following the letters r and s mean negative integers.) By (3.11) and (3.12) we have

$$d\theta_{j-k} \underset{j-p-1}{\equiv} -\frac{1}{2}\sum_{u+v=j-k}[\theta_u, \theta_v] + \frac{1}{2}\sum_{l\leqq p}R_{j-k}^l(\theta_-\wedge\theta_-)\,.$$

Therefore it follows from (3.10) that

$$(3.14)\qquad \mathscr{L}_{\rho(X_k)*}\theta_j \underset{j-p-1}{\equiv} -[X_k, d\theta_{j-k}]+f^{p+1}(d\theta_{j-p-1})$$

$$\underset{j-p-1}{\equiv} \frac{1}{2}\sum_{u+v=j-k}\Big[X_k, [\theta_u, \theta_v]\Big]-\frac{1}{2}\sum_{l\leqq p}\Big[X_k, R^l_{j-k}(\theta_-\wedge\theta_-)\Big]$$

$$-\frac{1}{2}\sum_{r+s=j-p-1} f^{p+1}\Big([\theta_r, \theta_s]\Big).$$

Furthermore we see from (3.10) that

$$(3.15)\qquad \sum_{u+v=j}[\mathscr{L}_{\rho(X_k)*}\theta_u, \theta_v]$$

$$\underset{j-p-1}{\equiv} -\sum_{u+v=j}\Big[[X_k, \theta_{u-k}], \theta_v\Big]+\sum_{u+v=j}\Big[f^{p+1}(\theta_{u-p-1}), \theta_v\Big].$$

We have

$$(3.16)\qquad \frac{1}{2}\sum_{u+v=j-k}\Big[X_k, [\theta_u, \theta_v]\Big]-\sum_{u+v=j}\Big[[X_k, \theta_{u-k}], \theta_v\Big]\underset{j-p-1}{\equiv} 0,$$

$$(3.17)\qquad \sum_{u+v=j}\Big[f^{p+1}(\theta_{u-p-1}), \theta_v\Big]\underset{j-p-1}{\equiv}\sum_{r+s=j-p-1}\Big[f^{p+1}(\theta_r), \theta_s\Big]$$

If $l>p-k$, we have

$$R^l_{j-k}(\theta_-\wedge\theta_-)=\sum_{r+s=j-k-l-1}R^l(\theta_r\wedge\theta_s)\underset{j-p-1}{\equiv} 0,$$

whence

$$(3.18)\qquad \sum_{l\leqq p}\Big[X_k, R^l_{j-k}(\theta_-\wedge\theta_-)\Big]\underset{j-p-1}{\equiv}\sum_{l\leqq p}\Big[X_k, R^{l-k}_{j-k}(\theta_-\wedge\theta_-)\Big].$$

Similarly if $l>p-k$, we have

$$R^l_j\Big(\rho(X_k)\,\theta_-\wedge\theta_-\Big)=\sum_{r+s=j-l-1}R^l\Big([X_k, \theta_{r-k}]\wedge\theta_s\Big)\underset{j-p-1}{\equiv} 0,$$

whence

$$(3.19)\qquad \sum_{l\leqq p}R^l_j\Big(\rho(X_k)\,\theta_-\wedge\theta_-\Big)\underset{j-p-1}{\equiv}\sum_{l\leqq p}R^{l-k}_j\Big(\rho(X_k)\,\theta_-\wedge\theta_-\Big).$$

Moreover recalling the definitions of the operator ∂ and the functions $(R^{l-k})^{X_k}$, we obtain

$$(3.20)\qquad \frac{1}{2}(\partial f^{p+1})_j(\theta_-\wedge\theta_-)=\sum_{r+s=j-p-1}\Big[f^{p+1}(\theta_r), \theta_s\Big]$$

$$-\frac{1}{2}\sum_{r+s=j-p-1}f^{p+1}\Big([\theta_r, \theta_s]\Big),$$

$$(3.21)\qquad \frac{1}{2}\Big((R^{l-k})^{X_k}\Big)_j(\theta_-\wedge\theta_-)=-\frac{1}{2}\Big[X_k, R^{l-k}_{j-k}(\theta_-\wedge\theta_-)\Big]$$

$$+R^{l-k}_j\Big(\rho(X_k)\,\theta_-\wedge\theta_-\Big).$$

From (3. 13)$\sim$(3. 21) it follows immediately that

$$\sum_{l \leqq p} (\mathscr{L}_{\rho(X_k)*} R^l_j)(\theta_- \wedge \theta_-)$$
$$\underset{j-p-1}{\equiv} \sum_{l \leqq p} \left((R^{l-k})^{X_k} \right)_j (\theta_- \wedge \theta_-) + (\partial f^{p+1})_j (\theta_- \wedge \theta_-) ,$$

meaning the $\mathscr{L}_{\rho(X_k)*} R^l = (R^{l-k})^{X_k}$, $l<p$, and

$$\mathscr{L}_{\rho(X_k)*} R^p = (R^{p-k})^{X_k} + \partial f^{p+1} .$$

We have thereby completed the proof of Lemma 3. 19.

3. 10. Proof of Lemma 3 .14. Since $Y \in \mathfrak{q}^p$, we have $(\mathscr{L}_{Y*}\theta_p)(A^*) = -\theta_p([Y, A]^*)$ for every $A \in \mathfrak{u}^p$ (cf. the proof of Lemma 3. 18). Hence we obtain

$$\mathscr{L}_{Y*}\theta_p \underset{-1}{\equiv} 0 .$$

By (p. 2) we have $\mathscr{L}_{Y*}\theta_j \underset{j-p-1}{\equiv} 0$, $j \leqq p-1$, and by Lemma 3. 5 we have $\mathscr{L}_{Y*}\theta_j = -Y(\theta_{j-p-1})$, $j<0$. From these facts we see that there is a unique function $f^{p+1}: B^p \to C^{p+1,1}_+$ such that

$$\mathscr{L}_{Y*}\theta_j \underset{j-p-2}{\equiv} (-Y + f^{p+1})(\theta_{j-p-1}) , \qquad j \leqq p . \tag{3. 22}$$

Lemma 3. 20. $\mathscr{L}_{Y_*} R^p = \partial(-Y + f^{p+1})$.

Now Lemma 3. 14 can be obtained from (3. 22) and Lemma 3. 20 in the following manner: We have $Y = Y' + \rho(X_{p+1}) = Y' - \pi_-(\partial X_{p+1})$, and $\partial\pi_-(\partial X_{p+1}) = -\partial\pi_+(\partial X_{p+1})$. Therefore from Lemma 3. 20 we obtain

$$\mathscr{L}_{Y*} R^p = -\partial Y + \partial\pi_-(\partial X_{p+1}) + \partial f^{p+1}$$
$$= -\partial Y' + \partial\left(f^{p+1} - \pi_+(\partial X_{p+1}) \right) .$$

Hence it follows as before that $f^{p+1} = \pi_+(\partial X_{p+1})$ and $\mathscr{L}_{Y*} R^p = -\partial Y'$. Moreover if $0 \leqq j \leqq p$, we have

$$f^{p+1}(\theta_{j-p-1}) = \pi_+(\partial X_{p+1})(\theta_{j-p-1}) = -[X_{p+1}, \theta_{j-p-1}] .$$

Consequently we see from (3. 22) that

$$\mathscr{L}_{Y*}\theta_j \underset{j-p-2}{\equiv} -[X_{p+1}, \theta_{j-p-1}] - Y'(\theta_{j-p-1}) , \qquad j \leqq p .$$

Thus Lemma 3. 14 follows.

We now proceed to the proof of Lemma 3. 20, which is analogous to that of Lemma 3. 19. First of all we obtain

$$d\mathscr{L}_{Y*}\theta_j + \sum_{u+v=j} [\mathscr{L}_{Y*}\theta_u, \theta_v]$$
$$\underset{j-p-1}{\equiv} \frac{1}{2} \sum_{l \leqq p} (\mathscr{L}_{Y*} R^l_j)(\theta_- \wedge \theta_-) - \sum_{l \leqq p} R^l_j (Y\theta_- \wedge \theta_-) , \qquad j \leqq p-1 .$$

Putting $g^{p+1}=-Y+f^{p+1}$, we further obtain the following equalities:

$$d\mathscr{L}_{Y^*}\theta_j \underset{j-p-1}{\equiv} g^{p+1}(d\theta_{j-p-1}) \underset{j-p-1}{\equiv} -\frac{1}{2}\sum_{r+s=j-p-1} g^{p+1}\big([\theta_r, \theta_s]\big),$$
$$\sum_{u+v=j}[\mathscr{L}_{Y^*}\theta_u, \theta_v] \underset{j-p-1}{\equiv} \sum_{r+s=j-p-1}[g^{p+1}(\theta_r), \theta_s],$$
$$R^l_j(Y\theta_-\wedge\theta_-) = \sum_{r+s=j-l-1} R^l(Y\theta_{r-p-1}\wedge\theta_s) \underset{j-p-1}{\equiv} 0,$$

From these equalities it follows immediately that

$$\sum_{l\leqq p}(\mathscr{L}_{Y^*}R^l_j)(\theta_-\wedge\theta_-) \underset{j-p-1}{\equiv} (\partial g^{p+1})_j(\theta_-\wedge\theta_-),$$

meaning that $\mathscr{L}_{Y^*}R^l=0$, $l<p$, and

$$\mathscr{L}_{Y^*}R^p = \partial g^{p+1}.$$

We have thus proved Lemma 3. 20.

3. 11. Proof of Lemma 3. 15. As before we have

$$\mathscr{L}_{Z^*}\theta_p \underset{-1}{\equiv} 0.$$

By (p. 2) we have $\mathscr{L}_{Z^*}\theta_j \underset{j-p-1}{\equiv} 0$, $j\leqq p-1$, and by Lemma 3. 5 we have $\mathscr{L}_{Z^*}\theta_j \underset{j-p-2}{\equiv} 0$, $j<0$. From these facts we see that there is a unique function f^{p+1}: $B^p\to C^{p+1,1}_+$ such that

$$\mathscr{L}_{Z^*}\theta_j \underset{j-p-2}{\equiv} f^{p+1}(\theta_{j-p-1}), \qquad j\leqq p. \tag{3. 23}$$

LEMMA 3. 21. $\mathscr{L}_{Z^*}R^p=\partial f^{p+1}$.

The proof of this lemma is quite similar to that of Lemma 3. 20, and therefore it is omitted. Lemma 3. 15 follows easily from (3. 23) and Lemma 3. 21.

§ 4. The existence of normal connections of type $\mathfrak{G}$

The main aim of this section is to accomplish the proof of Theorem 2. 7, (1). Let $(P^\sharp, \xi)$ be a $G^\sharp_0$-structure of type $\mathfrak{M}$ on a manifold M. By Theorem 3. 7 $(P^\sharp, \xi)$ is reduced to a unique $\tilde{G}$-structure of type $(\mathfrak{M}, \mu-1)$, $(\tilde{P}, \tilde{\xi})$, on M in a natural manner.

4. 1. The set of normal pre-μ-systems. Let $\theta^{(\mu)}=\{\theta_j\}_{j\leqq\mu-1}$ be a system of $\mathfrak{g}_j$-valued 1-forms θ_j on $\tilde{P}$ which is compatible with $\tilde{\xi}$, *i.e.*, $\theta_j=\tilde{\xi}_j$, $j<0$. Then we recall that $\theta^{(\mu)}$ is a pre-μ-system in $(\tilde{P}, \tilde{\xi})$ if it satisfies the following three conditions (μ. a), (μ. b), and (μ. c):

(μ. a) The system $\theta^{(\mu-1)}=\{\theta_j\}_{j\leqq\mu-2}$ is a normal $(\mu-1)$-system in $(\tilde{P}, \tilde{\xi})$, *i.e.*, for any $j\leqq\mu-2$ θ_j satisfies the following conditions:

(μ-1. 1) $\theta_j(\rho(X_r)^*)=\delta_{jr}X_r,\ X_r\in\mathfrak{g}_r,\ 0\leqq r\leqq\mu-1,$

(μ-1. 2) i) $R^*{}_a\theta_j=\mathrm{Ad}(a^{-1})\theta_j,\ a\in G_0,$

ii) $\mathscr{L}_{\rho(X_r)^*}\theta_j\underset{j-\mu}{\equiv}-[X_r,\theta_{j-r}],\ X_r\in\mathfrak{g}_r,\ 0\leqq r\leqq\mu-1.$

(μ. b) $\theta_{\mu-1}(\rho(X_r)^*)=\delta_{\mu-1,r}X_r,\ X_r\in\mathfrak{g}_r,\ 0\leqq r\leqq\mu-1.$

(μ. c) $R^*{}_a\theta_{\mu-1}=\mathrm{Ad}(a^{-1})\theta_{\mu-1},\ a\in G_0.$

Let $\theta^{(\mu)}$ be a pre-μ-system in $(\tilde{P},\tilde{\xi})$. Then we have the structure equation:

$$\Theta^\mu{}_j\underset{j-\mu}{\equiv}\frac{1}{2}\sum_{l\leqq\mu-1}R^l{}_j(\theta_-\wedge\theta_-)\,,\qquad j\leqq\mu-2\,.$$

We recall that $\theta^{(\mu)}$ is normal if the curvature $\{R^l\}_{l\leqq\mu-1}$ satisfies the following conditions:

i) $R^l=0$ *for* $l<0$,

ii) $\partial^*R^l=0$ *for* $0\leqq l\leqq\mu-1$.

LEMMA 4. 1. Let $\theta^{(\mu)}$ be a normal pre-μ-system in $(\tilde{P},\tilde{\xi})$. Let $1\leqq k\leqq\mu-1$, and $X_k\in\mathfrak{g}_k$. Then there is a unique function $g_{\rho(X_k)}:\tilde{P}\to\mathfrak{g}_k$ such that

$$\mathscr{L}_{\rho(X_k)^*}\theta_j\underset{j-\mu-1}{\equiv}-[X_k,\theta_{j-k}]+[g_{\rho(X_k)},\theta_{j-\mu}]\,,\qquad j\leqq\mu-1\,.$$

PROOF. The proof of this lemma is quite similar to that of Lemma 3. 13 (see 3. 9). First of all we see that there is a unique function $f^\mu:\tilde{P}\to C^{\mu,1}_+=C^{\mu,1}$ such that

$$\mathscr{L}_{\rho(X_k)^*}\theta_j\underset{j-\mu-1}{\equiv}-[X_k,\theta_{j-k}]+f^\mu(\theta_{j-\mu})\,,\qquad j\leqq\mu-1\,.$$

Then we can show that

$$\mathscr{L}_{\rho(X_k)^*}R^{\mu-1}=(R^{\mu-1-k})^{X_k}+\partial f^\mu\,.$$

Since $\theta^{(\mu)}$ is normal, both $R^{\mu-1}$ and $(R^{\mu-1-k})^{X_k}$ take values in $Z^{\mu-1,2}_*$. Hence $\partial f^\mu=0$. Therefore it follows from Lemma 1. 14 that there is a unique function $g_{\rho(X_k)}:\tilde{P}\to\mathfrak{g}_\mu$ such that $f^\mu=-\partial g_{\rho(X_k)}$, proving Lemma 4. 1.

Let Δ be the set of all normal pre-μ-systems in $(\tilde{P},\tilde{\xi})$. Let $\alpha=\{\alpha_j\}_{j\leqq\mu-1}$, $\beta=\{\beta_j\}_{j\leqq\mu-1}\in\Delta$. By (2) of Lemma 3. 12 there is a unique function $f_{\alpha\beta}:\tilde{P}\to\mathfrak{g}_\mu$ such that

$$\alpha_j\underset{j-\mu-1}{\equiv}\beta_j+[f_{\alpha\beta},\beta_{j-\mu}]\,,\qquad j\leqq\mu-1\,. \tag{4. 1}$$

Let $1\leqq k\leqq\mu-1$, and $X_k\in\mathfrak{g}_k$. By Lemma 4. 1 there is a unique function $g^\alpha_{\rho(X_k)}:\tilde{P}\to\mathfrak{g}_\mu$ such that

$$\mathscr{L}_{\rho(X_k)^*}\alpha_j\underset{j-\mu-1}{\equiv}-[X_k,\alpha_{j-k}]+[g^\alpha_{\rho(X_k)},\alpha_{j-\mu}]\,,\qquad j\leqq\mu-1\,. \tag{4. 2}$$

We can easily prove the following

LEMMA 4.2. $f_{\alpha\beta}+f_{\beta\gamma}=f_{\alpha\gamma}$.

LEMMA 4.3. (1) $R^*_a f_{\alpha\beta}=\mathrm{Ad}(a^{-1})f_{\alpha\beta}$, $a\in G_0$.

(2) $\mathscr{L}_{\rho(X_k)^*}f_{\alpha\beta}=g^\alpha_{\rho(X_k)}-g^\beta_{\rho(X_k)}$.

PROOF. (1) can be obtained by applying R^*_a to (4.1) and using ($\mu-1$. 2), i), and (μ. c) (cf. the proof of Lemma 3.10). Similarly (2) can be obtained by applying $\mathscr{L}_{\rho(X_k)^*}$ to (4.1) and using (4.2).

LEMMA 4.4. (1) $R^*_a g^\alpha_{\rho(X_k)}=\mathrm{Ad}(a^{-1})\, g^\alpha_{\rho(\mathrm{Ad}(a)X_k)}$, $a\in G_0$.

(2) *If* $1\leqq r,\ s\leqq\mu-1$, *and if* $X_r\in\mathfrak{g}_r$ *and* $Y_s\in\mathfrak{g}_s$, *then we have*:

$$\mathscr{L}_{\rho(X_r)^*}g^\alpha_{\rho(Y_s)}-\mathscr{L}_{\rho(Y_s)^*}g^\alpha_{\rho(X_r)}-g^\alpha_{\rho([X_r,Y_s])}=\delta_{\mu,r+s}[X_r,Y_s]\,.$$

PROOF. (1) can be obtained by applying R^*_a to (4.2) and using (μ-1. 2), i), (μ, c) and the following equalities:

$$\begin{aligned}R^*_a\mathscr{L}_{\rho(X_k)^*}\alpha_j&=\mathscr{L}_{(R_{a^{-1}})_*\rho(X_k)^*}R^*_a\alpha_j\\&=\mathscr{L}_{\rho(\mathrm{Ad}(a)X_k)^*}R^*_a\alpha_j\,.\end{aligned}$$

(2) Applying $\mathscr{L}_{\rho(X_r)^*}$ to the equalities

$$\mathscr{L}_{\rho(Y_s)^*}\alpha_j\underset{j-\mu-1}{\equiv}-[Y_s,\alpha_{j-s}]+[g^\alpha_{\rho(Y_s)},\alpha_{j-\mu}]\,,$$

we obtain

$$\begin{aligned}\mathscr{L}_{\rho(X_r)^*}\mathscr{L}_{\rho(Y_s)^*}\alpha_j&\underset{j-\mu-1}{\equiv}-[Y_s,\mathscr{L}_{\rho(X_r)^*}\alpha_{j-s}]\\&\qquad+[\mathscr{L}_{\rho(X_r)^*}g^\alpha_{\rho(Y_s)},\alpha_{j-\mu}]\\&\underset{j-\mu-1}{\equiv}\Big[Y_s,[X_r,\alpha_{j-r-s}]\Big]\\&\qquad+[\mathscr{L}_{\rho(X_r)^*}g^\alpha_{\rho(Y_s)},\alpha_{j-\mu}]\,.\end{aligned}$$

Since

$$\mathscr{L}_{\rho(X_r)^*}\mathscr{L}_{\rho(Y_s)^*}-\mathscr{L}_{\rho(Y_s)^*}\mathscr{L}_{\rho(X_r)^*}=\mathscr{L}_{\rho([X_r,Y_s])^*}\,,$$

it follows that

$$\begin{aligned}\mathscr{L}_{\rho([X_r,Y_s])^*}\alpha_j&\underset{j-\mu-1}{\equiv}-\Big[[X_r,Y_s],\alpha_{j-r-s}\Big]\\&\qquad+[\mathscr{L}_{\rho(X_r)^*}g^\alpha_{\rho(Y_s)}-\mathscr{L}_{\rho(Y_s)^*}g^\alpha_{\rho(X_r)},\alpha_{j-\mu}].\end{aligned}$$

On the other hand if $r+s<\mu$, we have

$$\begin{aligned}\mathscr{L}_{\rho([X_r,Y_s])^*}\alpha_j&\underset{j-\mu-1}{\equiv}-\Big[[X_r,Y_s],\alpha_{j-r-s}\Big]\\&\qquad+[g^\alpha_{\rho([X_r,Y_s])},\alpha_{j-\mu}]\,.\end{aligned}$$

Furthermore if $r+s>\mu$, we have $[X_r, Y_s]=0$, and if $r+s=\mu$, we have $\rho([X_r, Y_s])=0$. Thus (2) follows.

4.2. The principal fibre bundle P. Consider the kernel $G''=\exp \mathfrak{g}_\mu$ of the homomorphism $\rho: G'\to\tilde{G}$. For any $\alpha, \beta\in\Delta$ we define a map $\sigma_{\alpha\beta}$: $\tilde{P}\to G''$ by

$$\sigma_{\alpha\beta}=\exp f_{\alpha\beta}\,.$$

Then by Lemma 4.2 we see that the system $\{\sigma_{\alpha\beta}\}$ gives a system of transition functions, *i.e.*,

$$\sigma_{\alpha\beta}\sigma_{\beta\gamma}=\sigma_{\alpha\gamma}\,.$$

Hence the system $\{\sigma_{\alpha\beta}\}$ defines a principal fibre bundle P over the base space $\tilde{P}$ with structure group G''. And there corresponds to every α the canonical trivialization

$$\psi_\alpha:\ P\to\tilde{P}\times G''\,.$$

Let ρ denote the projection $P\to\tilde{P}$. Then ψ_α may be expressed as follows:

$$\psi_\alpha(z)=\big(\rho(z),\ \sigma_\alpha(z)\big)\,,\qquad z\in P\,.$$

For any α, β we have

$$\sigma_\alpha(z)=\sigma_{\alpha\beta}\big(\rho(z)\big)\,\sigma_\beta(z)\,,\qquad z\in P\,,$$

and putting $\psi_{\alpha\beta}=\psi_\alpha\circ\psi_\beta^{-1}$, we obtain

$$\psi_{\alpha\beta}(x, u)=\big(x, \sigma_{\alpha\beta}(x)\,u\big)\,,\qquad (x, u)\in\tilde{P}\times G''\,.$$

We shall show that the group G' acts on P to the right in a natural manner, so that P becomes a principal fibre bundle over the base space M and $\rho: P\to\tilde{P}$ becomes a homomorphism of P onto $\tilde{P}$ corresponding to the homomorphism $\rho: G'\to\tilde{G}$.

First of all for any $z\in P$ and $a\in G_0$ we define $za\in P$ by

$$\psi_\alpha(za)=\big(\rho(z)a,\ a^{-1}\sigma_\alpha(z)\,a\big)\,.$$

By (1) of Lemma 4.3 we have $\sigma_{\alpha\beta}(xa)=a^{-1}\sigma_{\alpha\beta}(x)a$, $x\in\tilde{P}$, showing that za does not depend on the choice of α. Clearly the group G_0 acts on P by the rule $P\times G_0\ni(z, a)\to za\in P$. For any $X\in\mathfrak{g}_0$ we denote by X^* the vector field on P induced from the 1-parameter group of transformations $z\to z\cdot\exp tX$.

Let us define subspaces $\mathfrak{n}$ and $\mathfrak{p}$ of $\mathfrak{g}'=\sum_{j=0}^{\mu}\mathfrak{g}_j$ respectively by

$$\mathfrak{n}=\sum_{j=1}^{\mu}\mathfrak{g}_j\,,$$

$$\mathfrak{p}=\sum_{j=1}^{\mu-1}\mathfrak{g}_j\,.$$

Then $\mathfrak{n}$ is an ideal of $\mathfrak{g}'$, and we have:

$$\mathfrak{g}'=\mathfrak{g}_0+\mathfrak{n}\qquad\text{(direct sum)}\,,$$

$$\mathfrak{n}=\mathfrak{p}+\mathfrak{g}_\mu\qquad\text{(direct sum)}\,.$$

For any $A\in\rho(\mathfrak{n})=\rho(\mathfrak{p})$ and $\alpha\in\Delta$ let us define a function $g_A^\alpha:\tilde{P}\to\mathfrak{g}_\mu$ by

$$g_A^\alpha=\sum_{r=1}^{\mu-1}g_{A_r}^\alpha\,,$$

where A_r is the $\rho(\mathfrak{g}_r)$-component of A with respect to the decomposition $\rho(\mathfrak{n})=\sum_{r=1}^{\mu-1}\rho(\mathfrak{g}_r)$.

Let $X\in\mathfrak{n}$. Then we define a vector field X^* on P by

$$(\psi_{\alpha*}X^*)_{(x,u)}=\left(\rho(X)^*{}_x,\left(g_{\rho(X)}^\alpha(x)+X''\right)_u\right),\qquad (x,u)\in\tilde{P}\times G''\,,$$

where X'' stands for the $\mathfrak{g}_\mu$-component of X with respect to the decomposition $\mathfrak{n}=\mathfrak{p}+\mathfrak{g}_\mu$. (Note that we are identifying $\mathfrak{g}_\mu$ with the Lie algebra of all left invariant vector fields on G''.) We show that X^* does not depend on the choice of α. Indeed let $(Y,Z)\in T(\tilde{P})_x\times T(G'')_u$. Then we have

$$(\psi_{\beta\alpha})_*(Y,Z)=\left(Y,(Yf_{\beta\alpha}+\tilde{Z})_u\right),$$

where $\tilde{Z}$ is the unique element of $\mathfrak{g}''$ with $\tilde{Z}_u=Z$. Hence using (2) of Lemma 4.3, we obtain

$$\begin{aligned}(\psi_{\beta\alpha})_*\left(\rho(X)^*{}_x,\left(g_{\rho(X)}^\alpha(x)\right)_u\right)&=\left(\rho(X)^*{}_x,\left(\rho(X)^*{}_xf_{\beta\alpha}+g_{\rho(X)}^\alpha(x)\right)_{\sigma_{\beta\alpha}(x)u}\right)\\&=\left(\rho(X)^*{}_x,\left(g_{\rho(X)}^\beta(x)\right)_{\sigma_{\beta\alpha}(x)u}\right),\end{aligned}$$

which proves our assertion.

Let us now calculate the integral curves of X^*. For any $A\in\rho(\mathfrak{n})$ and $\alpha\in\Delta$ we define a function $\bar{g}_A^\alpha:\tilde{P}\to\mathfrak{g}_\mu$ by

$$\bar{g}_A^\alpha(x)=\int_0^1 g_A^\alpha(x\cdot\exp tA)\,dt\,,\qquad x\in\tilde{P}\,,$$

and define a map $\varepsilon_A^\alpha:\tilde{P}\to G''$ by

$$\varepsilon_A^\alpha=\exp\bar{g}_A^\alpha\,.$$

LEMMA 4.5. *Let $X\in\mathfrak{n}$ and $z\in P$. Let $z_t=z_X(t)$ be the maximal integral curve of X^* with $z_0=z$. Then z_t is defined for any $t\in\boldsymbol{R}$, and*

$$\psi_\alpha(z_t)=\left(\rho(z)\cdot\exp t\rho(X),\ \sigma_\alpha(z)\cdot\varepsilon^\alpha_{t\rho(X)}\left(\rho(z)\right)\cdot\exp tX''\right), \qquad t\in\boldsymbol{R}.$$

PROOF. Put $\psi_\alpha(z_t)=(x_t, u_t)$, which is a maximal integral curve of $\psi_{\alpha*}X^*$. Hence we have

$$\frac{dx_t}{dt}=\rho(X)^*_{x_t},$$

$$\frac{du_t}{dt}=\left(g^\alpha_{\rho(X)}(x_t)+X''\right)_{u_t}.$$

Since $x_0=\rho(z)$, it follows that

$$x_t=\rho(z)\cdot\exp t\rho(X).$$

If we define a curve v_t of $\mathfrak{g}_\mu$ by $\exp v_t=u_t$, it also follows that

$$\frac{dv_t}{dt}=g^\alpha_{\rho(X)}\left(\rho(z)\cdot\exp t\rho(X)\right)+X'',$$

whence

$$\begin{aligned} v_t &= v_0+\int_0^t g^\alpha_{\rho(X)}\left(\rho(z)\cdot\exp t\rho(X)\right)dt+tX'' \\ &= v_0+\bar{g}_{t\rho(X)}\left(\rho(z)\right)+tX''. \end{aligned}$$

Since $u_0=\sigma_\alpha(z)$, we therefore obtain

$$u_t=\sigma_\alpha(z)\cdot\varepsilon^\alpha_{t\rho(X)}\left(\rho(z)\right)\cdot\exp tX'',$$

proving Lemma 4. 5. (From the discussion above it is clear that z_t is defined for any $t\in\boldsymbol{R}$.)

We denote by N the connected Lie subgroup of G' generated by the ideal $\mathfrak{n}$ of $\mathfrak{g}'$, which is nothing but the subgroup $\exp\mathfrak{g}_1\cdots\exp\mathfrak{g}_\mu$ of G' (cf. Lemma 1. 7). From the very definition of $\mathfrak{n}$ we can easily verify that the exponential map exp maps $\mathfrak{n}$ diffeomorphically onto N. (This fact also follows from the facts that $\mathfrak{n}$ is nilpotent and that N is simply connected.)

Let $z\in P$ and $a\in N$. Taking account of Lemma 4. 5, we define $za\in P$ by $za=z_X(1)$, *i. e.*,

$$\psi_\alpha(za)=\left(\rho(z)\exp\rho(X),\ \sigma_\alpha(z)\cdot\varepsilon^\alpha_{\rho(X)}\left(\rho(z)\right)\cdot\exp X''\right),$$

where X is the unique element of $\mathfrak{n}$ with $a=\exp X$. In particular if $a\in G''$, we have $\psi_\alpha(za)=(\rho(z), \sigma_\alpha(z)a)$, and hence the group G'' acts on P by the rule $P\times G''\ni(z, a)\rightarrow za\in P$. Note that this action is nothing but the action of G'' on the principal fibre bundle P over $\tilde{P}$.

By Lemma 1.7 every element a of G' can be written uniquely in the form:

$$a = a_0 \cdot b\,,$$

where $a_0 \in G_0$ and $b \in N$. This being said, for any $z \in P$ and $a \in G'$ we define $za \in P$ by

$$za = (za_0)\, b\,.$$

Every element X of $\mathfrak{g}'$ can be written uniquely in the form:

$$X = X_0 + Y\,,$$

where $X_0 \in \mathfrak{g}_0$ and $Y \in \mathfrak{n}$. Then we define a vector field X^* on P by

$$X^* = X^*{}_0 + Y^*\,.$$

Furthermore we denote by $\tilde{\pi}$ the projection $\tilde{P} \rightarrow M$, and define a map π: $P \rightarrow M$ by

$$\pi = \tilde{\pi} \circ \rho\,.$$

These being prepared, we shall prove the following

THEOREM 4.6. (1) *The group G' acts on P by the rule $P \times G' \ni (z, a) \rightarrow za \in P$.*

(2) *With respect to this action of G' P is a principal fibre bundle over the base space M with structure group G' with projection π.*

(3) *The projection ρ: $P \rightarrow \tilde{P}$ is a homomorphism corresponding to the homomorphism ρ: $G' \rightarrow \tilde{G}$.*

(4) *For any $X \in \mathfrak{g}'$ X^* is the vector field on P induced from the* 1-*parameter group of transformations $z \rightarrow z \cdot \exp tX$.*

We first prove the following

LEMMA 4.7. $[X^*, Y^*] = [X, Y]^*$, $X, Y \in \mathfrak{n}$.

PROOF. By (2) of Lemma 4.4 we have

$$\mathscr{L}_{\rho(X)^*} g^\alpha_{\rho(Y)} - \mathscr{L}_{\rho(Y)^*} g^\alpha_{\rho(X)} - g^\alpha_{\rho([X,Y])} = [X, Y]''\,.$$

Therefore it follows that

$$\begin{aligned}
[\psi_{\alpha^*} X^*, \psi_{\alpha^*} Y^*]_{(x,u)} &= \Big(\big[\rho(X)^*, \rho(Y)^*\big]_x, \big(\rho(X)^*{}_x g^\alpha_{\rho(Y)} - \rho(Y)^*{}_x g^\alpha_{\rho(X)}\big)_u\Big) \\
&= \Big(\rho\big([X, Y]\big)^*{}_x, \big(g^\alpha_{\rho([X,Y])}(x) + [X, Y]''\big)_u\Big) \\
&= \big(\psi_{\alpha^*}[X, Y]^*\big)_{(x,u)}\,.
\end{aligned}$$

Hence we obtain $[X^*, Y^*] = [X, Y]^*$, proving Lemma 4.7.

LEMMA 4.8. *The group N acts on P by the rule $P\times N\ni(z, a)\to za\in P$.*

PROOF. Fix any point x_0 of $\tilde{P}$, and consider the orbit $x_0\cdot\rho(N)$ of $\rho(N)$ through x_0. Put $Q=\rho^{-1}(x_0\cdot\rho(N))$, which is a submanifold of P diffeomorphic with $\rho(N)\times G''$ and hence with a Euclidean space. It is clear that for every $X\in\mathfrak{n}$ the vector field X^* is tangent to Q. This being said, we denote by $\tilde{X}$ the restriction of X^* to Q. Then by Lemma 4.7 we have $[\tilde{X}, \tilde{Y}]=\widetilde{[X, Y]}$, $X, Y\in\mathfrak{n}$. Furthermore it is easy to see that for every $z\in Q$ the assignment $X\to\tilde{X}_z$ gives a linear isomorphism of $\mathfrak{n}$ onto $T(Q)_z$. Since Q is simply connected, we know from these facts that there is a map $\varphi: Q\to N$ such that

$$\varphi_*\tilde{X}_z=X_{\varphi(z)}\,,\qquad X\in\mathfrak{n}\,,\qquad z\in Q\,.$$

(For example, see [5]). Clearly φ is a local diffeomorphism. If $z\in Q$ and $X\in\mathfrak{n}$, we know from Lemma 4.5 that $z\cdot\exp tX$ is an integral curve of $\tilde{X}$. Since $\varphi(z)\cdot\exp tX$ is an integral curve of X, it follows that $\varphi(z\cdot\exp tX)=\varphi(z)\cdot\exp tX$. Hence we have shown that

$$\varphi(za)=\varphi(z)\,a\,,\qquad z\in Q\,,\qquad a\in N\,.$$

If $z\in Q$, $a\in N$, and $Y\in\mathfrak{n}$, we therefore see that $\varphi(z\cdot(a\cdot\exp tY))=\varphi(z)\cdot a\cdot\exp tY$ is an integral curve of Y, meaning that $z\cdot(a\cdot\exp tY)$ is an integral curve of $\tilde{Y}$. Hence $z\cdot(a\cdot\exp tY)=(za)\cdot\exp tY$. We have thereby shown that

$$(za)\,b=z(ab)\,,\qquad z\in Q\,,\qquad a,\ b\in N\,,$$

proving Lemma 4.8.

LEMMA 4.9. $(zb)\,a=z(ba)$, $z\in P$, $b\in N$, $a\in G_0$.

PROOF. By (1) of Lemma 4.4 we can easily verify that

$$\bar{g}^{\alpha}_{\mathrm{Ad}(a^{-1})A}(xa)=\mathrm{Ad}\,(a^{-1})\,\bar{g}^{\alpha}_{A}(x)\,,\qquad x\in\tilde{P}\,,\qquad a\in G_0\,,\qquad A\in\rho(\mathfrak{n})\,,$$

whence

$$\varepsilon^{\alpha}_{\mathrm{Ad}(a^{-1})A}(xa)=a^{-1}\varepsilon^{\alpha}_{A}(x)\,a\,.$$

Therefore if $z\in P$, $a\in G_0$, and $X\in\mathfrak{n}$, we obtain:

$$\begin{aligned}\psi_\alpha\Big((z\cdot\exp X)\cdot a\Big)&=\Big(\rho(z\cdot\exp X)\cdot a,\ a^{-1}\sigma_\alpha(z\exp X)\,a\Big)\\&=\Big(\rho(z)\cdot\rho(\exp X)\cdot a,\ a^{-1}\sigma_\alpha(z)\,\varepsilon^{\alpha}_{\rho(X)}\Big(\rho(z)\Big)\cdot\exp X''\cdot a\Big)\\&=\Big(\rho(za)\,\rho\Big(\exp\mathrm{Ad}\,(a^{-1})\,X\Big),\ \sigma_\alpha(za)\,\varepsilon^{\alpha}_{\rho(\mathrm{Ad}(a^{-1})X)}\Big(\rho(za)\Big)\,\exp\mathrm{Ad}\,(a^{-1})\,X''\Big)\\&=\psi_\alpha\Big((za)\cdot\exp\mathrm{Ad}\,(a^{-1})\,X\Big)=\psi_\alpha\Big(z\cdot(\exp X\cdot a)\Big)\,.\end{aligned}$$

(Note that $\exp X \cdot a = a \cdot \exp \mathrm{Ad}(a^{-1}) X$ and $\mathrm{Ad}(a^{-1}) X \in \mathfrak{n}$.) Hence we have shown that $(z \cdot \exp X) a = z \cdot (\exp X \cdot a)$, proving Lemma 4.9.

From Lemmas 4.8 and 4.9 we can easily derive the following

LEMMA 4.10. *The group G' acts on P by the rule $P \times G' \ni (z, a) \to za \in P$.*

LEMMA 4.11. (1) $\rho(za) = \rho(z)\rho(a)$, $z \in P$, $a \in G'$.

(2) *The group G' freely acts on P, i.e., if $z \in P$, $a \in G'$, and $za = z$, then $a = e$.*

(3) *Let $z, w \in P$. Then $\pi(z) = \pi(w)$ if and only if there is an $a \in G'$ such that $w = za$.*

PROOF. (1) is clear. (2) We have $\rho(za) = \rho(z)\rho(a) = \rho(z)$, whence $\rho(a) = e$, *i.e.*, $a \in G''$. Hence $a = e$, proving (2). (3) If $w = za$, we have $\pi(w) = \tilde{\pi}(\rho(w)) = \tilde{\pi}(\rho(z)\rho(a)) = \tilde{\pi}(\rho(z)) = \pi(z)$. Conversely suppose that $\pi(w) = \pi(z)$. Since $\tilde{\pi}(\rho(w)) = \tilde{\pi}(\rho(z))$, there is an $a' \in G'$ such that $\rho(w) = \rho(z)\rho(a') = \rho(za')$. Hence there is an $a'' \in G''$ such that $w = (za')a'' = z(a'a'')$.

We have thus proved (1)$\sim$(3) of Theorem 4.6. (4) of Theorem 4.6 is now clear.

4.3. The systems $\bar{\theta}$. Let P be the principal fibre bundle over the base space M with structure group G' which was constructed in the previous section. Our task from now on is to show that P is endowed with a normal connection of type $\mathfrak{G}$.

For any $\alpha \in \varDelta$ we define a function $f_\alpha : P \to \mathfrak{g}_\mu$ by

$$\sigma_\alpha = \exp f_\alpha .$$

Then we have

$$f_\alpha = \rho^* f_{\alpha\beta} + f_\beta , \qquad \alpha, \beta \in \varDelta .$$

Let $\theta \in \varDelta$. For any $j \leqq \mu - 1$ we define a $\mathfrak{g}_j$-valued 1-form $\bar{\theta}_j$ on P by

$$\bar{\theta}_j = \rho^* \theta_j - [f_\theta, \rho^* \theta_{j-\mu}] ,$$

and denote by $\bar{\theta}$ the system $\{\bar{\theta}_j\}_{j \leqq \mu-1}$. Clearly we have

$$\bar{\theta}_j = \rho^* \theta_j = \rho^* \xi_j , \qquad j < 0 .$$

Hereafter the symbols $\underset{k}{\equiv}$ will be considered with respect to the system $\{\bar{\theta}_j\}_{j<0}$.

LEMMA 4.12. *The system $\bar{\theta} = \{\bar{\theta}_j\}_{j \leqq \mu-1}$ has the following properties:*

(1) $\qquad \bar{\theta}_j(X^*_r) = \delta_{jr} X_r , \qquad X_r \in \mathfrak{g}_r ,\ 0 \leqq r \leqq \mu .$

(2) i) $\quad R^*_a \bar{\theta}_j = \mathrm{Ad}(a^{-1}) \bar{\theta}_j , \qquad a \in G_0 ,$

ii) $\mathscr{L}_{X_r^*}\bar{\theta}_j \underset{j-\mu-1}{\equiv} -[X_r, \bar{\theta}_{j-r}]$, $\quad X_r \in \mathfrak{g}_r$, $\quad 0 \leqq r \leqq \mu$.

PROOF. (1) We first remark that for every $X \in \mathfrak{g}'$ X^* and $\rho(X)^*$ are ρ-related, *i. e.*, $\rho_* X^*{}_z = \rho(X)^*{}_{\rho(z)}$, $z \in P$. Let $0 \leqq r \leqq \mu-1$, and let $X_r \in \mathfrak{g}_r$. Then using ($\mu-1$. 1) and (μ. b), we obtain

$$\begin{aligned}\bar{\theta}_j(X^*{}_r) &= \rho^*\Big(\theta_j\big(\rho(X_r)^*\big)\Big) - \Big[f_\theta, \rho^*\Big(\theta_{j-\mu}\big(\rho(X_r)^*\big)\Big)\Big] \\ &= \delta_{jr} X_r .\end{aligned}$$

Now let $X_\mu \in \mathfrak{g}_\mu$. Since $\rho(X_\mu)=0$, we clearly have $\bar{\theta}_j(X_\mu^*)=0$. We have thus proved (1).

(2) Let $a \in G_0$. Then we have $R^*{}_a f_\theta = \mathrm{Ad}(a^{-1}) f_\theta$ and $R_a \circ \rho = \rho \circ R_a$. Therefore using ($\mu-1$. 2), i) and ($\mu$. c), we obtain $R^*{}_a \bar{\theta}_j = \mathrm{Ad}(a^{-1})\bar{\theta}_j$. In particular we have $\mathscr{L}_{X^*{}_0}\bar{\theta}_j = -[X_0, \bar{\theta}_j]$, $X_0 \in \mathfrak{g}_0$. Now let $1 \leqq r \leqq \mu-1$, and let $X_r \in \mathfrak{g}_r$. By Lemma 4. 5 we have $\sigma_\theta(z \cdot \exp tX_r) = \sigma_\theta(z) \cdot \varepsilon^\theta_{t\rho(X_r)}(\rho(z))$, $z \in P$, whence $\mathscr{L}_{X^*{}_r} f_\theta = \rho^* g^\theta_{\rho(X_r)}$. Therefore using (4. 2), we obtain

$$\begin{aligned}\mathscr{L}_{X^*{}_r}\bar{\theta}_j &= \rho^*(\mathscr{L}_{\rho(X_r)^*}\theta_j) - [\mathscr{L}_{X^*{}_r} f_\theta, \rho^*\theta_{j-\mu}] \\ &\quad - \Big[f_\theta, \rho^*(\mathscr{L}_{\rho(X_r)^*}\theta_{j-\mu})\Big] \\ &\underset{j-\mu-1}{\equiv} -[X_r, \rho^*\theta_{j-r}] + [\rho^* g^\theta_{\rho(X_r)}, \rho^*\theta_{j-\mu}] \\ &\quad -[\rho^* g^\theta_{\rho(X_r)}, \rho^*\theta_{j-\mu}] + \Big[f_\theta, [X_r, \rho^*\theta_{j-r-\mu}]\Big] \\ &\underset{j-\mu-1}{\equiv} -[X_r, \bar{\theta}_{j-r}] .\end{aligned}$$

Finally let $X_\mu \in \mathfrak{g}_\mu$. Then we have $\sigma_\theta(z \cdot \exp tX_\mu) = \sigma_\theta(z) \cdot \exp tX_\mu$, whence $\mathscr{L}_{X_\mu^*} f_\theta = X_\mu$. Hence we obtain

$$\begin{aligned}\mathscr{L}_{X_\mu^*}\bar{\theta}_j &= -[\mathscr{L}_{X_\mu^*} f_\theta, \rho^*\theta_{j-\mu}] = -[X_\mu, \rho^*\theta_{j-\mu}] \\ &\underset{j-\mu-1}{\equiv} -[X_\mu, \bar{\theta}_{j-\mu}] .\end{aligned}$$

We have thus proved (2).

Let $\{R^l\}_{l \leqq \mu-1}$ be the curvature of the normal pre-μ-system θ in $(\tilde{P}, \tilde{\xi})$. For any $j \leqq \mu-2$ we define a $\mathfrak{g}_j$-valued 2-form Θ^μ_j on P by

$$\bar{\Theta}^\mu_j = d\bar{\theta}_j + \frac{1}{2}\sum_{u+v=j} [\bar{\theta}_u, \bar{\theta}_v] .$$

Put $\bar{\theta}_- = \sum_{j<0} \bar{\theta}_j$.

LEMMA 4. 13. $\bar{\Theta}^\mu_j \underset{j-\mu}{\equiv} \dfrac{1}{2}\sum_{l \leqq \mu-1} (\rho^* R^l_j)(\bar{\theta}_- \wedge \bar{\theta}_-)$, $j \leqq \mu-2$.

PROOF. The form $\bar{\theta}_j$ may be expressed as follows:

$$\bar{\theta}_j = \rho^*\theta_j + (\partial f_\theta)(\rho^*\theta_{j-\mu}).$$

Therefore we deduce from the proof of Lemma 3.9 that

$$\bar{\Theta}^{\mu}_j \underset{j-\mu}{\equiv} \rho^*\Theta^{\mu}_j + \frac{1}{2}(\partial\partial f_\theta)(\bar{\theta}_- \wedge \bar{\theta}_-).$$

Hence the lemma follows from the structure equation for the normal pre-μ-system θ.

LEMMA 4.14. *For any $\alpha, \beta \in \Delta$ we have*

$$\bar{\alpha}_j \underset{j-\mu-1}{\equiv} \bar{\beta}_j, \qquad j \leqq \mu-1.$$

PROOF. This fact follows immediately from (4.1) and the fact that $f_\alpha = \rho^* f_{\alpha\beta} + f_\beta$.

4.4. An existence theorem for normal connections of type $\mathfrak{G}$. By Lemma 4.14 we know that if $\alpha, \beta \in \Delta$, then $\bar{\alpha}_j = \bar{\beta}_j$, $j \leqq 0$. This being said, we denote by η_j the 1-form $\bar{\alpha}_j$ for any $j \leqq 0$, and denote by η the system $\{\eta_j\}_{j\leqq 0}$. Note that $\eta_j = \rho^*\tilde{\xi}_j$ if $j<0$.

In the subsequent two paragraphs we shall prove the following

THEOREM 4.15. *There is a unique normal connection of type $\mathfrak{G}$, ω, in P which is compatible with η, i.e., $\eta_j = \omega_j$, $j \leqq 0$.*

Let ω be the unique connection whose existence is assured by the theorem. Then we have $\omega_- = \rho^*\tilde{\xi}$, indicating that the pair (P, ω) induces the given $G^\#_0$-structure of type $\mathfrak{M}$, $(P^\#, \xi)$. Accordingly (2) of Theorem 2.7 follows from Theorem 4.15. The pair (P, ω) will be called the normal connection of type $\mathfrak{G}$ associated with $(P^\#, \xi)$.

4.5. Normal p-systems ($p \geqq \mu$). Let $p \geqq \mu$. Let $\omega^{(p)} = \{\omega_j\}_{j\leqq p-1}$ be a system of $\mathfrak{g}_j$-valued 1-forms ω_j, $j \leqq p-1$, on P. Assume that $\omega^{(p)}$ is compatible with η, *i.e.*, $\omega_j = \eta_j$, $j \leqq 0$. Then we say that $\omega^{(p)}$ is a p-system in (P, η) if it satisfies the following conditions:

(p. 1) $\qquad \omega_j(X^*_r) = \delta_{jr} X_r, \qquad X_r \in \mathfrak{g}_r, \qquad 0 \leqq r \leqq \mu.$

(p. 2) i) $\quad R^*_a \omega_j = \mathrm{Ad}(a^{-1})\,\omega_j, \qquad a \in G_0,$

ii) $\quad \mathscr{L}_{X^*_r}\omega_j \underset{j-p-1}{\equiv} -[X_r, \omega_{j-r}], \qquad X_r \in \mathfrak{g}_r, \qquad 0 \leqq r \leqq \mu,$

where the symbols $\underset{j-p-1}{\equiv}$ are considered with respect to the system $\{\omega_j\}_{j<0}$.

Let $\omega^{(p)}$ be a p-system in (P, η). Let $q \leqq p$ and $j \leqq q-2$. We define a $\mathfrak{g}_j$-valued 2-form Ω^q_j on P by

$$\Omega^q_j = d\omega_j + \frac{1}{2}\sum_{\substack{u+v=j,\\ u,v\leqq q-1}} [\omega_u, \omega_v].$$

In the following the symbols $\underset{k}{\equiv}$ will be considered with respect to the system $\{\omega_j\}_{j<0}$. Similarly to the proof of Lemma 3.6 we can show that there are unique functions $K^l: P \to C^{l,2}$, $l \leqq p-1$, such that

$$\Omega_j^p \underset{j-p}{\equiv} \frac{1}{2} \sum_{l \leqq p-1} K_j^l(\omega_- \wedge \omega_-) , \qquad j \leqq p-2 ,$$

where $\omega_- = \sum_{j<0} \omega_j$. The system of the equations above will be called the structure equation, and the system of functions, $\{K^l\}_{l \leqq p-1}$, will be called the curvature. Clearly the structure equation induces the equations

$$\Omega_j^q \underset{j-q}{\equiv} \frac{1}{2} \sum_{l \leqq q-1} K_j^l(\omega_- \wedge \omega_-) , \qquad j \leqq q-2 , \qquad q \leqq p .$$

Finally we say that the p-system $\omega^{(p)}$ is normal if the curvature satisfies the following conditions:

i) $\qquad K^l = 0 \quad \text{for} \quad l < 0 ,$

ii) $\qquad \partial^* K^l = 0 \quad \text{for} \quad 0 \leqq l \leqq p-1 .$

From Lemmas 4.12 and 4.13 we know that for any $\theta \in \Delta$, $\bar{\theta}$ gives a normal μ-system in (P, η).

LEMMA 4.16. *For any $p \geqq \mu$, (P, η) admits a normal p-system.*

This lemma will be proved in the next paragraph.

LEMMA 4.17. *Let $p \geqq \mu$. Let $\omega^{(p)}$ and $\omega'^{(p)}$ be two normal p-systems in (P, η). Then we have*

$$\omega'_j \underset{j-p-1}{\equiv} \omega_j , \qquad j \leqq p-1 .$$

PROOF. First consider the case where $p = \mu$. In the same manner as in the proof of Lemma 3.12, (2), we can find a unique function $g_\mu : P \to \mathfrak{g}_\mu$ such that

$$\omega'_j \underset{j-\mu-1}{\equiv} \omega_j + [g_\mu, \omega_{j-\mu}] , \qquad j \leqq \mu-1 .$$

In particular we have $\omega'_0 = \omega_0 + [g_\mu, \omega_{-\mu}]$. Since $\omega_0 = \omega'_0 = \eta_0$, it follows that $[g_\mu, \omega_{-\mu}] = 0$. Hence $g_\mu = 0$, proving the lemma for $p = \mu$. (Let $X \in \mathfrak{g}_\mu$ be such that $[X, \mathfrak{g}_{-\mu}] = 0$. Then we have $X = 0$. Indeed $B(X, \mathfrak{g}_{-\mu}) = B(X, [\mathfrak{g}_{-\mu}, E]) \subset B([X, \mathfrak{g}_{-\mu}], E) = o$, whence $X = 0$.) Next consider the case where $p \geqq \mu+1$. For any $q \geqq \mu+1$ we have $C_+^{q,1} = C^{q,1}$ and $\mathfrak{g}_q = 0$. Consequently it follows from Lemma 1.14 that the map $\partial : C_+^{q,1} \to C^{q-1,2}$ is injective. Therefore starting from the equalities

$$\omega'_j \underset{j-\mu-1}{\equiv} \omega_j , \qquad j \leqq \mu-1 ,$$

and reasoning similarly to the prood of Lemma 3. 9, we can prove the equalities

$$\omega'_j \underset{j-p-1}{\equiv} \omega_j, \qquad j \leqq p-1,$$

which completes the proof of the lemma.

Utilizing Lemmas 4. 16 and 4. 17, we shall now prove Theorem 4. 15. Let $\omega^{(3\mu)}=\{\omega_j\}_{j\leqq 3\mu-1}$ and $\omega'^{(3\mu)}=\{\omega'_j\}_{j\leqq 3\mu-1}$ be two normal 3μ-systems in (P, η). Since $\omega_j=\omega'_j=0$ if $j<-\mu$ or $j>\mu$, it follows from Lemma 4. 17 that

$$\omega'_j = \omega_j, \qquad -\mu \leqq j \leqq \mu.$$

This being remarked, we define a $\mathfrak{g}$-valued 1-form ω on P by

$$\omega = \sum_{j=-\mu}^{\mu} \omega_j.$$

From the very conditions for a 3μ-system it is easy to see that the system $\{\omega_j\}_{-\mu\leqq j\leqq\mu}$ satisfies the following:

(1) $\qquad \omega_j = \eta_j, \qquad j \leqq 0.$

(2) $\qquad \omega_j(X^*_r)=\delta_{jr}X_r, \qquad X_r\in\mathfrak{g}_r, \qquad 0\leqq r\leqq\mu.$

(3) i) $\quad R^*_a\omega_j = \mathrm{Ad}\,(a^{-1})\,\omega_j, \qquad a\in G_0,$

ii) $\quad L_{X^*_r}\omega_j = -[X_r, \omega_{j-r}], \qquad X_r\in\mathfrak{g}_r, \qquad 0\leqq r\leqq\mu.$

These clearly mean that ω is a connection of type $\mathfrak{G}$ in P compatible with η.

Let $\{K^l\}_{l\leqq 3\mu-1}$ be the curvature of $\omega^{(3\mu)}$. $\omega^{(3\mu)}$ being normal, we have:

i) $\quad K^l=0 \quad \text{for} \quad l<0,$

ii) $\quad \partial^* K^l=0 \quad \text{for} \quad 0\leqq l\leqq 3\mu-1.$

Furthermore it is easy to see that the structure equation for $\omega^{(3\mu)}$ yields the equations

$$d\omega_j+\frac{1}{2}\sum_{\substack{r+s=j,\\ -\mu\leqq r,\, s\leqq\mu}}[\omega_r, \omega_s] = \frac{1}{2}\sum_{l\leqq 3\mu-1}K^l_j(\omega_-\wedge\omega_-), \qquad -\mu\leqq j\leqq\mu.$$

Putting $K=\sum_{l\leqq 3\mu-1} K^l$, we therefore see that

$$d\omega+\frac{1}{2}[\omega, \omega] = \frac{1}{2}K(\omega_-\wedge\omega_-)$$

and hence K is the curvature of ω. In this way we have proved that ω is a normal connection of type $\mathfrak{G}$ in P compatible with η.

Let ω' be another normal connection of type $\mathfrak{G}$ in P compatible with η.

Then the system $\omega'^{(3\mu)}=\{\omega'_j\}_{j\leqq 3\mu-1}$ gives a normal 3μ-system in (P,η) (see 2. 3). Hence ω and ω' coincide, as we have remarked above. We have thus proved Theorem 4. 15.

4. 6. Proof of Lemma 4. 16. Let $p\geqq\mu$. Let $\omega^{(p+1)}=\{\omega_j\}_{j\leqq p}$ be a system of $\mathfrak{g}_j$-valued 1-forms ω_j, $j\leqq p$, on P. Then we say that $\omega^{(p+1)}$ is a pre-$(p+1)$-system in (P,η) if it satisfies the following conditions:

$(p+1.$ a) $\qquad \omega^{(p)}=\{\omega_j\}_{j\leqq p-1}$ is a p-system in (P,η).

$(p+1.$ b) $\qquad \omega_p(X^*_r)=\delta_{pr}X_r\,,\qquad X_r\in\mathfrak{g}_r\,,\qquad 0\leqq r\leqq\mu\,.$

$(p+1.$ c) $\qquad R^*_a\omega_p=\mathrm{Ad}\,(a^{-1})\,\omega_p\,,\qquad a\in G_0\,.$

Let $\omega^{(p+1)}$ be a pre-$(p+1)$-system in (P,η). For any $j\leqq p-1$ we define a $\mathfrak{g}_j$-valued 2-form Ω_j^{p+1} on P by

$$\Omega_j^{p+1}=d\omega_j+\frac{1}{2}\sum_{\substack{u+v=j,\\ u,v\leqq p}}[\omega_u,\omega_v]\,.$$

As before it can be shown that there are unique functions $K^l\colon P\rightarrow C^{l,2}$, $l\leqq p$, such that

$$\Omega_j^{p+1}\underset{j-p-1}{\equiv}\frac{1}{2}\sum_{l\leqq p}K_j^l(\omega_-\wedge\omega_-)\,,\qquad j\leqq p-1\,.$$

The system of the equations above will be called the structure equation (for $\omega^{(p+1)}$), and the system of functions, $\{K^l\}_{l\leqq p}$, will be called the curvature (of $\omega^{(p+1)}$). The structure equation for $\omega^{(p+1)}$ induces the equations

$$\Omega_j^p\underset{j-p}{\equiv}\frac{1}{2}\sum_{l\leqq p-1}K_j^l(\omega_-\wedge\omega_-)\,,\qquad j\leqq p-2\,,$$

which together form the structure equation for the p-system $\omega^{(p)}$. In the same manner as in the proof of Lemma 3. 10, it can be shown that

$$R^*_aK^p=(K^p)^a\,,\qquad a\in G_0\,.$$

Finally we say that the pre-$(p+1)$-system $\omega^{(p+1)}$ is normal if the curvature $\{K^l\}_{l\leqq p}$ satisfies the following conditions:

i) $K^l=0$ for $l<0$,

ii) $\partial^*K^l=0$ for $0\leqq l\leqq p-1$.

LEMMA 4. 18. *A normal pre-$(p+1)$-system is a normal $(p+1)$-system.*

PROOF. Let $\omega^{(p+1)}$ be a normal pre-$(p+1)$-system in (P,η). Let $1\leqq k\leqq\mu$, and let $X_k\in\mathfrak{g}_k$. Then we must show that

$$\mathcal{L}_{X^*_k}\omega_j\underset{j-p-2}{\equiv}-[X_k,\omega_{j-k}]\,,\qquad j\leqq p\,.$$

The proof of this fact is quite similar to that of Lemma 3.13 (see 3.9). First of all we see that there is a unique function $f^{p+1}: P \to C_{+}^{p+1,1} = C^{p+1,1}$ such that

$$\mathscr{L}_{X^*_k}\omega_j \underset{j-p-2}{\equiv} -[X_k, \omega_{j-k}] + f^{p+1}(\omega_{j-p-1}), \qquad j \leqq p.$$

Then we can show that

$$\mathscr{L}_{X^*_k}K^p = (K^{p-k})^{X_k} + \partial f^{p+1}.$$

Since $\omega^{(p+1)}$ is normal, and since the map $\partial: C^{p+1,1} \to C^{p,2}$ is injective, it follows that $f^{p+1}=0$, proving the lemma.

Let us now prove Lemma 4.16, which is carried out by induction on the integer $p \geqq \mu$ (cf. the proof of Lemma 3.11). As we have remarked before, (P, η) admits a normal μ-system. Thus we assume that for some $p \geqq \mu$ (P, η) admits a normal p-system, say $\omega^{(p)} = \{\omega_j\}_{j \leqq p-1}$.

We take a connection α (in the usual sense) in P, and denote by ω_p the $\mathfrak{g}_p$-component of ω with respect to the decomposition $\mathfrak{g}' = \sum_{j \geqq 0} \mathfrak{g}_j$. Then we have

$$\omega_p(X^*_r) = \delta_{rp} X_r, \qquad X_r \in \mathfrak{g}_r, \qquad 0 \leqq r \leqq \mu,$$

$$R^*_a \omega_p = \mathrm{Ad}(a^{-1})\, \omega_p, \qquad a \in G_0,$$

indicating that the system $\omega^{(p+1)} = \{\omega_j\}_{j \leqq p}$ formed by $\{\omega_j\}_{j \leqq p-1}$ and ω_p gives a pre-$(p+1)$-system in (P, η). We shall modify $\omega^{(p+1)}$ to obtain a normal pre-$(p+1)$-system.

The space $C^{p,2}$ is decomposed as follows:

$$C^{p,2} = Z_{*}^{p,2} + \partial C^{p+1,1}.$$

Let $\{K^l\}_{l \leqq p}$ be the curvature of $\omega^{(p+1)}$. K^p taking values in $C^{p,2}$, we denote by L^p the $\partial C^{p+1,1}$-component of K^p. Since $C_{+}^{p+1,1} = C^{p+1,1} = Z_{*}^{p+1,1} + \partial \mathfrak{g}_{p+1} = Z_{*}^{p+1,1}$, and since $H^{p+1,1}(\mathfrak{G}) = 0$, we see that there is a unique function $f^{p+1}: P \to C_{+}^{p+1,1}$ such that $L^p = -\partial f^{p+1}$. Since $R^*_a K^p = (K^p)^a$, $a \in G_0$, it follows from Lemma 1.11 that

$$R^*_a f^{p+1} = (f^{p+1})^a, \qquad a \in G_0.$$

Using the function f^{p+1}, we now modify $\omega^{(p+1)}$ as follows:

$$\omega'_j = \omega_j + f^{p+1}(\omega_{j-p-1}), \qquad j \leqq p.$$

Let $\omega'^{(p+1)} = \{\omega'_j\}_{j \leqq p}$. Then in the same manner as in the proof of Lemma 3.11, we see that $\omega'^{(p+1)}$ is a normal pre-$(p+1)$-system in (P, η). By Lemma 4.18 $\omega'^{(p+1)}$ is a normal $(p+1)$-system in (P, η), thus completing the proof of Lemma 4.16.

4.7. REMARK. Let $(P^{\#}, \xi)$ (resp. $(P'^{\#}, \xi')$) be a $G^{\#}_0$-structure of type $\mathfrak{M}$ on a manifold M (resp. on M'), and (P, ω) (resp. (P', ω')) the normal connection of type $\mathfrak{G}$ on M (resp. on M') associated with $(P^{\#}, \xi)$ (resp. with $(P'^{\#}, \xi')$). Then we know that every isomorphism $\varphi: (P, \omega) \to (P', \omega')$ induces an isomorphism $\varphi^{\#}: (P^{\#}, \xi) \to (P'^{\#}, \xi')$ in a natural manner (see 2.3). Conversely we remark that every isomorphism $\varphi^{\#}: (P^{\#}, \xi) \to (P'^{\#}, \xi')$ induces an isomorphism $\varphi: (P, \omega) \to (P', \omega')$ in a natural manner.

Although this fact follows from (2) of Theorem 2.7 which will be proved in the next section, we shall give a direct proof of it from now on. By Theorem 3.7 $(P^{\#}, \xi)$ (resp. $(P'^{\#}, \xi')$) is reduced to a unique $\tilde{G}$-structure of type $(\mathfrak{M}, \mu-1)$, $(\tilde{P}, \tilde{\xi})$ (resp. $(\tilde{P}', \tilde{\xi}')$), on M (resp. on M') in a natural manner. Clearly the image $\varphi^{\#}(\tilde{P})$ of $\tilde{P}$ by $\varphi^{\#}$ defines a $\tilde{G}$-structure of type $(\mathfrak{M}, \mu-1)$, from which follows that $\varphi^{\#}(\tilde{P}) = \tilde{P}'$, and hence $\varphi^{\#}$ induces an isomorphism $\tilde{\varphi}: (\tilde{P}, \tilde{\xi}) \to (\tilde{P}', \tilde{\xi}')$. Let Δ (resp. Δ') be the set of all normal pre-μ-systems in $(\tilde{P}, \tilde{\xi})$ (resp. in $(\tilde{P}', \tilde{\xi}')$). For every $\alpha' = \{\alpha'_j\}_{j \leqq \mu-1} \in \Delta'$ let $\tilde{\varphi}^* \alpha'$ denote the system $\{\tilde{\varphi}^* \alpha'_j\}_{j \leqq \mu-1}$. Then it is clear that $\tilde{\varphi}^* \alpha' \in \Delta$, the assignment $\alpha' \to \tilde{\varphi}^* \alpha'$ gives an injective map of Δ' onto Δ, and

$$f_{\alpha\beta} = \tilde{\varphi}^* f'_{\alpha'\beta'}, \qquad \alpha', \beta' \in \Delta',$$

where $\alpha = \tilde{\varphi}^* \alpha'$ and $\beta = \tilde{\varphi}^* \beta'$. Hence $\tilde{\varphi}$ gives rise to an isomorphism $\varphi: P \to P'$ as G''-bundles in a natural manner. Moreover we clearly have

$$g^{\alpha}_{\rho(X)} = \tilde{\varphi}^* g'^{\alpha'}_{\rho(X)}, \qquad \alpha' \in \Delta', \qquad X \in \mathfrak{p}.$$

Hence it follows that φ gives an isomorphism $P \to P'$ as G'-bundles, and $\overline{\tilde{\varphi}^* \alpha'} = \varphi^* \bar{\alpha}'$. Consequently we see that $\varphi^* \omega'$ gives a normal connection of type $\mathfrak{G}$ in P compatible with η, and hence $\varphi^* \omega' = \omega$ by Theorem 4.15, proving our remark.

§ 5. The uniqueness of normal connections of type $\mathfrak{G}$

In this section we shall prove (2) of Theorem 2.7.

5.1. The $\tilde{G}$-structures corresponding to normal connections of type $\mathfrak{G}$.

LEMMA 5.1. *Let (P, ω) be a normal connection of type $\mathfrak{G}$ on a manifold M, and $(\tilde{P}, \tilde{\xi})$ the corresponding $\tilde{G}$-structure. Then $(\tilde{P}, \tilde{\xi})$ is of type $(\mathfrak{M}, \mu-1)$.*

Consider the kernel $G'' = \exp \mathfrak{g}_\mu$ of the homomorphism $\rho: G' \to \tilde{G}$. Then P is a principal fibre bundle over the base space $\tilde{P}$ with structure group G''. The proof of Lemma 5.1 is preceded by the following

LEMMA 5.2. *There is a cross section $\psi: \tilde{P} \to P$ having the following properties:*

1) $\psi(z\cdot a)=\psi(z)\cdot a,\ z\in\tilde{P},\ a\in G_0$.

2) $\psi(z\cdot\rho(a))\equiv\psi(z)\cdot a \pmod{G''},\ z\in\tilde{P},\ a\in G'$.

PROOF. Let Q be a reduction of P to G_0. (Such a reduction necessarily exists, because the homogeneous space G'/G_0 is diffeomorphic with the space $\sum_{j=1}^{\mu}\mathfrak{g}_j$ (Lemma 1.7).) Let ρ be the projection $P\to\tilde{P}$. Clearly ρ maps Q injectively into $\tilde{P}$, and hence the image $\rho(Q)$ of Q by ρ gives a G_0-subbundle of $\tilde{P}$. If we put

$$\tilde{N}=\exp\mathfrak{g}_1\cdots\exp\mathfrak{g}_{\mu-1},$$

we see from Lemma 1.7 that every element z of $\tilde{P}$ can be written uniquely in the form:

$$z=\rho(y\cdot b),$$

where $y\in Q$ and $b\in\tilde{N}$. Now define a map $\psi:\tilde{P}\to P$ by

$$\psi(z)=y\cdot b.$$

Then we can easily verify that ψ has the required properties.

Proof of Lemma 5.1. Let ψ be a cross section $\tilde{P}\to P$ having the properties in Lemma 5.2. For any $j\leqq\mu-1$ we put $\theta_j=\psi^*\omega_j$. We shall show that the system $\theta^{(\mu)}=\{\theta_j\}_{j\leqq\mu-1}$ is a normal pre-μ-system in $(\tilde{P},\xi)$, which implies that $(\tilde{P},\xi)$ is of type $(\mathfrak{M},\mu-1)$. First of all we have $\psi^*\omega_-=\xi$, because $\omega_-=\rho^*\xi$ and $\rho\circ\psi=1$. Hence $\theta^{(\mu)}$ is compatible with the basic form ξ.

Let $0\leqq r\leqq\mu-1$, and $X_r\in\mathfrak{g}_r$. From property 2) for ψ we easily see that

$$\psi_*\left(\rho(X_r)^*{}_z\right)=(X_r)^*_{\psi(z)}+Y^*_{\psi(z)},\qquad z\in\tilde{P},$$

where Y is a suitable element of $\mathfrak{g}_\mu$ depending on X_r and z. Since $\omega_j(X^*{}_r)=\delta_{jr}X_r$, it follows that

$$\theta_j\left(\rho(X_r)^*\right)=\delta_{jr}X_r,\qquad X_r\in\mathfrak{g}_r,\qquad 0\leqq r\leqq\mu-1. \tag{5.1}$$

By property 1) for ψ we have $\psi\circ R_a=R_a\circ\psi$, $a\in G_0$. We have $R^*{}_a\omega_j=\mathrm{Ad}\,(a^{-1})\,\omega_j$, $a\in G_0$. Hence we obtain

$$R^*{}_a\theta_j=\mathrm{Ad}\,(a^{-1})\,\theta_j,\qquad a\in G_0. \tag{5.2}$$

Let K be the curvature of ω. By Lemma 2.5 we have

$$\Omega''_j\underset{j-\mu}{\equiv}\frac{1}{2}\sum_{l\leqq\mu-1}K^l_j(\omega_-\wedge\omega_-),\qquad j\leqq\mu-2,$$

whence

(5.3) $$\Theta^{\mu}_{j} \underset{j-\mu}{\equiv} \frac{1}{2} \sum_{l \leqq \mu-1} (\psi^{*} K^{l}_{j}) (\theta_{-} \wedge \theta_{-}) , \qquad j \leqq \mu-2 .$$

Let $0 \leqq r \leqq \mu-1$, and $X_r \in \mathfrak{g}_r$. Let $j \leqq \mu-2$. Then using (5.1), we have

$$\mathscr{L}_{\rho(X_r)^*} \theta_j = \rho(X_r)^* \lrcorner\, d\theta_j = \rho(X_r)^* \lrcorner\, \Theta^{\mu}_{j} - [X_r, \theta_{j-r}] ,$$

and from (5.3) we obtain

$$\rho(X_r)^* \lrcorner\, \Theta^{\mu}_{j} \underset{j-\mu}{\equiv} 0 .$$

Therefore it follows that

(5.4) $$\mathscr{L}_{\rho(X_r)^*} \theta_j \underset{j-\mu}{\equiv} -[X_r, \theta_{j-r}] , \qquad X_r \in g_r , \qquad 0 \leqq r \leqq \mu-1 .$$

Now we see from (5.1), (5.2) and (5.4) that $\theta^{(\mu)}$ gives a pre-μ-system in $(\tilde{P}, \tilde{\xi})$. Thus equations (5.3) together form the structure equation for $\theta^{(\mu)}$. Since ω is normal, it is clear that $\theta^{(\mu)}$ is normal, proving our assertion.

5.2. Some lemmas on normal connections of type $\mathfrak{G}$. Let P be a principal fibre bundle over a base manifold M with structure group G', where $\dim M = \dim \mathfrak{m}$. Suppose that there are given two normal connections of type $\mathfrak{G}$, ω and ω', in P such that

$$\omega'_{-} = \omega_{-} .$$

We remark that for each $p \geqq \mu$ the system $\omega^{(p)} = \{\omega_j\}_{j \leqq p-1}$ satisfies all the conditions for normal p-systems stated in 4.5 (except that $\omega^{(p)}$ is compatible with η). The same remark holds for the system $\omega'^{(p)} = \{\omega'_j\}_{j \leqq p-1}$. Therefore we have the following two lemmas (cf. Lemmas 3.9, and 4.17).

LEMMA 5.3. *There is a unique function $g_\mu : P \to \mathfrak{g}_\mu$ such that*

$$\omega'_j \underset{j-\mu-1}{\equiv} \omega_j + [g_\mu, \omega_{j-\mu}] , \qquad j \leqq \mu-1 .$$

LEMMA 5.4. *Assume that the function g_μ vanishes. Then for each $p \geqq \mu$ we have*

$$\omega'_j \underset{j-p-1}{\equiv} \omega_j , \qquad j \leqq p-1 .$$

Hence the two connections ω and ω' coincide.

LEMMA 5.5. *The function g_μ in Lemma 5.3 satisfies the equality*

$$R^*_a g_\mu = \mathrm{Ad}\,(a^{-1})\, g_\mu , \qquad a \in G' .$$

PROOF. By Lemma 1.7 it suffices to prove that

$$R^*_a g_\mu = \mathrm{Ad}\,(a^{-1})\, g_\mu , \qquad a \in G_0$$

and

$$\mathscr{L}_{X_r^*} g_\mu = 0\,, \qquad X_r \in \mathfrak{g}_r\,, \qquad 1 \leqq r \leqq \mu\,,$$

which can be obtained in a similar way to the proof of Lemma 4. 3.

5. 3. The uniqueness of normal connections of type $\mathfrak{G}$. Let (P, ω) (resp. (P', ω')) be a normal connection of type $\mathfrak{G}$ on a manifold M (resp. on M'), $(\tilde{P}, \tilde{\xi})$ (resp. $(\tilde{P}', \tilde{\xi}')$) the corresponding $\tilde{G}$-structure, and $(P^\sharp, \xi)$ (resp. $(P'^\sharp, \xi')$) the corresponding $G^\sharp{}_0$-structure of type $\mathfrak{M}$. Suppose that there is given an isomorphism $\varphi^\sharp: (P^\sharp, \xi) \to (P'^\sharp, \xi')$.

By Lemma 5. 1 we know that both $(\tilde{P}, \tilde{\xi})$ and $(\tilde{P}', \tilde{\xi}')$ are of type $(\mathfrak{M}, \mu-1)$. Therefore we have $\varphi^\sharp(\tilde{P}) = \tilde{P}'$, as we have already seen in 4. 7, and hence $\varphi^\sharp$ induces an isomorphism $\tilde{\varphi}: (\tilde{P}, \tilde{\xi}) \to (\tilde{P}', \tilde{\xi}')$.

Let ρ (resp. ρ') denote the projection $P \to \tilde{P}$ (resp. $P' \to \tilde{P}'$). It is easy to see that there is a bundle isomorphism $\varphi: P \to P'$ which induces $\tilde{\varphi}$, *i. e.*, $\rho' \circ \varphi = \tilde{\varphi} \circ \varphi$. (This follows from the fact that there are reductions Q and Q' of P and P' respectively to G_0 such that $\tilde{\varphi}(\rho(Q)) = \rho'(Q')$ (cf. the proof of Lemma 5. 2). Clearly $\varphi^* \omega'$ is a normal connection of type $\mathfrak{G}$ in P. Since $\omega_- = \rho^* \tilde{\xi}$, $\omega'_- = \rho'^* \tilde{\xi}'$, $\tilde{\varphi}^* \tilde{\xi}' = \tilde{\xi}$, and $\rho' \circ \varphi = \tilde{\varphi} \circ \rho$, we obtain

$$\varphi^* \omega'_- = \omega_-\,.$$

Thus we may apply the arguments in 5. 2 to the two connections ω and $\varphi^* \omega'$.

LEMMA 5. 6. *There is a unique bundle isomorphism $\varphi: P \to P'$ which satisfies the following conditions*:

1) φ *induces* $\tilde{\varphi}$.

2) $\varphi^* \omega'_j \underset{j-\mu-1}{\equiv} \omega_j\,, \qquad j \leqq \mu - 1\,.$

By Lemma 5. 4 the second condition means that $\varphi^* \omega' = \omega$ or in other words, φ gives an isomorphism $(P, \omega) \to (P', \omega')$. Thus (2) of Theorem 2. 7 follows from Lemma 5. 6.

Proof of Lemma 5. 6. We first prove the existence. Let us consider a bundle isomorphism $\varphi: P \to P'$ which induces $\tilde{\varphi}$. By Lemma 5. 3 there is a unique function $g_\mu: P \to \mathfrak{g}_\mu$ such that

$$\varphi^* \omega'_j \underset{j-\mu-1}{\equiv} \omega_j + [g_\mu, \omega_{j-\mu}]\,, \qquad j \leqq \mu - 1\,, \tag{5. 5}$$

and by Lemma 5. 5 g_μ satisfies the following equality

$$g_\mu(z \cdot a) = \mathrm{Ad}\,(a^{-1})\, g_\mu(z)\,, \qquad z \in P\,, \qquad a \in G'\,.$$

Putting $\sigma(z) = \exp g_\mu(z)$, we define a map $\varphi_1: P \to P'$ by

$$\varphi_1(z) = \varphi(z) \cdot \sigma(z)\,, \qquad z \in P\,.$$

Since $\sigma(z\cdot a)=a^{-1}\sigma(z)\,a$, $z\in P$, $a\in G'$, we see that φ_1 gives a bundle isomorphism $P\to P'$. Furthermore it is clear that φ_1 induces $\tilde{\varphi}$. Let $z\in P$ and $X\in T(P)_z$. Then we have

$$\varphi_{1*}X=(R_{\sigma(z)})_*\varphi_*X+Y^*_{\varphi_1(z)}\,,$$

where Y is a suitable element of $\mathfrak{g}_\mu$ depending on z and X. Let $j\leqq\mu-1$. By condition (*C*. 3) for ω' we have

$$R^*_{\sigma(z)}\omega'_j=\omega'_j-\left[g_\mu(z),\omega'_{j-\mu}\right],$$

and hence

$$\begin{aligned}(\varphi^*{}_1\omega'_j)(X)&=(R^*_{\sigma(z)}\omega'_j)(\varphi_*X)\\&=(\varphi^*\omega'_j)(X)-\left[g_\mu(z),(\varphi^*\omega'_{j-\mu})(X)\right].\end{aligned}$$

Therefore we obtain

$$\varphi^*{}_1\omega'_j=\varphi^*\omega'_j-[g_\mu,\varphi^*\omega'_{j-\mu}]\,.$$

This equality together with (5. 5) gives

$$\varphi^*{}_1\omega'_j\underset{j-\mu-1}{\equiv}\omega_j\,,\qquad j\leqq\mu-1\,,$$

which proves the existence.

Let us now prove the uniqueness. Let φ_2 be any bundle isomorphism $P\to P'$ which satisfies the conditions in Lemma 5. 6. Since $\rho'(\varphi_2(z))=\tilde{\varphi}(\rho(z))=\rho'(\varphi_1(z))$, $z\in P$, we see that there is a unique function $g_\mu\colon P\to\mathfrak{g}_\mu$ such that $\varphi_2(z)=\varphi_1(z)\cdot\sigma(z)$, $z\in P$, where $\sigma(z)=\exp g_\mu(z)$. Then we see from the discussions above that

$$\varphi^*{}_2\omega'_j=\varphi^*{}_1\omega'_j-[g_\mu,\varphi^*{}_1\omega'_{j-\mu}]\,,\qquad j\leqq\mu-1\,.$$

Since

$$\varphi^*{}_1\omega'_j\underset{j-\mu-1}{\equiv}\omega_j\underset{j-\mu-1}{\equiv}\varphi^*{}_2\omega'_j\,,\qquad j\leqq\mu-1\,,$$

it follows that $[g_\mu,\omega_{j-\mu}]\underset{j-\mu-1}{\equiv}0$, $j\leqq\mu-1$. This means that $g_\mu=0$ and hence $\varphi_1=\varphi_2$, proving the uniqueness.

We have thereby completed the proof of Lemma 5. 6.

Bibliography

[1] É. Cartan: Les sous-groupes des groupes continus de transformations, Ann. Sci. École Norm. Sup. 25 (1908), 57–194.

[2] É. Cartan: Les systèmes de Pfaff à cinq variables et les équations aux derivées partielles du second ordre, Ann. Sci. École Norm. Sup. 27 (1910), 109–192.

[3] S. KOBAYASHI and K. NOMIZU: Foundations of Differential Geometry I, Interscience Publishers, 1963.

[4] B. KOSTANT: Lie algebra cohomology and the generalized Borel-Weil theorem, Ann. of Math. 74 (1961), 329–387.

[5] S. STERNBERG: Lectures on Differential Geometry, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[6] N. TANAKA: On the equivalence problems associated with a certain class of homogeneous spaces, J. Math. Soc. Japan 17 (1965), 103–139.

[7] N. TANAKA: On differential systems, graded Lie algebras and pseudo-groups, J. Math. Kyoto Univ. 10 (1970), 1–82.

[8] N. TANAKA: On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japan. J. Math. 2 (1976), 131–190.

[9] A. TRESSE: Détermination des invariants ponctuels de l'équation différetielle ordinaire du second ordre $y''=\omega(x, y, y')$, Preischr. Fürstlich Jablon. Ges., Leipzig, Hirzel, 1896.

[10] Seminaire Sophus Lie, École Norm. Sup., 1954–1955.

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