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A note on linearly compact modules

Dedicated to Professor Goro Azumaya on his 60th birthday

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Let R, S be rings[1] and ${}_RM_S$ be an $R-S$-bimodule such that ${}_RM$ is linearly compact[2] as left R-module. In this note we consider the conditions under which M_S, as right S-module, to be injective. Thus we have the following theorem which generalize Theoreme 2 in [1].

THEOREM. *Let ${}_RM_S$ be an R-S-bimodule such that ${}_RM$ is linearly compact. Then the following statements are equivalent*:

(1) *M_S is injective.*

(2) *M_S is absolutely pure, that is, every homomorphism of a finitely generated submodule of S_S^m to M_S is extended to that of S_S^m, where m is arbitaray natural number and S_S^m is a direct sum of m-fold copies of S_S.*

(3) *M_S is semi S-injective, that is, every homomorphism of a finitely generated right ideal of S to M_S is extended to that of S.*

(4) ${}_S\mathrm{Hom}_R(M, Q)$ *is flat for every injevtive left R-module Q with essential socle.*

(5) ${}_S\mathrm{Hom}_R(M, K)$ *is flat for every injective cogenerator K with essential socle.*

(6) ${}_S\mathrm{Hom}_R(M, K_0)$ *is flat for some injective cogenerator K_0 with essential socle.*

In case where $S=\mathrm{End}({}_RM)$, *the endomorphism ring of ${}_RM$, the above statements* (1)$\sim$(6) *are equivalent also to*

(7) *${}_RM$ cogenerates the cokernel of every homomorphism ${}_RM^m \to {}_RM^n$, where m, n are arbitrary natural numbers. (Here one can set* $m=1$).

In order to prove the theorem we need the following

LEMMA[3]. *Let A_S be a finitely generated right S-module, ${}_RM_S$ be an*

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1) In what follows it is assumed that all rings have an identity element and all modules are unital.

2) A left R-module is called linearly compact if every finitely solvable system of congruences $x \equiv m_\alpha (\mathrm{mod}\, M_\alpha)$, $\alpha \in \Lambda$, is solvable where $m_\alpha \in M$ and M_α are submodules of M.

3) Cf. [1], Lemma 2, also [4], Lemma 3.5.

R-S-bimodule such that $_RM$ is linearly compact, and, $_RQ$ be an injective left R-module with essential socle. Then $A\underset{S}{\otimes}\mathrm{Hom}_R(M, Q)$ *and* $\mathrm{Hom}_R(\mathrm{Hom}_S(A, M), Q)$ *are isomorphic under the isomorphism* θ_A:

$$\theta_A:\ A\underset{S}{\otimes}\mathrm{Hom}_R(M, Q)\ni\sum_i a_i\otimes f_i\longrightarrow$$
$$\Big(\mathrm{Hom}_S(A, M)\ni g\longrightarrow\sum_i f_i\big(g(a_i)\big)\in Q\Big),$$
$$a_i\in A,\qquad f_i\in\mathrm{Hom}_R(M, Q).$$

PROOF. At first we show that the mapping θ_A is a monomorphism. Let $A=\sum_{i=1}^n a_iS$ and $\sum_{i=1}^n a_i\otimes f_i$ be an element of the kernel of θ_A : $\theta_A(\sum_{i=1}^n a_i\otimes f_i)=0$. This implies that $\sum_{i=1}^n f_i(g(a_i))=0$ for all $g\in\mathrm{Hom}_S(A, M)$. Let $\mathscr{U}:=\left\{(s_1, s_2, \cdots, s_n)\in S^n\,\middle|\,\sum_{i=1}^n a_is_i=0\right\}$ and $\mathscr{U}^{\perp}:=\left\{(x_1, x_2, \cdots, x_n)\in M^n\,\middle|\,\sum_{i=1}^n x_is_i=0 \text{ for all } (s_1, s_2, \cdots, s_n)\in\mathscr{U}\right\}$. Let $(y_1, y_2, \cdots, y_n)$ be an element of $\mathscr{U}^{\perp}$. Then the mapping $g: A\ni\sum_{i=1}^n a_is_i\rightarrow\sum_{i=1}^n y_is_i\in M$ is a (well defined) homomorphism of A_S into M_S such that $g(a_i)=y_i$, $i=1, 2, \cdots, n$. It follows that $\sum_{i=1}^n f_i(y_i)=0$. Let φ be the homorphism of $_RM^n$ into $_RQ$ defined by $\varphi((x_1, x_2, \cdots, x_n))=\sum_{i=1}^n f_i(x_i)$. Then we have Ker $\varphi\supseteq\mathscr{U}^{\perp}$. Since $_RM$, whence $_RM^n$, is linearly compact and $M^n/\mathrm{Ker}\,\varphi\cong\varphi(M^n)\subseteq Q$ is cofinitely generated[4], there exist a finite number of elements $\mathscr{S}_j:=(s_1, s_2, \cdots, s_n)$, $j=1, 2, \cdots, l$, of $\mathscr{U}$ such that $\ker\varphi\supseteq\bigcap_{j=1}^l\mathscr{S}_j^{\perp}$, where $\mathscr{S}_j^{\perp}:=\left\{(x_1, x_2, \cdots, x^n)\in M^n\,\middle|\,\sum_{i=1}^n x_is_i^{(j)}=0\right\}$[5]. Then, since $_RQ$ is injective, the well defined mapping $((\mathscr{X}, \mathscr{S}_1), (\mathscr{X}, \mathscr{S}_2), \cdots, (\mathscr{X}, \mathscr{S}_l))\rightarrow\sum_{i=1}^n f_i(x_i)\in Q$, where $\mathscr{X}=(x_1, x_2, \cdots, x_n)\in M^n$ and $(\mathscr{X}, \mathscr{S}_j)=\sum_{i=1}^n x_is_i^{(j)}$, is extended to a homorphism of $_RM^l$ into $_RQ$. Thus there exist homomorphisms $g_1, g_2, \cdots, g_l\in\mathrm{Hom}_R(M, Q)$ such that $\sum_{i=1}^n f_i(x_i)=\sum_{j=1}^n g_j\Big(\sum_{i=1}^n x_is_i^{(j)}\Big)=\sum_{i=1}^n\Big(\sum_{j=1}^l s_i^{(j)}g_j\Big)(x_i)$ for $(x_1, x_2, \cdots, x_n)\in M^n$. It follows that $f_i=\sum_{j=1}^l s_i^{(j)}\cdot g_j$, $i=1, 2, \cdots, n$. Then we have $\sum_{i=1}^n a_i\otimes f_i=\sum_{i=1}^n a_i\otimes\Big(\sum_{j=1}^l s_i^{(j)}\cdot g_j\Big)=\sum_{j=1}^l\Big(\sum_{i=1}^n a_is_i^{(j)}\Big)\otimes g_j=0$. Thus θ_A is a monomorphism. Next we show that θ_A is an epimorphism. Let α be a element of $\mathrm{Hom}_R(\mathrm{Hom}_S(A, M)$,

4) Cf. [2], Propositions 1, 3, 5.
5) Cf. [1], Theorem 6.

$Q)$. Then, since ${}_RQ$ is injective, the well defined mapping

$$\big(g(a_1), g(a_2), \cdots, g(a_n)\big) \longrightarrow \alpha(g) \in Q\,, \qquad \text{where } g \in \mathrm{Hom}_S(A, M)\,,$$

is extended to a homorphism of ${}_RM^n$ to ${}_RQ$. It follows that there exist $f_1, f_2, \cdots, f_n \in \mathrm{Hom}_R(M, Q)$ such that $\alpha(g) = \sum_{i=1}^{n} f_i(g(a_i))$, $g \in \mathrm{Hom}_S(A, M)$. This implies that $\alpha = \theta_A\Big(\sum_{i=1}^{n} a_i \otimes f_i\Big)$. Thus θ_A is an epimorphism.

Proof of the theorem. The implications (1)$\Rightarrow$(2)$\Rightarrow$(3) are clear. (3)$\Rightarrow$(4). Let $\mathfrak{a}$ be a finitely generated right ideal of S and

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\ \iota\ } S \xrightarrow{\ \nu\ } S/\mathfrak{a} \longrightarrow 0$$

be the canonical exact sequence. Then, since M_S is semi S-injective and ${}_RQ$ is injective, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathfrak{a} \otimes_S \mathrm{Hom}_R(M, Q) & \longrightarrow & S \otimes_S \mathrm{Hom}_R(M, Q) & \longrightarrow & S/\mathfrak{a} \otimes_S \mathrm{Hom}_R(M, Q) & \longrightarrow & 0 \\
\Big\downarrow \theta_{\mathfrak{a}} & & \Big\downarrow \theta_S & & \Big\downarrow \theta_{S/\mathfrak{a}} & & \\
0 \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_S(\mathfrak{a}, M), Q) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S, M), Q) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S/\mathfrak{a}, M), Q) & \longrightarrow & 0\,.
\end{array}$$

Here, by the above lemma, the vertical arrows are all isomorphisms. It follows that $\iota \otimes 1_{\mathrm{Hom}_R(M,Q)} : \mathfrak{a} \otimes_S \mathrm{Hom}_R(M, Q) \to S \otimes_S \mathrm{Hom}_R(M, Q)$ is a monomorphism. Thus ${}_S\mathrm{Hom}_R(M, Q)$ is a flat left S-module. (4)$\Rightarrow$(5)$\Rightarrow$(6) are clear. (6)$\Rightarrow$(3). Suppose that ${}_S\mathrm{Hom}_R(M, K_0)$ is flat. Let $\mathfrak{a}$ be a finitely generated right ideal of S and

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\ \iota\ } S \xrightarrow{\ \nu\ } S/\mathfrak{a} \longrightarrow 0$$

be the canonical exact sequence. Then we have the following exact sequence:

$$0 \longrightarrow \mathfrak{a} \otimes_S \mathrm{Hom}_R(M, K_0) \longrightarrow S \otimes_S \mathrm{Hom}_R(M, K_0) \longrightarrow S/\mathfrak{a} \otimes_S \mathrm{Hom}_R(M, K_0) \longrightarrow 0\,.$$

It follows by the above lemma that we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_S(\mathfrak{a}, M), K_0) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_S(S, M), K_0) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_S(S/\mathfrak{a}, M), K_0) \longrightarrow 0\,.$$

Since ${}_RK_0$ is an injective cogenerator, this means that the sequence

$$0 \longrightarrow \mathrm{Hom}_S(S/\mathfrak{a}, M) \xrightarrow{\mathrm{Hom}\,(\nu, 1_M)} \mathrm{Hom}_S(S, M) \xrightarrow{\mathrm{Hom}\,(\nu, 1_M)} \mathrm{Hom}_S(\mathfrak{a}, M) \longrightarrow 0$$

is exact. It follows that M_S is semi S-injective. (3)$\Rightarrow$(1). Let $\mathfrak{r}$ be a right ideal of S and φ be a homomorphism of $\mathfrak{r}$ into M_S. For each element s of $\mathfrak{r}$, since M_S is semi S-injective, there is an element m_s of M such that $\varphi(s)=m_s s$. Then the system of congruences:

$$x \equiv m_s\left(\mathrm{mod}\ \mathrm{Ann}_M(s)\right), \qquad s\in\mathfrak{r},$$

where $\mathrm{Ann}_M(s):=\{x\in M| xs=0\}$, is finitely solvable, because again M_S is semi S-injective. Since ${}_RM$ is linearly compact there exists a solution m_0 for the system of congruences. Then we have $m_0 s=m_s s=\varphi(s)$ for all $s\in\mathfrak{r}$. Thus M_S is injective. The equivalence (7)$\Leftrightarrow$(2) is obtained in [5] ([5], 1.8 Satz).

From our theorem we have the following

COROLLARY[6]. *Let R be a ring such that ${}_RR$ is linearly compact. Then the following statements are equivalent:*

(1) *R_R is injective.*
(2) *R_R is absolutely pure.*
(3) *R_R is semi R-injective.*
(4) *Every injective left R-module with essential socle is flat.*
(5) *Every cofinitely generated injective left R-module is flat.*
(6) *Every colocal[7] injective left R-module is flat.*
(7) *Every injective cogenerator with essential socle is flat.*
(8) *There is a flat injective cogenerator with essential socle.*
(9) *${}_RR$ cogenerates the cokernel of every homomorphism $R^m\to R^n$, where m, n are arbitrary natural numbers. (Here one can set $m=1$).*

PROOF. Since R is semi-perfect[8], the number of non-isomorphic simple left R-modules is finite. Thus there is a cofinitely generated injective cogenerator. Then it is easy to see that the equivaleces (5)$\Leftrightarrow$(6)$\Leftrightarrow$(8) hold. The rest of the proof follows direct from the theorem.

6) Cf. [1], Théorème 2.
7) Cf. [3], Satz 1′.
8) Cf. [2], Corollary to Theorem 5.

References

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