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Cauchy problems for the operator of Tricomi's type

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§ 1. Introduction.

In this paper we shall consider the well-posedness of the Cauchy problem for the weakly hyperbolic partial differential operator of the second order such that:

$$P(x, D)=D_{x_0}^{2}-x_0 A(x, x_{x'})+P_1(x, D_x), \tag{1.1}$$

where $D_x=\left(\frac{1}{i}\frac{\partial}{\partial x_0}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_n}\right)$, $A(x, D_{x'})$ is an elliptic operator whose principal symbol is positive and P_1 is an arbitrary first order term. We treat $P(x, D)$ in the closed half space $\overline{R_+^{n+1}}=\{x; x=(x_0, x'), x'=(x_1, \cdots, x_n), x_0\geqslant 0\}$ and assume that the coefficients of $P(x, D)$ are constant for large $|x'|$. Note that the principal symbol $P_2(x, \xi)$ of P has no critical point with respect to (x, ξ) i.e., $grad_{(x,\xi)} P_2(x^0, \xi^0)\neq 0$ for any $(x^0, \xi^0), \xi^0\neq 0$.

Oleinik proved the well-posedness of (1.1) by the energy estimate ([11]). More generally, Ivrii proved that if the principal symbol of the weakly hyperbolic operator has no critical point, then the Cauchy problem is well-posed for an aribitrary lower order term and called these operators completely regularly hyperbolic ([5], [6]). He also proved the above fact by the energy estimate. However, in this paper, using Airy function we shall construct the fundamental solution of (1.1) and give somewhat sharp results.

Airy function was used in the construction of the parameterix for the exterior mixed problem for hyperbolic equations at a diffractive point ([2], [4], [8], [13]). The situation is similar if the boundary surface is replaced by the initial one, because the bicharacteristic strip $(x(t), \xi(t))$ of P_2 is tangent to the hyper-surface $x_0=0$ in $\overline{R_{+,x}^{n+1}}\times(R_{\xi}^{n+1}\backslash 0)$ of order one.

Let us formulate our problem more precisely as follows: for given data $f(x)$, $v_0(x')$ and $v_1(x')$, we shall consider the Cauchy problem with initial surface $x_0=0$ such that;

$$\begin{cases} P(x, D)\, u=f(x) & \text{in } [0, T]\times R^n, \\ D_{x_0}^{j} u|_{x_0=0}=v_j(x') & \text{in } R^n (j=0, 1), \end{cases} \tag{1.2}$$

where T is some positive number.

Now the purpose of the present paper is to construct the following parameterices of (1. 2).

THEOREM 1. 1. *There exist operators* $G^k(x_0, t)$, $k=0, 1$, *with parameters* $0\leqslant t\leqslant x_0\leqslant T$ *such that*

$$\left(G^k(x_0, t)\, V\right)(x')\in C^\infty\left([0, T]\times[0, T]\times R^n\right) \quad \textit{for} \quad V\in C_0^\infty(R^n)\,,$$

$$\left(G^k(x_0, t)\, V\right)(x')\in C^\infty\left([0, T]\times[0, T]\,;\ \mathscr{D}'(R^n)\right) \quad \textit{for} \quad V\in \mathcal{E}'(R^n)$$

and

$$(1.3)\qquad \begin{cases} P(x, D)\left(G^k(x_0, t)\, V\right) = R^k(x_0, t)\, V\,, \\ G^k(x_0, t)\, V\Big|_{x_0=t} = \left(\delta_0^k\, I + R_0^k(t)\right) V\,, \\ D_{x_0} G^k(x_0, t)\, V\Big|_{x_0=t} = \left(\delta_1^k\, I + R_1^k(t)\right) V\,, \end{cases}$$

where δ_j^k *is the Kornecker delta,* $R^k(x_0, t)$ *and* $R_j^k(t)$ *are operators with* C^∞-*kernels of* (x', y') *depending smoothly on* (x_0, t) *and* t *respectively.*

From the properties of these parametrices we can directly show the propergation of singularities along null bicharacteristic strips of P_2 (see Lemma 3. 3) and prove the next theorem.

We denote by $\mathscr{H}_s([0, T]\times R^n)$ the function space:

$$\mathscr{H}_s\left([0, T]\times R^n\right) = \left\{u\,;\ D_{x_0}^k u(x_0, \cdot)\in L^2\left([0, T]\,;\ H_{s-k}(R^n)\right),\ k\leqslant s\right\},$$

where $H_s(R^n)$ is the Sobolev space. Then we obtain the following

THEOREM 1. 2. *Let* $f\in\mathscr{H}_s([0, T]\times R^n)$, $v_0\in H_{s+1}(R^n)$ *and* $v_1\in H_{s+\frac{1}{3}}(R^n)$ *such that for some compact set* K *in* R^n, supp $f\subset[0, T]\times K$, supp $v_0\cup$ supp $v_1\subset K$. *Then there exists a unique solution* $u\in\mathscr{H}_{s+1}([0, T]\times R^n)$ *of* (1. 2) *which satisfies the following estimate*:

$$(1.4)\qquad \begin{cases} \sum_{k\leqslant s+1}\left\|D_{x_0}^k u(t, \cdot)\right\|_{s+1-k} \leqslant C\left\|v_0\right\|_{s+1} + C\left\|v_1\right\|_{s+\frac{1}{3}} \\ \qquad + C\sum_{k\leqslant s-1}\left\|D_{x_0}^k f(t, \cdot)\right\|_{s-k} + C\sum_{k\leqslant s}\int_0^t\left\|D_{x_0}^k f(\tau, \cdot)\right\|_{s-k}\tau^{-\frac{1}{2}}d\tau\,, \end{cases}$$

where s *is an arbitrary positive number and* $\|\cdot\|_s$ *is the norm of* $H_s(R^n)$ *and* C *depends only on* K *and* s.

In Theorem 1 of [5], Ivrii obtained the following estimate

$$\sum_{k\leqslant s+1}\left\|D_{x_0}^k u(t, \cdot)\right\|_{s+1-k} \leqslant C\sum_{k\leqslant s+1}\left\|D_{x_0}^k u(t, \cdot)\right\|_{s+2-k}$$
$$+ C\sum_{k\leqslant s}\left\{\int_0^t\left\|D_{x_0}^k Pu(\tau, \cdot)\right\|_{s-k}\tau^{\frac{\kappa}{2}-1}d\tau\right\}^{\frac{1}{\kappa}},$$

where $u\in\mathscr{H}_{s+1}([0, T]\times R^{n})$ and κ is any number such that $1\leqslant\kappa\leqslant 2$. The estimate (1.4) corresponds to the case $\kappa=1$, but in his estimate the initial data are taken by $v_j\in H_{s+2-j}$ $(j=0, 1)$. Furthermore our consideration used here will be also applicable to non-hyperbolic boundary-value problems (see Melrose [9] p. 7, Osher [12]. p. 504).

§ 2. Phase functions and transport equations.

Let the principal symbol P_2 of P be $\xi_0^2-x_0A(x, \xi')$, and consider the eikonal equation $P_2(x, \varphi_x)=0$. To solve it we must notice that on the surface $x_0=0$ the characteristic equation $P_2(x, \xi_0, \xi')=0$ in ξ_0 has double roots, and the bicharacteristic curve $x(t)$ of Hamilton-Jacobi equations of P_2 starting from the initial surface has a singularity of cusp type there. Thus we can not apply the usual method for strictly hyperbolic operators to this case. However for the bicharacteristic strip $(x(t), \xi(t))$ of P_2 there is no singular point, because P_2 dose not have critical points and it is tangent to the hyper-surface $x_0=0$ in $\overline{R^{n+1}_{+,x}}\times(R^{n+1}_{\xi}\backslash 0)$ of the first order as mentioned in § 1.

Noting these facts we have the next

Lemma 2.1. *There exist real functions* $\theta(x, \xi')$, $\rho(x, \xi')\in C^{\infty}$ $([0, T]\times R^{n}_{x'}\times R^{n}_{\xi'}\backslash 0)$ *of homogeneous in* ξ' *of order* 1 *and* $\frac{2}{3}$ *respectively such that* $\varphi_{\pm}(x, \xi')=\theta(x, \xi')\pm\frac{2}{3}\rho^{\frac{3}{2}}(x, \xi')$ *is a solution of the eikonal equation*

$$(2.1)\qquad P_2(x, \varphi_{\pm_x})=0\,,$$

$$(2.2)\qquad \theta(x, \xi')=\langle x', \xi'\rangle+0(x_0^2)$$

and

$$(2.3)\qquad \rho(0, x', \xi')=0\,,\qquad \rho_{x_0}(0, x', \xi')\gneqq 0\,.$$

Proof. Let $P_2=(\xi_0-x_0^{\frac{1}{2}}a(x, \xi'))(\xi_0+x_0^{\frac{1}{2}}a(x, \xi'))$ where $a=\sqrt{A}$, and consider the Hamilton-Jacobi equations relative to $\pm x_0^{\frac{1}{2}}a(x, \xi')$ such that

$$(2.4)_{\pm}\qquad \begin{cases} \dfrac{dx'^{\pm}}{dx_0}=\pm x_0^{\frac{1}{2}}a_{\xi'}(x_0, x'^{\pm}, p'^{\pm})\,, \\ \dfrac{dp'^{\pm}}{dx_0}=\mp x_0^{\frac{1}{2}}a_{x'}(x_0, x'^{\pm}, p'^{\pm}) \qquad \text{and}\quad (x'^{\pm}, p'^{\pm})\big|_{x=0}=(y', \xi')\,. \end{cases}$$

By the change of the variable $x_0=s^2$, $(2.4)_{\pm}$ can be rewritten in the form

$$(2.5)_{\pm}\qquad \begin{cases} \dfrac{dx'^{\pm}}{ds}=\pm 2s^2a_{\xi'}(s^2, x'^{\pm}, p'^{\pm})\,, \\ \dfrac{dp'^{\pm}}{ds}=\mp 2s^2a_{x'}(s^2, x'^{\pm}, p'^{\pm}) \qquad \text{and}\quad (x'^{\pm}, p'^{\pm})\big|_{s=0}=(y', \xi')\,. \end{cases}$$

Now for $(x'^+(s), p'^+(s))$ the solution of $(2.5)_+$, let $(\tilde{x}'(s), \tilde{p}'(s))=(x'^+(-s), p'^+(-s))$. Then $(\tilde{x}', \tilde{p}')$ becomes a solution of $(2.5)_-$ and hence we solve only $(2.5)_+$.

Let $x'^+(s\,;\,y', \xi')$ be the solution of $(2.5)_+$ and $y'^+(s\,;\,x', \xi')$ its inverse mapping. Then $dy'^+/ds(s\,;\,x', \xi')=0(s^2)$. In fact if we differentiate $y'^+(s\,;\,x'^+(s), \xi')=\text{const.}$ with respect to s, we obtain that

$$0=\frac{d}{ds}y'^+\left(s\,;\,x'^+(s), \xi'\right)=\frac{\partial y'^+}{\partial x'}\frac{dx'^+}{ds}+\frac{dy'^+}{ds}(s\,;\,x'^+, \xi')\,.$$

Hence it follows from $(2.5)_+$

$$\frac{dy'^+}{ds}=-\frac{\partial y'^+}{\partial x'}\frac{dx'^+}{ds}=0(s^2)\,.$$

By Taylor expansion it holds

$$\begin{aligned} y'(s\,;\,x', \xi') &= x'+s^3\alpha(s, x', \xi') \\ &= x'+s^3\left(\alpha_1(s^2, x', \xi')+s\alpha_2(s^2, x', \xi')\right) \\ &= x'+s^4\alpha_2(s^2, x', \xi')+s^3\alpha_1(s^2, x', \xi')\,. \end{aligned}$$

Since the solution of (2.1) is obtained by taking $\varphi_+(x, \xi')=\langle y'^+(x_0, x', \xi'), \xi'\rangle$, we have merely to put

$$\theta(x, \xi')=\left\langle x'+x_0^2\alpha_2(x_0, x', \xi'), \xi'\right\rangle=\langle x', \xi'\rangle+x_0^2\langle\alpha_2, \xi'\rangle\,,$$

$$\rho(x, \xi')=\left(\frac{2}{3}\right)^{\frac{2}{3}}x_0\left(\left\langle\alpha_1(x_0, x', \xi', \xi'), \xi'\right\rangle\right)^{\frac{2}{3}}\,.$$

Thus setting $\varphi_+=\theta+\dfrac{2}{3}\rho^{\frac{3}{2}}$ we have $\varphi_{+x_0}-x_0^{\frac{1}{2}}a(x, \varphi_{+x'})=0$.

To prove (2.3), rewrite the eikonal equation as follows:

$$\theta_{x_0}+\frac{2}{3}x_0^{\frac{1}{2}}\langle\alpha_1, \xi'\rangle+x_0^{\frac{3}{2}}\langle\alpha_{1x_0}, \xi'\rangle=x_0^{\frac{1}{2}}a(x, \varphi_{+x'})\,.$$

Now multiply $x_0^{-\frac{1}{2}}$ both side of this equation and tend x_0 to zero, then we obtain

$$\frac{2}{3}\langle\alpha_1, \xi'\rangle\Big|_{x_0=0}=a(0, x', \xi')\gneqq 0\,.$$

This proves the lemma.

For given θ, ρ we shall construct amplitude functions g, h as follows:

$$g(x, \xi')\sim\sum_{j=0}^{\infty}g_{-j}(x, \xi')\,, \qquad \operatorname*{ord}_{\xi'}g_{-j}=-j\,, \quad g|_{x_0=0}=1\,,$$

$$h(x, \xi')\sim\sum_{j=0}^{\infty}h_{-j}(x, \xi')\,, \qquad \operatorname*{ord}_{\xi'}h_{-j}=-\frac{1}{3}-j$$

and

$$P(x, D)\int_{C_\pm}\left(g(x, \xi')-\tau h(x, \xi')\right) e^{i\left(\frac{\tau^3}{3}-\tau\rho+\theta\right)}\, d\tau = O(|\xi'|^{-\infty}), \tag{2.6}$$

where $C_\pm$ are complex contours such that;

$$C_\pm = \begin{cases} |t| e^{\left(\frac{\pi}{2}\pm\frac{\pi}{3}\right)i} & \text{for } t\to\pm\infty, \\ |t| e^{-\frac{\pi}{2}i} & \text{for } t\to\mp\infty. \end{cases}$$

We remark that g and h will be taken independent of $C_\pm$.

Let $A_\pm(x)$ be the integrals

$$A_\pm(x) = \int_{C_\pm} \exp\left(i\left(\frac{\tau^3}{3}-\tau x\right)\right) d\tau. \tag{2.7}$$

Then for the usual Airy function $Ai(x)$, it holds

$$A_\pm(x) = 2\pi e^{\pm\frac{2}{3}\pi i} Ai\left(e^{\pm\frac{2}{3}\pi i}(-x)\right) \tag{2.8}$$

and the integrals of the left hand side (2. 6) can be rewritten in the form

$$\left(g(x, \xi')\, A_\pm\left(\rho(x, \xi')\right) - ih(x, \xi')\, A'_\pm\left(\rho(x, \xi')\right)\right) e^{i\theta}, \tag{2.9}$$

where $A'_\pm(x)$ is the derivative of $A_\pm(x)$ ([10]).

The constructions of g and h are analogous to Eskin's ([2]), but for the completeness of the description we shall give them in the following (see also § 3, § 4 of [2]).

By taking $a(x, \xi', \tau) = g(x, \xi') - \tau h(x, \xi')$ the equation (2. 6) follows from the equation:

$$e^{-i\left(\frac{\tau^3}{3}-\tau\rho+\theta\right)} P(x, D)\left\{a(x, \xi', \tau)\, e^{i\left(\frac{\tau^3}{3}-\tau\rho+\theta\right)}\right\} = O(|\xi'|^{-\infty}). \tag{2.10}$$

Let $\phi(x, \xi', \tau) = \dfrac{\tau^3}{3} - \tau\rho + \theta$ than we have

$$\begin{aligned} &e^{-i\phi} P(x, D)\left(a(x, \xi', \tau)\, e^{i\phi}\right) \\ &= P_2(x, \phi_x)\, a_0 + \left\{\frac{1}{i}\sum_{k=0}^{n} \frac{\partial P_2}{\partial \xi_k}(x, \phi_x)\frac{\partial a_0}{\partial x_k}\right. \\ &- \frac{1}{2i}\sum_{j,k=0}^{n}\frac{\partial^2 P_2}{\partial\xi_j\partial\xi_k}(x, \phi_x)\, \phi_{x_j x_k} a_0 \\ &\left. + P_1(x, \phi_x)\, a_0 + P_2(x, \phi_x)\, a_{-1}\right\} + \cdots, \end{aligned} \tag{2.11}$$

where $a_{-j} = g_{-j} - \tau h_{-j}$ and $P_1(x, \xi)$ is the principal symbol of $P_1(x, D)$.

Now let $B(x, \xi', \tau)$ be a polynomial in τ, smooth with respect to (x, ξ'),

$\xi' \neq 0$ and satisfy the homogeneity;

$$B(x, k\xi', k^{\frac{1}{3}}\tau) = k^m B(x, \xi', \tau) \qquad \text{for} \quad k>0 .$$

If $B(x, \xi', \pm\sqrt{\rho})=0$ then there exists a polynomial $B_1(x, \xi', \tau)$ such that

$$B(x, \xi', \tau) = i(\tau^2-\rho)\, B_1(x, \xi', \tau) .$$

On the other hand $i(\tau^2-\rho)=\dfrac{\partial}{\partial\tau}(i\phi(x, \xi', \tau))$, so after integration by parts we see

$$\int_{C_\pm} i(\tau^2-\rho)\, B_1 e^{i\phi}\, d\tau = \int_{C_\pm} \frac{\partial B_1}{\partial\tau}\, e^{i\phi}\, d\tau ,$$

where the order of $\dfrac{\partial B_1}{\partial\tau}$ is $m-1$.

Note that $P_2(x, \phi_x)|_{\tau=\mp\sqrt{\rho}}=P_2(x, \varphi_{\pm x})=0$. Hence from the above remarks there exists a function $Q(x, \xi')$ and $P_2(x, \phi_x)=i(\tau^2-\rho)\, Q(x, \xi')$.

Thus the highest order term of (2.11) becomes as follows:

$$(2.12) \qquad \frac{1}{i}\sum_{k=0}^{n}\frac{\partial P_2}{\partial\xi_k}(x, \theta_x-\tau\rho_x)\frac{\partial a_0}{\partial x_k}-\frac{1}{2i}\sum_{j,k=0}^{n}\frac{\partial^2 P_2}{\partial\xi_j\partial\xi_k}(x, \theta_x-\tau\rho_x)(\theta_{x_jx_k}-\tau\rho_{x_jx_k})$$
$$\times a_0+P_1(x, \theta_x-\tau\rho_x)\, a_0-\frac{\partial}{\partial\tau}\Big(Q(x, \xi')\, a_0\Big) .$$

Substituting $\mp\sqrt{\rho}$ for τ in (2.12), we obtain the following transport equation:

$$(2.13) \qquad \frac{1}{i}\sum_{k=0}^{n}\frac{\partial P_2}{\partial\xi_k}(x, \theta_x+\sqrt{\rho}\,\rho_x)\left(\frac{\partial g_0}{\partial x_k}\pm\sqrt{\rho}\,\frac{\partial h_0}{\partial x_k}\right)-\frac{1}{2i}\sum_{j,k=0}^{n}\frac{\partial^2 P_2}{\partial\xi_j\partial\xi_k}(\theta_x+\sqrt{\rho}\,\rho_x)$$
$$\times(\theta_{x_jx_k}\pm\sqrt{\rho}\,\rho_{x_jx_k})\,(g_0+\sqrt{\rho}\,h_0)+P_1(x, \theta_x+\sqrt{\rho}\,\rho_x)\,(g_0\pm\sqrt{\rho}\,h_0)$$
$$+Q(x, \xi')\, h_0 .$$

Now let $a_0^\pm=g_0\pm\sqrt{\rho}\,h_0$ then the equation (2.13) is represented as follows:

$$(2.14)_{\pm 0} \qquad \frac{1}{i}\sum_{k=0}^{n}\frac{\partial P_2}{\partial\xi_k}(x, \theta_x, \pm\sqrt{\rho}\,\rho_x)\frac{\partial a_0^\pm}{\partial x_k}+c(x, \xi', \pm\sqrt{\rho})a_0^\pm=0 ,$$

where

$$c(x, \xi', \pm\sqrt{\rho}) = -\frac{1}{2i}\sum_{j,k=0}^{n}\frac{\partial^2 P_2}{\partial\xi_j\partial\xi_k}(x, \theta_x\pm\sqrt{\rho}\,\rho_x)\,(\theta_{x_jx_k}\pm\sqrt{\rho}\,\rho_{x_jx_k})$$
$$+P_1(x, \theta_x\pm\sqrt{\rho}\,\rho_x) .$$

And here the signs $\pm$ are taken according to $\pm\sqrt{\rho}$ but independant of contours $C_\pm$.

In fact differentiating the both sides of the equation $P_2(x, \theta_x - \tau\rho_x) = i(\tau^2-\rho)\, Q(x, \xi')$ with respect to τ and replacing τ by $\mp\sqrt{\rho}$ we see that

$$Q(x, \xi') = \pm \frac{1}{i} \sum_{k=0}^{n} \frac{\partial P_2}{\partial \xi_k}(x, \theta_x \pm \sqrt{\rho}\, \rho_x) \frac{\rho_{x_k}}{2\sqrt{\rho}}\,.$$

Thus from the equation

$$\frac{\partial a_0^\pm}{\partial x_k} = \frac{\partial g_0}{\partial x_k} \pm \sqrt{\rho}\, \frac{\partial h_0}{\partial x_k} \pm \frac{\rho_{x_k}}{2\sqrt{\rho}} h_0\,,$$

$(2, 14)_{\pm 0}$ follows from (2. 13). By the same arguments we have the following transport equations for $j>0$:

$$(2.\,14)_{\pm j} \qquad \frac{1}{i} \sum_{k=0}^{n} \frac{\partial P_2}{\partial \xi_k}(x, \theta_x \pm \sqrt{\rho}\, \rho_x) \frac{\partial a_{-j}^\pm}{\partial x_k} + c(x, \xi', \pm\sqrt{\rho})\, a_{-j}^\pm = f_{-j}^\pm\,,$$

where $f_{-j}^\pm$ are determined by θ, ρ, $a_{-k}^\pm$ $(0 \leqslant k \leqslant j-1)$.

Now remarking that $\rho_{x_0} \neq 0$ when $x_0=0$, we make a change of variables such that $(x_0, x') \to (\rho, x')$ and find that

$$\sum_{k=0}^{n} \frac{\partial P_2}{\partial \xi_k}(x, \theta_x \pm \sqrt{\rho}\, \rho_x) \frac{\partial a_{-j}^\pm}{\partial x_k} = \sum_{k=1}^{n} \frac{\partial P_2}{\partial \xi_k} \frac{\partial \hat{a}_{-j}^\pm}{\partial x_k} + \sum_{k=0}^{n} \frac{\partial P_2}{\partial \xi_k} \rho_{x_k} \frac{\partial \hat{a}_{-j}^\pm}{\partial \rho}\,.$$

Finally from the facts that

$$\sum_{k=0}^{n} \frac{\partial P_2}{\partial \xi_k}(x, \theta_x)\, \rho_{x_k} = 0$$

and

$$\sum_{j,k=0}^{n} \frac{\partial^2 P_2}{\partial \xi_j \partial \xi_k}(x, \theta_x)\, \rho_{x_j} \rho_{x_k} \neq 0$$

when $x_0=0$, we see that

$$\sum_{k=0}^{n} \frac{\partial P_2}{\partial \xi_k}(x, \theta_x \pm \sqrt{\rho}\, \rho_x)\, \rho_{x_k} = \pm c_0(x, \xi', \pm\sqrt{\rho}) \frac{\sqrt{\rho}}{2}\,, \qquad c_0 \neq 0\,.$$

Hence we can again make a change of variables $x'=x'$, $\rho=t^2$ and for a fixed ξ' we may rewrite $(2.\,14)_{+j}$ in the from:

$$(2.\,15)_{+j} \qquad \frac{1}{i} c_0^+(x', t) \frac{\partial \hat{a}_{-j}^+}{\partial t} + \frac{1}{i} \sum_{k=1}^{n} c_k^+(x', t) \frac{\partial \hat{a}_{-j}^+}{\partial x_k} + c_{n+1}^+(x', t)\, \hat{a}_{-j}^+ = \hat{f}_{-j}^+$$

for $t \geqslant 0$, where $c_k^+(x', t)$ $(k=0, \cdots, n+1)$ are C^∞-functions except the zero section and $c_0^+(x', t) \neq 0$. Therefore we obtain solutions $\hat{a}_{-j}^\pm$ of $(2.\,15)_{+j}$. Now decompose $\hat{a}_{-j}^\pm$ into odd and even functions as follows;

$$\hat{a}_{-j}^\pm(x', t) = \hat{g}_{-j}(x', t^2) + t\hat{h}_{-j}(x', t^2)\,.$$

If t is replaced by $-t$ in $\hat{a}^{\pm}_{-j}$, it satisfies (2. 14)$_-$. Thus we obtain desired $g(x, \xi')$, $h(x, \xi')$, if we give initial conditions $\hat{a}^{+}_{0}(x', 0)=1$, $\hat{a}^{\pm}_{-j}(x', 0)=0$ $(j>0)$ and put $g_{-j}(x, \xi')=\hat{g}_{-j}(x', \rho)$, $h_{-j}(x, \xi')=\hat{h}_{-j}(x', \rho)$.

§ 3. Constructions of parametrices for the Chauchy problem.

Making use of g and h obtained in § 2 and Airy functon, we shall define parametrices $G^k(x_0, t)$ of P containing the parameters $0 \leqslant t \leqslant x_0 \leqslant T$.

We abbreviate θ and ρ as follows:

$$\theta(x_0) = \theta(x_0, x', \xi'), \qquad \hat{\theta}(t) = \theta(t, y', \xi'),$$

$$\rho(x_0) = \rho(x_0, x', \xi'), \qquad \hat{\rho}(t) = \rho(t, y', \xi'),$$

$$\varphi_\pm(x_0) = \theta(x_0) \pm \frac{2}{3}\rho^{\frac{3}{2}}(x_0) \quad \text{and} \quad \hat{\varphi}_\pm(t) = \hat{\theta}(t) + \frac{2}{3}\hat{\rho}^2(t).$$

Now using the above notation, define the operators $G_\pm(x_0, t)$ for $V \in C_0^\infty(R^n)$ such that:

$$\text{(3. 1)} \qquad \Big(G_\pm(x_0, t)\, V\Big)(x') = \int e^{i(\theta(x_0)-\hat{\theta}(t))}\Big(g(x, \xi')\frac{A_\pm(\rho(x_0))}{A'_\pm(\hat{\rho}(t))} - ih(x, \xi')\frac{A'_\pm(\rho(x_0))}{A'_\pm(\hat{\rho}(t))}\Big)V(y')\, dy'\, d\xi'.$$

From (2. 6) and (2. 10) it is evident that

$$P(x, D)\, G_\pm(x_0, t) \equiv 0 \qquad (mod\ C^\infty).$$

To examine the symbols of (3. 1) more precisely we prepare some lemmas.

Let $a(x_0, x', \xi')$ be a function in $C^\infty([0, T] \times R^n \times R^n \backslash 0)$ for $T>0$. We denote $a \in S^m_{1,0,\frac{2}{3}}([0, T] \times R^n \times R^n)$, if we have the following estimates:

$$\text{(3. 2)} \qquad \left|\partial^\alpha_{\xi'}\partial^\beta_{x'}\partial^\gamma_{x_0} a(x_0, x', \xi')\right| \leqslant C_{\alpha,\beta,\gamma}\, |\xi'|^{m-|\alpha|+\frac{2}{3}\gamma} \quad \text{for} \quad |\xi'| \geqslant 1.$$

LEMMA 3. 1. *It holds that*

$$\left.\begin{array}{ll} \text{(3. 3)}_\pm & A_\pm\big(\rho(x, \xi')\big), \\ \text{(3. 3)}'_\pm & 1/A'_\pm\big(\rho(x, \xi')\big) \end{array}\right\} \in S^0_{1,0,\frac{2}{3}}\big([0, T] \times R^n \times R^n\big) \quad \text{for} \quad \rho(x, \xi') \leqslant 2$$

and that

$$\left.\begin{array}{ll} \text{(3. 4)}_\pm & A_\pm\big(\rho(x, \xi')\big) \times \exp\Big(\mp i\frac{2}{3}\rho^{\frac{3}{2}}(x, \xi')\Big), \\ \text{(3. 4)}'_\pm & 1/A'_\pm\big(\rho(x, \xi')\big) \times \exp\Big(\pm i\frac{2}{3}\rho^{\frac{3}{2}}(x, \xi')\Big) \end{array}\right\} \in S^0_{1,0,\frac{2}{3}}\big([0, T] \times R^n \times R^n\big) \quad \text{for} \quad \rho(x, \xi') \geqslant 1.$$

PROOF. From (2. 3) we have $C_1 x_0|\xi'|^{\frac{2}{3}} \leqslant \rho \leqslant C_2 x_0|\xi'|^{\frac{2}{3}}$ for some positive C_1, C_2. By virture of this relation and the fact that the k-th derivative $A^{(k)}(\rho)$ is bounded if $\rho \leqslant 2$, $(3.3)_\pm$ and $(3.3)'_\pm$ can be proved by simple calculations.

To prove $(3.4)_\pm$, $(3.4)'_\pm$ we will make use of the saymptotic expansion of Airy function as follows:

$$(3.5) \qquad Ai(z) \sim \exp\left(-\frac{2}{3} z^{\frac{3}{2}}\right) z^{-\frac{1}{4}} \left(\sum_{\nu=0}^{\infty} a_\nu z^{-\frac{3}{2}\nu}\right) \text{ with } a_0 \neq 0\,,$$
$$\text{for } |z| \geqslant \varepsilon \text{ and } -\pi+\varepsilon \leqslant arg\ z \leqslant \pi-\varepsilon\,,$$

where ε is a small positive number (see [10]).

Now assume that $\rho \geqslant 1$ and let $F(\rho) = A_+(\rho) \exp\left(-i \frac{2}{3} \rho^{\frac{3}{2}}\right)$. From (2. 8) and (3. 5) it holds that $F^{(k)}(\rho) = O(\rho^{-\frac{1}{4}-k})$. Let z be one of variables x', ξ' or x_0 then we find that

$$\partial_z^\alpha \left(F(\rho)\right) = \sum_{|\alpha_1|+\cdots+|\alpha_k|=|\alpha|} F^{(k)}(\rho)\, \partial_z^{\alpha_1} \rho \partial_z^{\alpha_2} \rho \cdots \partial_z^{\alpha_k} \rho\,.$$

Since $C_1 x_0|\xi'|^{\frac{2}{3}} \leqslant \rho \leqslant C_2 x_0|\xi'|^{\frac{2}{3}}$, we have

$$(3.6) \qquad \left|\partial_{x'}^\beta \partial_{\xi'}^\alpha \left(F(\rho)\right)\right| = \left|\sum_{\substack{|\alpha_1|+\cdots+|\alpha_k|=|\alpha| \\ |\beta'|+|\beta''|=|\beta|}} \partial_{x'}^{\beta'} \left(F^{(k)}(\rho)\right) \partial_{x'}^{\beta''} (\partial_{\xi'}^{\alpha_1} \rho \cdots \partial_{\xi'}^{\alpha_k} \rho)\right|$$
$$\leqslant \sum_{\substack{k \leqslant |\alpha| \\ j \leqslant |\beta|}} O(\rho^{-\frac{1}{4}-j-k})\, t^{j+k} |\xi'|^{\frac{2}{3}(j+k)-|\alpha|}$$
$$\leqslant O(\rho^{-1/4}) |\xi'|^{-|\alpha|}\,.$$

Therefore we see that

$$(3.7) \qquad \left|\partial_{\xi'}^\alpha \partial_{x'}^\beta \partial_{x_0}^\gamma \left(F(\rho)\right)\right| = \left|\partial_{\xi'}^\alpha \partial_{x'}^\beta \left(\sum_{\gamma_1+\cdots+\gamma_k=\gamma} F^{(k)}(\rho)\, (\partial_{x_0}^{\gamma_1} \rho \cdots \partial_{x_0}^{\gamma_k} \rho)\right)\right|$$
$$= \left|\sum_{\substack{k \leqslant \gamma \\ \alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}} C_{\alpha',\beta'} \left(\partial_{\xi'}^{\alpha'} \partial_{x'}^{\beta'} \left(F^{(k)}(\rho)\right)\right) \partial_{\xi'}^{\alpha''} \partial_{x'}^{\beta''} (\partial_{x_0}^{\gamma_1} \rho \cdots \partial_{x_0}^{\gamma_k} \rho)\right|$$
$$\leqslant O(\rho^{-\frac{1}{4}}) |\xi'|^{\frac{2}{3}\gamma-|\alpha|}\,.$$

This proves $(3.4)_+$. The same argument holds for $(3.4)_-$ and $(3.4)'_\pm$.

Let $S^m_{1,0,\frac{2}{3}}([0, T] \times R^n_{x'} \times R^n_{y'} \times R^n_{\xi'})$ be the set of functions such that;

$$\left|\partial_{\xi'}^\alpha \partial_{x'}^{\beta_1} \partial_{y'}^{\beta_2} \partial_t^\gamma a(t, x', y', \xi')\right| \leqslant C_{\alpha,\beta_1,\beta_2,\gamma} |\xi'|^{m-|\alpha|+\frac{2}{3}\gamma} \quad \text{for} \quad |\xi'| \geqslant 1\,,$$

where $a \in C^\infty([0, T] \times R^n_{x'} \times R^n_{y'} \times R^n_{\xi'} \backslash 0)$.

LEMMA 3. 2. *Let $\rho(t) = \rho(t, x', \xi')$. It holds that*

$$(3.8)_\pm \qquad t\frac{A_\pm(\rho(t))}{A'_\pm(\hat{\rho}(t))}\in S^{-\frac{1}{3}}_{1,0,\frac{2}{3}}\Big([0, T]\times R^n_{x'}\times R^n_{y'}\times R^n_{\xi'}\Big) \quad \textit{for} \quad \rho(t)\leqslant 2\,,$$

$$(3.9)_\pm \qquad t\frac{A_\pm(\rho(t))}{A'_\pm(\hat{\rho}(t))}\exp\Big(\mp i\frac{2}{3}\Big(\rho^{\frac{3}{2}}(t)-\hat{\rho}^{\frac{3}{2}}(t)\Big)\Big)$$
$$\in S^{-\frac{1}{3}}_{1,0,\frac{2}{3}}\Big([0, T]\times R^n_{x'}\times R^n_{y'}\times R^n_{\xi'}\Big) \quad \text{for} \quad \rho(t)\geqslant 1\,,$$

and these functions are estimated by $O(t^{\frac{1}{2}}|\xi'|^{-\frac{1}{3}})$ *when* $\rho(t)\leqslant 2$, $\rho(t)\geqslant 1$ *respectively.*

PROOF. When $\rho(t)\leqslant 2$, from (2.3) there exists a positive number C' such that $t|\xi'|^{\frac{2}{3}}\leqslant C'$. Now $(3.8)_\pm$ follows from $(3.3)_\pm$ and $(3.3)'_\pm$, because $\rho(t)$ and $\hat{\rho}(t)$ have equivalent growth as the coefficients of $P(x, D)$ are constant for large $|x'|$.

When $\rho(t)\geqslant 1$, let $F(\rho)$ be the function introduced in the previous proof and $\hat{F}(\hat{\rho})=1/A'_+(\hat{\rho})\exp\Big(i\frac{2}{3}\hat{\rho}^{\frac{3}{2}}\Big)$. By the same calculation as (3.7) we find

$$\Big|\partial^\alpha_{\xi'}\partial^{\beta_1}_{x'}\partial^{\beta_2}_{y'}\partial^\gamma_t\Big(tF(\rho)\,\hat{F}(\hat{\rho})\Big)\Big| =$$
$$\Big|t\partial^\alpha_{\xi'}\partial^{\beta_1}_{x'}\partial^{\beta_2}_{y'}\partial^\gamma_t\Big(F(\rho)\,\hat{F}(\hat{\rho})\Big)+\gamma\partial^\alpha_{\xi'}\partial^{\beta_1}_{x'}\partial^{\beta_2}_{y'}\partial^{\gamma-1}_t\Big(F(\rho)\,\hat{F}(\hat{\rho})\Big)\Big|$$
$$\leqslant tO(\rho^{-\frac{1}{2}})|\xi'|^{-|\alpha|+\frac{2}{3}\gamma}+O(\rho^{-\frac{1}{2}})|\xi'|^{-|\alpha|+\frac{2}{3}(\gamma-1)}$$
$$\leqslant 0(t^{\frac{1}{2}})|\xi'|^{-\frac{1}{3}-|\alpha|+\frac{2}{3}\gamma}\,.$$

The same argument holds for the sign $-$. This proves the lemma,

Let $\chi(\sigma)$ be a function in $C^\infty(R^+)$ such that $\chi(\sigma)\equiv 1$ for $\sigma\leqq 1$, $\chi(\sigma)\equiv 0$ for $\sigma\geqslant 2$. By the analogous method in Lemma 3.1 we conclude that $\chi(\rho(x_0))$, $1-\chi(\rho(x_0))\in S^0_{1,0,\frac{2}{3}}$. From Lemma 3.1, multiplying these cut off functions to the amplitude of the integral (3.1), we can adopt $\theta(x_0)$ or $\varphi_\pm(x_0)$ as the phase function for $G_\pm(x_0, t)$ when $\rho(x_0)\leqslant 2$ or $\rho(x_0)\geqslant 1$ respectively. As the same argument holds for $\hat{\theta}(t)$and $\hat{\varphi}_\pm(t)$, we divide $G_\pm(x_0, t)$ into four parts according to the cases that $\rho(x_0)$ or $\rho(t)$ are large or small. Taking admissible phase functions corresponding to each part, we regard them as Fourier integral operators. Now by the oscillatory integral method (see [3]) we have that

$$(3.10) \qquad \Big(G_\pm(x_0, t)\,V\Big)(x')\in C^\infty\Big([0, T]\times[0, T]\times R^n\Big) \quad \text{for} \quad V\in C(R^n)$$

and

$$(3.11) \qquad \Big(G_\pm(x_0, t)\,V\Big)(x')\in C^\infty\Big([0, T]\times[0, T]\,;\,\mathscr{D}'(R^n)\Big) \quad \text{for} \quad V\in\mathscr{E}'(R^n)\,.$$

Let $\psi_\pm(x_0, t, x', y', \xi')=\varphi_\pm(x_0)-\hat{\varphi}_\pm(t)$ and define

$$C^{\pm}_{x_0,t}=\{(x',y',\xi')\,;\ \psi_{\pm\xi'}(x_0,t,x',y',\xi')=0\}.$$

Let $\Lambda^{\pm}_{x_0,t}$ be the image of $C^{\pm}_{x_0,t}$ under the mapping

$$C^{\pm}_{x_0,t}\to\Lambda^{\pm}_{x_0,t}=\{(x',\eta',y',\zeta')\,;\ \eta'=\psi_{\pm},\ \zeta'=\psi_{\pm}\}\subset\left(T^*(R^n)\backslash 0\right)\times\left(T^*(R^n)\backslash 0\right),$$

then the following lemma shows the behavior of the wave front set of $G_{\pm}(x_0,t)$.

LEMMA 3.3.

$$WF\left(\left(G_{\pm}(x_0,t)\,V\right)(\cdot)\right)\subset\Lambda^{\pm}_{x_0,t}\circ WF(V).$$

PROOF. Since to find wave fronts one needs only to examine large ξ', $\psi_{\pm}$ may be taken as phase functions. Indeed in the domain $\rho(x_0)\leqslant 2$ and $\hat\rho(t)\geqslant 1$, we use the phase function $\theta(x_0)-\hat\varphi_{\pm}(t)$ and the cut off function $\chi(\rho(x_0))\,(1-\chi(\hat\rho(t)))$. However this cut off function vanishes for large ξ' if $x_0\neq 0$. In the remained domains where one of $\rho(x_0)$ or $\hat\rho(t)$ is less than 2, the same argument remains valid. Thus noting $\varphi_{\pm}(0)=\theta(0)$ we obtain the lemma by usual method. For example see Theorem 5.1 of [7]. The proof is complete.

Similary we have for any but fixed initial $t\geqslant 0$

$$WF\left(\left(G_{\pm}(x_0,t)\,V\right)(x')\right)\subset\Lambda^{\pm}_t\circ WF(V),$$

where $\Lambda^{\pm}_t\subset(T^*((0,T)\times R^n)\backslash 0)\times(T^*(R^n)\backslash 0)$ is a Lagrangean generated by $\psi_{\pm}(x_0,t,x',y',\xi')$ with respect to (x_0,x',y',ξ').

Now we shall show that the operators $G_{\pm}(x_0,t)V|_{x_0=t}$ and $D_{x_0}G_{\pm}(x_0,t)V|_{x_0=t}$ can be represented by pseudo-differential operators with the parameter t.

Let $a(t,x',y',\xi')$ be a symbol in $S^m_{1,0,\frac{2}{3}}$ and χ the cut off function mentioned above.

LEMMA 3.4. *Let $L_{\pm}(t)$ be an operator defined for $V\in C_0^\infty(R^n)$ as follows:*

$$\left(L_{\pm}(t)V\right)(x')=\int e^{i\psi_{\pm}(t,x',y',\xi')}\,a(t,x',y',\xi')\left(1-\chi\left(\rho(t,x',\xi')\right)\right)V(y')\,dy'\,d\xi'$$

where $\psi_{\pm}(t,x',\xi')=\varphi_{\pm}(t,x',\xi')-\varphi_{\pm}(t,y',\xi')$.
Then there exist symbols $b_{\pm}(t,x',y',\eta')\in S^m_{1,0,\frac{2}{3}}$ such that

$$\left(L_{\pm}(t)\,V\right)(x')=(2\pi)^{-n}\int e^{i\langle x'-y',\eta'\rangle}\,b_{\pm}(t,x',y',\eta')\,V(y')\,dy'\,d\eta'.$$

PROOF. Let $\eta(\sigma,x',y',\xi')\in S^1_{1,0}([0,T^2]\times R^n_{x'}\times R^n_{y'}\times R^n_{\xi'})$. Note that if $\rho(t)\geqslant 1$

then it holds $\eta(\sqrt{t}, x', y', \xi') \in S^1_{1,0,\frac{2}{3}}([0, T] \times R^n_{x'} \times R^n_{y'} \times R^n_{\xi'})$, because $t^{-1} \leqslant C'|\xi'|^{-\frac{2}{3}}$ for some positive number C'. Therefore the assertion can be proved easily with the aid of Lemma 7.2 and Theorem 7.6 of [7].

In the domain $\rho(t) \leqslant 2$, the above lemma is shown obviously if $\psi_\pm$ is replaced by $\theta(t) - \hat{\theta}(t)$.

We denote by $\gamma^0_\pm(t)$, $\gamma^1_\pm(t)$ the restrictions of $(G_\pm(x_0, t) V)(x')$ and $(D_{x_0} G_\pm(x_0, t) V)(x')$ at $x_0 = t$ respectively:

$$\begin{cases} \gamma^0_\pm(t) V = G_\pm(x_0, t) V\Big|_{x_0=t}, \\ \gamma^1_\pm(t) V = D_{x_0} G_\pm(x_0, t) V\Big|_{x_0=t}. \end{cases}$$

Now it can be shown that they become pseudo-differential operators of order 0 and $\frac{2}{3}$ respectively.

Indeed, since the order of g and of h is 0 and $-\frac{1}{3}$ respectively, the statement for $\gamma^0_\pm(t)$ follows from Lemma 3.1 and Lemma 3.4. For the order of $\gamma^1_\pm(t)$ it may seem that its principal part is $\theta_{x_0}(t)\, g(t, x', \xi')$, which is of order 1 and vanishes at t=0. This prevents to consider the Cauchy data. However making use of Lemma 3.2 we can avoid this difficulty.

In fact by direct computations we have:

$$(3.12) \qquad D_{x_0} G_\pm(x_0, t) V\Big|_{x_0=t} = \int e^{i(\theta(t)-\hat{\theta}(t))} \Big\{ \Big(\theta_{x_0} g(t, x', \xi') + \rho_{x_0} \rho h(t, x', \xi') - i g_{x_0}(t, x', \xi') \Big) \frac{A_\pm(\rho(t))}{A'_\pm(\hat{\rho}(t))} + \Big(-i \rho_{x_0} g(t, x', \xi') - i \theta_{x_0} h(t, x', \xi') - h_{x_0}(t, x', \xi') \Big) \frac{A'_\pm(\rho(t))}{A'_\pm(\hat{\rho}(t))} \Big\} V(y')\, dy'\, d\xi',$$

where we have used the relation $A''_i(z) = z A i(z)$.

From (2.2), (2.3) it follows $\theta_{x_0}(t, x', \xi') = O(t)$, $\rho(t, x' \xi') = O(t)$ and hence from Lemma 3.2 we have that the symbols

$$\Big(\theta_{x_0} g(t, x', \xi') + \rho_{x_0} \rho h(t, x', \xi') \Big) \frac{A_\pm(\rho(t))}{A'_\pm(\hat{\rho}(t))} \times \begin{cases} 1, & \text{for } \rho(t) \leqslant 2, \\ \exp\Big(\mp i \frac{2}{3} \big(\rho^{\frac{3}{2}}(t) - \hat{\rho}^{\frac{3}{2}}(t) \big) \Big), & \text{for } \rho(t) \geqslant 1 \end{cases}$$

are contained in $S^{\frac{2}{3}}_{1,0,\frac{2}{3}}$ and vanish at $t = 0$.

Since the contributions from $A'_\pm \rho(t))/A'_\pm(\hat{\rho}(t))$ are terms of order 0 only, $\gamma^1_\pm(t)$ becomes a pseudo-differential operator of order $\frac{2}{3}$.

Now we denote by γ and $\tilde{\Lambda}$ matrices

$$\gamma=\begin{pmatrix}\gamma_+^0(t), & \gamma_-^0(t)\\ \gamma_+^1(t), & \gamma_-^1(t)\end{pmatrix}, \qquad \tilde{\Lambda}=\begin{pmatrix}\Lambda^{\frac{2}{3}}, & 0\\ 0, & I\end{pmatrix},$$

where $\Lambda^{\frac{2}{3}}$ is the pseudo-differential operator with the symbol $(1+|\xi'|^2)^{\frac{1}{3}}$. Then the principal symbol of $\tilde{\Lambda}\gamma$ is written in the following from at $t=0$:

$$\begin{pmatrix}|\xi'|^{\frac{2}{3}}e^{\frac{\pi}{3}i}Ai(0)/Ai'(0), & |\xi'|^{\frac{2}{3}}e^{-\frac{\pi}{3}i}Ai(0)/Ai'(0)\\ -i\rho_{x_0}, & -i\rho_{x_0}\end{pmatrix}$$

for (2. 8), (3. 12) and $g|_{t=0}=1$.

Hence if T is sufficently small $\tilde{\Lambda}\gamma$ is elliptic for any $t\in[0, T]$. Now let $K(t)$ be its parametrix. Define the operators $G^k(x_0, t)$ for $V\in C_0^\infty(R^n)$ as follows;

$$G^k(x_0, t)\,V=\big(G_+(x_0, t),\ G_-(x_0, t)\big)\,K(t)\begin{pmatrix}\delta_0^k\Lambda^{\frac{2}{3}}V\\ \delta_1^k V\end{pmatrix}. \tag{3. 13}$$

From the previous constructions $G^k(x_0, t)$ satisfies (1. 3). Thus Theorem 1. 1 is proved.

§ 4. Construction of the fundamental solution for the Cauchy problem.

We shall construct the fundamental solution from (1. 3) and derive the estimate (1. 4).

For $f\in C^\infty([0, T]; C_0^\infty(R^n))$ we define the operator G' as follows;

$$G'f(x_0, x')=\int_0^{x_0}\big(G^1(x_0, t)-R_0^1(t)-(x_0-t)\,R_1^1(t)\big)f(t, x')\,dt. \tag{4. 1}$$

Then from (1. 3) we obtain that

$$\begin{cases} PG'f=f-Wf,\\ D_{x_0}^j G'f\big|_{x_0=0}=0 \qquad \text{for}\quad j=0, 1,\end{cases}$$

where W is the operator with C^∞-kernel $k(x_0, t, x', y')$ such that

$$Wf(x_0, x')=\int_0^{x_0}\int k(x_0, t, x', y')f(t, y')\,dy'\,dt. \tag{4. 2}$$

To construct the inverse of $(I-W)$ by the Neumann series $\sum_{k=0}^{\infty}W^k$, we must insert suitable cut off functions. Let α_1, $\alpha_2\in C_0^\infty(R^n)$ such that for some compact sets $K_1\Subset K_2\Subset K_3$ in R^n, $\alpha_1\equiv1$ on K_1, $\operatorname{supp}\alpha_1\subset K_2$, $\alpha_2\equiv1$ on K_2 and supp $\alpha_2 \subset K_3$. Choose K_2 such that K_2 contains any bicharacteristic curve $x(t)$ for P_2 starting from some point in K_1 at $t=0$ if t is smaller than T. It follows from Lemma 3.3,

$$(1-\alpha_2)\, G'\alpha_1 \equiv 0 \qquad (\bmod\, C^\infty)\,.$$

Let $W'f=P(1-\alpha_2)\, G'\alpha_1 f+W\alpha_1 f$. Then W' has the same representation as W in (4.2) and

$$W'f\in C^\infty\Big([0, T]\,;\, C_0^\infty(K_3)\Big)\,, \tag{4.3}$$

$$P\alpha_2 G'\alpha_1 f=\alpha_1 f-W'f \qquad \text{for } f\in C^\infty\Big([0, T]\,;\, C_0^\infty(R^n)\Big)\,. \tag{4.4}$$

By (4.3) the equation $(I-W')f'=f$ can be solved in $[0, T]\times K_2$ for given f thus from (4.4) we see that for any $f\in C^\infty([0, T]\,;\, C_0^\infty(K_1))$ there exists a solution $u\in C^\infty([0, T]\,;\, C_0^\infty(R^n))$ such that

$$\begin{cases} Pu=f \quad \text{in} \quad [0, T]\times K_1\,, \\ D_{x_0}^j u\big|_{x_0=0}=0\,, \qquad j=0, 1\,. \end{cases} \tag{4.5}$$

In fact, for the solution f' of the equation $(I-W')f'=f$, let $u=\alpha_2 G'\alpha_1 f'$ then it satisfies (4.5).

Now we shall prove the uniqueness of the solution of (4.5). Let P^* be the adjoint operator of P then the principal symbol of P^* is as same as that of P. Moreover from the definition (3.1) the parameterices of P^* is also smooth with respect to $0\leqslant x_0\leqslant t\leqslant T$, thus the equation (4.5) can be solved for given $f\in C^\infty([0, T]\times K_1)$ in $[0, T]\times K_1$ with the initial surface $x_0=T$ instead of $x_0=0$. Note that P^* is strictly hyperbolic in $x_0>0$ and the doamin of influence of f is finite because the bicharcteristic curve $x(t)$ does not tangent to the surface $x_0=0$. Hence the solution of (4.5) with initial surface $x_0=T$ is smooth in $[0, T]\times K_1$ and vanishes identically for large $|x'|$. Therefore by the usual dual argument we obtain the uniqueness of the solution of (4.5).

Now choose K_1 sufficiently large for given $f\in C^\infty([0, T]\,;\, C_0^\infty(R^n))$ then from the above facts we see that there exists a uniqu solution $u\in C^\infty([0, T]\,;\, C_0^\infty(R^n))$ of (4.5) in $[0, T]\times R^n$ for any $f\in C^\infty([0, T]\,;\, C_0^\infty(R^n))$.

Furthermore from (1.3), we shall give the formula of the solution of (1.2) with no-zero initial data $(v_0, v_1)\in C_0^\infty(R^n)$.

Let K be a compact set in R^n such that K contains the set supp $v_0\cup$ supp v_1 and any bicharacteristic cuve $x(t)$ starting from some point in supp $v_0\cup$ supp v_1 at $t=0$ if t is smaller than T. Choose $\alpha\in C_0^\infty(R^n)$ such that $\alpha\equiv 1$ on K. Now we define the function u such that;

$$(4.6)\qquad u=\sum_{k=0}^{1}\alpha G^{k}(x_0,0)\,v_k-\left(\sum_{k=0}^{1}\alpha R_0^k(0)\,v_k+x_0\sum_{k=1}^{1}\alpha R_1^k(0)\,v_k\right)$$
$$+G'(I-W')^{-1}f',$$

where

$$f'=f-P\left\{\sum_{k=0}^{1}\alpha G^{k}(x_0,0)\,v_k-\left(\sum_{k=0}^{1}\alpha R_0^k(0)\,v_k+x_0\sum_{k=0}^{1}\alpha R_1^k(0)\,v_k\right)\right\}.$$

Then u satisfies the equation (1.2). (See III of [1])

Finally we shall prove the estimate (1.4).

Assume that $x_0\geqslant t$ then there exists a positive number C such that $\rho(x_0)\geqslant C\hat{\rho}(t)$. Recall that $t\leqslant C'|\xi|'^{-\frac{2}{3}}$ if $\hat{\rho}(t)\leqslant 2$. Thus substituting $\rho(x_0)$ or $\hat{\rho}(t)$ for ρ in (3.6), we have that

$$(4.7)\qquad \frac{A_\pm(\rho(x_0))}{A'_\pm(\hat{\rho}(t))}\,t^{\frac{1}{2}}\in S_{1,0}^{-\frac{1}{3}}\quad\text{for}\quad \hat{\rho}(t)\leqslant 2,\quad \rho(x_0)\leqslant C+1,$$

$$(4.7)'\qquad \frac{A_\pm(\rho(x_0))}{A'_\pm(\hat{\rho}(t))}\times\exp\left(\pm i\frac{2}{3}\rho^{\frac{3}{2}}(x_0)\right)t^{\frac{1}{2}}\in S_{1,0}^{-\frac{1}{3}}\quad\text{for}\quad \hat{\rho}(t)\leqslant 2,\quad \rho(x_0)\geqslant C$$

and

$$(4.7)''\qquad \frac{A_\pm(\rho(x_0))}{A'_\pm(\hat{\rho}(t))}\times\exp\left(\pm i\frac{2}{3}\left(\rho^{\frac{3}{2}}(x_0)-\rho^{\frac{3}{2}}(t)\right)\right)t^{\frac{1}{2}}\in S_{1,0}^{-\frac{1}{3}}$$
$$\text{for}\quad \hat{\rho}(t)\geqslant 1,\quad \rho(x_0)\geqslant C,$$

where $S_{1,0}^{-\frac{1}{3}}=S_{1,0}^{-\frac{1}{3}}(R^n_{x'}\times R^n_{y'}\times R^n_{\xi'})$, x_0 and t are considered as continuous parameters.

These properties also hold for $A'_\pm(\rho(x_0))/A'_\pm(\hat{\rho}(t))$ if we take the symbol class $S^0_{1,0}$ instead of $S^{-1/3}_{1,0}$.

Now we rewrite the first term of (4.1) as follows

$$(4.8)\qquad \int_0^{x_0}G^1(x_0,t)f(t,x')\,dt=\int_0^{x_0}\left(G^1(x_0,t)\,t^{\frac{1}{2}}\right)f(t,x')\,t^{-\frac{1}{2}}\,dt$$
$$=\int_0^{x_0}\left(G_+(x_0,t),\ G_-(x_0,t)\right)t^{\frac{1}{2}}K(t)\begin{pmatrix}0\\ f(t,x')\end{pmatrix}t^{-\frac{1}{2}}dt.$$

Choose $\chi,\hat{\chi}\in C^\infty(R_+)$ such that $\chi(\sigma)\equiv 1$ for $\sigma\leqslant C$, $\chi(\sigma)\equiv 0$ for $\sigma\geqslant C+1$ and $\hat{\chi}(\sigma)\equiv 1$ for $\sigma\leqslant 1$, $\hat{\chi}(\sigma)\equiv 0$ for $\sigma\geqslant 2$.

Noting $\chi(\rho(x_0))$, $\hat{\chi}(\hat{\rho}(t))\in S^0_{1,0}$, we decompose $G_\pm(x_0,t)$ into the parts of (4.7), (4.7)$'$ and (4.7)$''$ and regard their phase functions $\theta(x_0)-\hat{\theta}(t)$, $\varphi_\pm(x_0)-\hat{\theta}(t)$ and $\varphi_\pm(x_0)-\hat{\varphi}_\pm(t)$ respectively. Since $K(t)$ is a pseudo-differential operator of order $-\frac{2}{3}$, $G^1(x_0,t)\,t^{\frac{1}{2}}$ becomes the sum of Fourier integral operators of order -1. Consequently by the L^2-estimate of Fourier integral operators ([3]) (1. 4) follows from (4. 6) and (4. 8).

Indeed for $s\geqslant 0$, it holds that

$$\left\|u(t,\cdot)\right\|_{s+1}\leqslant C\left\|\sum_{k=0}^{1}G^k(t,0)\,v_k\right\|_{s+1}+C\int_0^t\left\|G^1(t,\tau)\,\tau^{\frac{1}{2}}f(\tau,\cdot)\right\|_{s+1}\tau^{-\frac{1}{2}}\,d\tau$$

$$\leqslant C\|\Lambda^{\frac{2}{3}}v_0\|_{s+1-\frac{2}{3}}+C\|v_1\|_{s+1-\frac{2}{3}}+C\int_0^t\left\|f(\tau,\cdot)\right\|_s\tau^{-\frac{1}{2}}\,d\tau$$

$$\leqslant C\|v_0\|_{s+1}+C\|v_1\|_{s+\frac{1}{3}}+C\int_0^t\left\|f(\tau,\cdot)\right\|_s\tau^{-\frac{1}{2}}\,d\tau$$

and from (4. 6)

$$\left\|D_{x_0}u(t,\cdot)\right\|_s\leqslant C\left\|\sum_{k=0}^{1}\left(D_{x_0}G^k(t,0)\,v_k\right)\right\|_s+C\int_0^t\left\|D_{x_0}G^1(t,\tau)\,\tau^{\frac{1}{2}}f(\tau,\cdot)\right\|_s\tau^{-\frac{1}{3}}\,d\tau$$

$$\leqslant C\|v_0\|_{s+1}+C\|v_1\|_{s+\frac{1}{3}}+C\int_0^t\left\|f(\tau,\cdot)\right\|_s\tau^{-\frac{1}{2}}\,d\tau$$

because $(D_{x_0}G_\pm)(t,\tau)$ is of order 1. To estimate higher derivatives of u, note that

$$PD_{x_0}u=D_{x_0}Pu+[P,D_{x_0}]\,u$$
$$=D_{x_0}f+B_2(x,D_{x'})\,u+B_1(x,D_x)\,u\,,$$

where B_1 is the first order differential operator in D_x and B_2 is the second order one in $D_{x'}$. Let $F=D_{x_0}f+B_2u+B_1u$ and apply the above argument to $(D_{x_0}u, F)$ instead of (u,f). Then we obtain that

$$\left\|D_{x_0}u(t,\cdot)\right\|_s+\left\|D_{x_0}^2u(t,\cdot)\right\|_{s-1}\leqslant C\left\|D_{x_0}u(0,\cdot)\right\|_s+C\left\|D_{x_0}^2u(0,\cdot)\right\|_{s+\frac{1}{3}-1}$$
$$+C\int_0^t\left\|F(\tau,\cdot)\right\|_{s-1}\tau^{-\frac{1}{2}}\,d\tau$$
$$\leqslant C\left\|u(0,\cdot)\right\|_{s+1}+C\sum_{k=1}^{1}\left\|D_{x_0}^{k+1}u(0,\cdot)\right\|_{s+\frac{1}{3}-k}+C\sum_{k=0}^{1}\int_0^t\left\|D_{x_0}^kf(\tau,\cdot)\right\|_{s-k}\tau^{-\frac{1}{2}}\,d\tau\,.$$

Iterating this procedure for positive number k less than s, we see that

$$\sum_{k\leqslant s+1}\left\|D_{x_0}^ku(t,\cdot)\right\|_{s+1-k}\leqslant C\left\|u(0,\cdot)\right\|_{s+1}+C\sum_{k\leqslant s}\left\|D_{x_0}^{k+1}u(0,\cdot)\right\|_{s+\frac{1}{3}-k}$$
$$+C\sum_{k\leqslant s}\int_0^t\left\|D_{x_0}^kf(\tau,\cdot)\right\|_{s-k}\tau^{-\frac{1}{2}}\,d\tau\,. \tag{4. 9}$$

Now to obtain the estimate (1. 4) we must estimate higher derivatives of u at $t=0$ in the right hand side of (4. 9).

Thus we shall prove that

$$\left\|D_{x_0}^{k+1}u(0,\cdot)\right\|_{s+\frac{1}{3}-k}\leqslant C\|v_0\|_{s+1}+C\|v_1\|_{s+\frac{1}{3}}+C\sum_{j\leqslant k-1}\left\|D_{x_0}^jf(0,\cdot)\right\|_{s-j}\,, \tag{4. 10}$$

for $k\geqslant 1$.
In fact, since for $k=1$ we have

$$D_{x_0}^2 u(0, x') = \left(x_0 A(x, D_{x'})\, u + P_1(x, D_x)\, u + f\right)\Big|_{x_0=0}$$
$$= P_1(0, x', D_x)\, u + f(0, x'),$$

we obtain that

$$\left\|D_{x_0}^2 u(0, \cdot)\right\|_{s+\frac{1}{3}-1} \leqslant C\|v_0\|_{s+\frac{1}{3}} + C\|v_1\|_{s+\frac{1}{3}-1} + C\left\|f(0, \cdot)\right\|_{s+\frac{1}{3}-1}$$
$$\leqslant C\|v_0\|_{s+1} + C\|v_1\|_{s+\frac{1}{3}} + C\left\|f(0, \cdot)\right\|_s .$$

Now assume that the statement holds for any $k'\leqslant k$. Then it follows that

$$\left\|D_{x_0}^{k+2} u(0, \cdot)\right\|_{s+\frac{1}{3}-(k+1)} \leqslant C\sum_{j=-1}^{k}\left\|D_{x_0}^{k-j} u(0, \cdot)\right\|_{s+\frac{1}{3}-(k-j)} +$$
$$C\left\|D_{x_0}^{k} f(0, \cdot)\right\|_{s+\frac{1}{3}-(k+1)}$$
$$\leqslant C\|v_0\|_{s+1} + C\|v_1\|_{s+\frac{1}{3}} + C\sum_{j\leqq k-1}\left\|D_{x_0}^{j} f(0, \cdot)\right\|_{s-j},$$

because

$$D_{x_0}^{k+2} u(0, \cdot) = D_{x_0}^{k}\left(x_0 A(x, D_{x'})\, u + P_1(x, D_x)\, u + f\right)\Big|_{x_0=0} .$$

Apply (4. 10) to (4. 9) then we obtain (1. 4). This proves Theorem 1. 2.

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