Hokkaido Mathematical Journal Vol. 8 (1979) p. 103-114

Finite groups admitting an automorphism of prime order I

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(Received April 18, 1978; Revised October 4, 1978)

1. Introduction

Let G be a finite group and q a prime. We say that G is q-closed if G has a normal Sylow q-subgroup and q-nilpotent if G has a normal q-complement. In this paper we prove the following theorem.

THEOREM. *Let G be a finite group. Assume that G admits an automorphism α of order p, p a prime. Assume further that $C_G(\alpha)$ is a cyclic q-group for some odd prime q distinct from p. Then G is q-closed or q-nilpotent. In particular G is solvable.*

B. Rickman [8] prove the case $q\geqslant 5$, so we prove the case $q=3$.

2. Preliminaries

All groups considered in this paper are assumed finite. Our notation corresponds to that of Gorenstein [5].

(2.1) *Let A be a π'-group of automorphism of the π-group G, and suppose G or A is solvable. Then for each prime p in π, we have*

(1) *A leaves invariant some S_p-subgroup of G.*

(2) *Any two A-invariant S_p-subgroups of G are conjugate by an element of $C_G(A)$.*

(3) *Any A-invariant p-subgroup of G is contained in an A-invariant S_p-subgroup of G.*

(4) *If H is any A-invariant normal subgroup of G, then $C_{G/H}(A)$ is the image of $C_G(A)$ in G/H.*

(2.2) (Thompson)

A p-group P posseses a characteristic subgroup C with the following properties;

(1) *$cl(C)\leqslant 2$ and $C/Z(C)$ is elementary abelian.*

(2) $[P, C]\subseteq Z(C)$.

(3) $C_P(C)=Z(C)$.

(4) *Every nontrivial p'-automorphism of P induces a nontrivial automorphism of C.*

(2.3) *If A is a p'-group of automorphisms of the p-group P with p odd which acts trivially on $\Omega_1(P)$, then $A=1$.*

(2.4) *Let P be a p-group of class at most 2 with p odd. Then $\Omega_1(P)$ is of exponent p.*

(2.5) (Clifford)

Let V/F be an irreducible G-module and let H be a normal subgroup of G. Then V is the direct sum of H-invariant subspaces V_i, $1\leqslant i\leqslant r$, which satisfy the following conditions;

(1) *$V_i=X_{i1}\oplus X_{i2}\oplus\cdots\oplus X_{it}$, where each X_{ij} is an irreducible H-submodule, $1\leqslant i\leqslant r$, t is independent of i, and X_{ij}, $X_{i'j'}$ are isomorphic H-modules if and only if $i=i'$.*

(2) *For x in G, the mapping $\pi(x)$; $V_i\rightarrow V_ix$, $1\leqslant i\leqslant r$, is a permutation of the set $S=\{V_1, \cdots, V_r\}$ and π induces a transitive permutation representation of G on S.*

(2.6) (Thompson)

Assume G is a finite group admitting a fixed point free automorphism of prime order. Then G is nilpotent.

(2.7) (Shult)

Let $G=NQP$ with $N\rhd G$, $Q\rhd QP$, $|P|$ is a prime, $|Q|$ is an odd and $(|Q|, |P|)=1$, $(|N|, |Q|)=1$. Assume further that $C_N(P)=1$. Then $[P, Q]\subseteq C_Q(N)$.

(2.8) (Thompson Transitivity Theorem)

Let G be a group in which the centralizer of every p-element is p-constrained. Then if $A\in SCN_3(P)$, $C_G(A)$ permutes transitively under conjugation the set of all maximal A-invariant q-subgroups of G for any prime $q\neq p$.

(2.9) *Let G be a group in which the centralizer of every p-element is p-constrained. Let P be an S_p-subgroup of G and let A be an element of $SCN_3(P)$. Then for any prime $q\neq p$, P normalizes some maximal A-invariant q-subgroup of G.*

(2.10) (Glauberman)

Let G be a group, and P be an S_p-subgroup of G. If $p\geqslant 5$, $P\neq 1$, and $N_G(P)/C_G(P)$ is a p-group, then G has a factor group of order p.

Suppose p is an odd prime and P is an S_p-subgroup of G. A normal subgroup T of P is said to control strong fusion in P if T has the following property.

"Whenever $W\subseteq P$, $g\in G$, and $W^g\subseteq P$, then there exist $c\in C_G(W)$ and $n\in N_G(T)$ such that $cn=g$."

Define the quadratic group for the prime p to be the semidirect product $Qd(p)$ of a two dimentional vector space V over $GF(p)$ by the special linear group $SL(V)$ on V. Let $F(p)$ be the normalizer of some S_p-subgroup of $Qd(p)$.

(2. 11) (Glauberman)

If $F(p)$ is not involved in $N_G(Z(J(P)))$, then $Z(J(P))$ controls strong fusion in P with respect to G.

(2. 12) (Glauberman)

Let G be a non-abelian simple group. Assume that S_4 is not involved in G Then. G is a JR-group, $L_2(q)$, $q\equiv 3, 5\ (8)$, $L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$.

(2. 13) (Signalizer functor theorem)

Let A be an elementary abelian p-subgroup of G of rank at least 3. If G possesses the solvable A-signalizer functor θ, then the subgroup $<\theta(C_G(a))|a\in A^{\#}>$ of G is a solvable p'-group.

(2. 14) (*Gorenstein, Walter*)

Let G be a group with $O(G)=1$ and $SCN_3(2)\neq\phi$. Assume further that the centralizer of every involution of G is 2-constrained. Then $O(C_G(x))=1$ for every involution x of G.

3. The structure of solvable groups satisfying the hypothesis of the theorem

LEMMA 3. 1. *Let G be a solvable group admitting an automorphism α prime order p fixing a cyclic q-group for some odd prime q distinct from p. Then G is q-closed or q-nilpotent.*

PROOF. Suppose false and G be a minimal counterexample. First of all we prove that $G=O_{q,q'}(G)\,C_G(\alpha)$. We may assume that $O_q(G)=1$. Let Q be a α-invariant S_q-subgroup of G. By (2. 7) we have that $[Q,\alpha]\subseteq C_G(O_{q'}(G))\subseteq O_{q'}(G)$. Hence $Q=C_Q(\alpha)$. Let Q_0 be a subgroup of Q and M be a α-invariant Hall q'-subgroup of $N_G(Q_0)$. Let $y\in N_G(Q_0)$ and $x\in Q_0$. Then $(y^{-1})^{\alpha}xy^{\alpha}=(y^{-1}xy)^{\alpha}=y^{-1}xy$, this implies that $[y^{\alpha}y^{-1}, x]=1$. Since $M=[M,\alpha]$, we have that $[M,Q_0]=1$. Hence $N_G(Q_0)/C_G(Q_0)$ is a q-group. Hence G has a normal q-complement and $G=O_{q,q'}(G)C_G(\alpha)$. Let U be a α-invariant Hall q'-subgroup of G. Assume $[O_q(G), U]=1$. Then G is q-nilpotent, a contradiction. So we have $[O_q(G), U]\neq 1$. Hence $C_{O_q(G)}(\alpha)\neq 1$. Next we prove that $\Phi(O_q(G))=1$. Assume $\Phi(O_q(G))\neq 1$. By the minimality of G, $G/\Phi(O_q(G))$ is q-closed or q-nilpotent. Assume $G/\Phi(O_q(G))$ is q-closed. Then G is q-closed, hence $G/\Phi(O_q(G))$ is q-nilpotent. Hence $[O_q(G), U]\subseteq\Phi(O_q(G))$, it follows that $[U, O_q(G)]=1$, a contradiction. Hence $\Phi(O_q(G))=1$. By the

Frattini argument, $G=O_q(G)\,N_G(U)$ since $G=O_{q,q'}(G)\,C_G(\alpha)$. Hence $C_{N_G(U)}(\alpha)\neq 1$. Let $\langle g\rangle=\Omega_1(C_G(\alpha))$, then $g\in N_G(U)$. By Theorem 5.2.3 of [5], $O_q(G)=[O_q(G), U]\times C_{O_q(G)}(U)$. Since $[g, U]\subseteq U\cap O_q(G)=1$, $[O_q(G), U, U]=1$, this implies $[O_q(G), U]=1$, a contradiction.

4. The proof of the theorem

Let G be a minimal counterexample to the Theorem and assume $q=3$.

LEMMA 4.1. *G is simple.*

PROOF. By minimality of G, G is characteristic simple. Hence $G=G_1\times\cdots\times G_n$ where the G_i is non-abelian simple. Any normal non-abelian simple subgroup of G coincide with one of the G_i $1\leqslant i\leqslant n$. Since $G_1^\alpha\rhd G$, $G_1^\alpha=G_i$ for some i. Assume that $G_1^\alpha=G_1$. Then by minimality of G, $G=G_1$, which implies the conclusion of the Lemma 4.1. Hence we may assume that $G_1^\alpha\neq G_1$. Since $G_1\times G_1^\alpha\times\cdots\times G_1^{\alpha^{p-1}}\subseteq G$, $C_G(\alpha)$ is non-solvable, which is a contradiction since $C_G(\alpha)$ is cyclic.

LEMMA 4.2. *Let $^\forall r\in\pi(G)-\{2,3\}$. Then for any r-subgroup R_0 of G, $N_G(R_0)/C_G(R_0)$ is a $\{3,r\}$-group whose S_3-subgroups are cyclic.*

PROOF. Let R be a α-invariant S_r-subgroup of G. Then $N_G(R)$ is solvable. Let V be a α-invariant Hall $\{3,r\}'$-subgroup of $N_G(R)$. Then $[V, R]=1$ since $C_{VR}(\alpha)=1$. Let Q_0 be a α-invariant S_3-subgroup of $N_G(R)$. By (2.7), $[Q_0,\alpha]\subseteq C_{Q_0}(R)$. Hence $N_G(R)$ $C_{Q_0}(\alpha)\,RC_G(R)$, which implies that $N_G(R)/RC_G(R)$ is a cyclic 3-group. Next we prove that $N_G(Z(J(R)))=N_G(R)$. Suppose false. If $N_G(Z(J(R)))$ is 3-nilpotent, then $N_G(Z(J(R)))=N_G(R)$, a contradiction. If $N_G(Z(J(R)))$ is 3-closed, then $R\subseteq N_G(Q)$, where Q is a α-invariant S_3-subgroup of G, so $Q_0\subseteq Q$. Then $N_G(R)/C_G(R)$ is a r-group since $[Q_0, R]\subseteq R\cap Q=1$. By (2.10) G is non-simple, a contradiction. So we have $N_G(Z(J(R)))=N_G(R)$. By (2.11) $Z(J(R))$ controls strong fusion in R since $F(r)$ is not involved in $N_G(Z(J(R)))$. Hence if $x\in N_G(R_0)$, then there exist $c\in C_G(R_0)$ and $n\in N_G(Z(J(R)))$ such that $x=cn$. Hence we have the conclusion of Lemma 4.2.

LEMMA 4.3. *Let X be a finite group. For each $r\in\pi(X)-\{2,3\}$, assume that $N_G(R_0)/C_G(R_0)$ is odd order for any r-subgroup R_0 of X and that $L_3(3)$ and $L_2(7)$ are not involved in X. Then X is solvable.*

PROOF. Let X be a minimal counterexample. If there exists a non-trivial proper normal subgroup K of X, then X/K and K is solvable since X/K and K satisfy the hypothesis of Lemma 4.3, this implies that X is solvable, a contradiction. So X is a minimal simple group since proper subgroups are solvable. By N-paper [11] X is $L_2(q)$, $Sz(2^n)$ or $L_3(3)$. By the hypothesis of Lemma 4.3, X is $L_2(q)$ $(q\neq 7)$ or $Sz(2^n)$. But $L_2(q)$ $(q\neq 7)$ and $Sz(2^n)$ have a r-group R_0 such that $N_G(R_0)/C_G(R_0)$ is even order for some $r\in\pi(X)-\{2, 3\}$, a contradiction. Hence X is solvable.

By Lemma 4.3 we may assume that $L_3(3)$ or $L_2(7)$ is involved in G. Let S be a α-invariant S_2-subgroup of G and Q be a α-invaiant S_3-subgroup of G. Let S_0 be a α-invariant subgroup of $N_G(Q)$.

LEMMA 4.4. *$N_G(Q)/C_G(Q)$ is a non-trivial elementary 2-group and $N_G(Q)$ is a maximal α-invariant subgroup of G.*

PROOF. Assume that $N_G(Z(J(Q)))\supsetneqq N_G(Q)$, then $N_G(Z(J(Q)))$ is 3-nilpotent. Hence $N_G(Z(J(Q)))$ is $F(3)$-free. By (2.11) $Z(J(Q))$ controls strong fusion in Q. Hence S_4 is not involved in G. By (2.12) G is a JR-group, $L_2(q)$, $q\equiv 3, 5\ (8)$, $L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$. But such simple groups have not an automorphism which satisfy the hypothesis of the Theorem, a contradiction. Hence we have that $N_G(Z(J(Q)))=N_G(Q)$. If $N_G(Q)$ is not a maximal α-invariant subgroup of G, then $N_G(Q)$ is 3-nilpotent. Hence $N_G(Z(J(Q)))$ is 3-nilpotent, a contradiction. Therefore $N_G(Q)$ is a maximal α-invariant subgroup of G. Assume that $N_G(Q)/C_G(Q)$ is odd order, then we have similarly prove that S_4 is not involved in G. Hence $N_G(Q)/C_G(Q)$ is even order. Let L be a α-invariant Hall $3'$-subgroup of $N_G(Q)$. Then L is nilpotent by (2.6). We set $\bar{Q}=Q/\Phi(Q)$. By Maschke's theorem $\bar{Q}=\bar{Q}_0\oplus\bar{Q}_1\oplus\cdots\oplus\bar{Q}_n$, where $\bar{Q}_i$ is $\langle\alpha\rangle L$-irreducible, $1\leqslant i\leqslant n$. We may assume that $C_{\bar{Q}_i}(\alpha)=1$ for $i=1, \cdots n$, since $C_{\bar{Q}}(\alpha)$ is cyclic. Hence $[L, \bar{Q}_i]=1$ for $i=1, \cdots, n$. By (2.5) $\bar{Q}_0$ is the direct sum of L-invariant subspace V_i, $1\leqslant i\leqslant r$, such that $V_i=X_{i1}\oplus\cdots\oplus X_{it}$, where each X_{ij} is an irreducible L-submodule, $1\leqslant i\leqslant t$, and X_{ij}, $X_{i'j'}$ are isomorphic L-module if and only if $i=i'$. Assume that $r=1$, then $Z(L/C_L(Q_0))$ is a α-invariant cyclic group of even order. Hence $C_G(\alpha)$ is even order, a contradiction. Since $\langle\alpha\rangle$ induces a transitive permutation of the set $\{V_1, \cdots, V_r\}$ by (2.5), we have $\bar{Q}_0=V_1\oplus V_1^{\alpha}\oplus\cdots\oplus V_1^{\alpha^{p-1}}$, where $V_1^{\alpha^j}$ coincides with one of the V_i, $1\leqslant i\leqslant r$, for $j=0, \cdots, p-1$. Since $C_{Q_0}(\alpha)$ is cyclic, $|V_1|=3$, this implies that $L/C_L(Q)$ is elementary 2-group. Hence $N_G(Q)/QC_G(Q)$ is an elementary 2-group.

LEMMA 4.5. $C_{N_G(S)}(\alpha)=1$. *In particular $N_G(S)$ is nilpotent and $\{2, 3\}$-group.*

PROOF. Suppose that $C_{N_G(S)}(\alpha)\neq 1$. We set $\Omega_1(C_G(\alpha))=\langle g\rangle$, then $g\in N_G(S)$. Let S_0 be a α-invariant S_2-subgroup of $N_G(Q)$, then by Lemma 4.4 $[S_0, Q]\neq 1$. By (2.2) there exists a characteristic subgroup C of Q such that class $C\leqslant 2$ and $[S_0, C]\neq 1$. By (2.3) $[S_0, \Omega_1(C)]\neq 1$, and $\Omega_1(C)$ is of exponent 3 by (2.4). If $g\notin\Omega_1(C)$, then $[S_0, \Omega_1(C)]=1$, a contradiction, hence

$g\in\Omega_1(C)$. On the other hand $[S_0, g]\subseteq S\cap Q=1$. $\langle\alpha\rangle S_0$ acts on $D=\Omega_1(C)/\Phi(\Omega_1(C))$. Since $\bar{g}\in C_D(S_0)$, α acts fixed point free on $D/C_D(S_0)$, hence $[S_0, D]\subseteq C_D(S_0)$, this implies that $[S_0, D]=1$, which implies $[S_0, \Omega_1(C)]=1$, a contracdiction. Hence $C_{N_G(S)}(\alpha)=1$. In particular $N_G(S)$ is nilpotent. Next assume that $N_G(S)$ is not $\{2, 3\}$-group, then there exists an element $r\in\pi(N_G(S))-\{2, 3\}$. Let R be a α-invariant S_r-subgroup of G. $N_G(S)=N_G(R)$ is nilpotent. By (2.10) G is non-simple, which is a contradiction.

Let P be a α-invariant S_{13}-subgroup of G and $\langle g\rangle=\Omega_1(C_G(\alpha))$.

LEMMA 4.6. *Assume $P\neq 1$, then the followings hold*;

(i) $g\in N_G(P)$,

(ii) $C_P(g)=1$.

PROOF. Assume $g\notin N_G(P)$, then $N_G(P)$ is nilpotent, which implies G is non-simple by (2.10), a contradiction. Next we prove that $C_P(g)=1$. Suppose false. We set $P_0=C_P(g)\neq 1$. Let M be a maximal α-invariant subgroup of G which contains $C_G(g)$, then M is 3-closed or 3-nilpotent. If M is 3-closed, then $P_0\subseteq N_G(Q)$, this implies that $N_G(S)=N_G(P)$ by Lemma 4.4, a contradiction. Hence M is 3-nilpotent and we deduce that $M=N_G(P)$. Assume that $g\in Z(Q)$, then $Q\subseteq N_G(P)$. Hence $[Q, \alpha]\subseteq C_Q(P)$, which implies that $[\Omega_1(Z(Q)), P_0]=1$. Since $N_G(Q)$ is a maximal α-invariant subgroup of G, $P_0\subseteq N_G(Q)$, a contradiction. Hence $g\notin Z(Q)$. This implies that $[Z(Q), P]=1$. Hence $P\subseteq N_G(Q)$, a contradiction.

LEMMA 4.7. *$C_G(x)$ is 13-nipotent for each $x\in P^{\#}$.*

PROOF. By taking a conjugation of x we may assume that $C_P(x)$ is a S_{13}-subgroup of $C_G(x)$. Let P_0 be a non-trivial 13-subgroup of $C_P(x)$. We set $P_1=\langle x\rangle P_0$. Assume that $N_{C_G(x)}(P_0)/C_{C_G(x)}(P_0)$ is not a 13-group. Then there exists an element y such that $y\in N_{C_G(x)}(P_0)-C_{C_G(x)}(P_0)$ and y is a 13'-element. This implies that $y\in N_G(P_1)-C_G(P_1)$. Assume that $N_G(Z(J(P)))\supsetneqq N_G(P)$, then $N_G(P)$ is nilpotent, a contradiction. Hence $N_G(Z(J(P)))=N_G(P)=C_{N_G(P)}(\alpha)PC_G(P)$. Since $F(13)$ is not involved in $N_G(Z(J(P)))$, $Z(J(P))$ controls strong fusion in P. Hence there exists $c\in C_G(P_1)$ and $n\in N_G(Z(J(P)))$ such that $y=cn$. Since $N_G(P)=C_{N_G(P)}(\alpha)PC_G(P)$, we may assume $n\in C_{N_G(P)}(\alpha)$. By Lemma 4.6 $n=1$ since $C_P(g)=1$, which contradicts the choice of y. Hence $N_{C_G(x)}(P_0)/C_{C_G(x)}(P_0)$ is a 13-group. Hence $C_G(x)$ is 13-nilpotent.

In particular $C_G(x)$ is 13-constrained for each $x\in P^{\#}$ by Lemma 4.7. Assume that $P\neq 1$ and $Z(P)$ is cyclic, then $p(=|\alpha|)$ is 2 or 3. Hence G is odd order or a 3'-group, a contradiction. Hence we may assume that $P=1$ or $Z(P)$ is a non-cyclic group.

1. The case $SCN_3(P)\neq\phi$

LEMMA 4.8. *$C_G(x)$ is a $\{2.3\}'$-group for each $x\in P^{\#}$.*

PROOF. Suppose false. Then there exists an element $x\in P^{\#}$ and r such that $r\in\pi(C_G(x))$, where $r=2$ or 3. Since $Z(P)$ is a non-cyclic group, we may assume that $x\in Z(P)$. Then P normalizes some S_r-subgroup of $C_G(x)$ since $C_G(x)$ is 13-nilpotent. Let $A\in SCN_3(P)$. By Transitivity Theorem $C_G(A)$ permutes transitively under conjugation the set of all maximal A-invariant r-subgroup. Then all maximal A-invariant r-subgroups are P-invariant since $C_G(A)\subseteq C_G(Z(P))\subseteq N_G(P)$. Since α permutes maximal P-invariant r-subgroups and the number of maximal P-invariant r-subgroups is coprime to 13, α invariants some maximal P-invariant r-subgroup. Let W be a $\langle\alpha\rangle P$-invariant r-subgroup. If $r=2$, then $N_G(P)$ is nilpotent since $N_G(P)=N_G(S)$, a contradiction. Next we assume $r=3$. Let M be a maximal α-invariant subgroup of G which contains $N_G(W)$. If M is 3-closed, then $P\subseteq N_G(Q)$, a contradiction. Hence M is 3-nilpotent and so $M=N_G(P)$. By (2.7) $[Z(Q),\alpha]\subseteq C_Q(P)$. Assume that $[Z(Q),\alpha]=1$, then $[S_0, Z(Q)]=1$. Since $g\in Z(Q)$, $[S_0, Q]=1$, a contradiction. Hence we may assume that $[Z(Q),\alpha]\neq 1$. Next we prove that $C_{Z(Q)}(S_0)=1$. Suppose false. Let M be a maximal α-invariant subgroup of G which contains $N_G(S_0)$. Since $C_{Z(Q)}(S_0)\subseteq M$ and $N_G(S)$ is nilpotent M is 3-closed. Hence $N_S(S_0)=S_0$, this implies $S=S_0$. Hence we see $S\subseteq N_G(Q)$, in particular $C_{Z(Q)}(S)\neq 1$. By Glauberman's weakly closed elements theorem [2] $C_{Z(Q)}(S)$ is weakly closed in Q with respect to G since $C_{Z(Q)}(S)\subseteq Z(N_G(J(Q)))$. Let $z\in\Omega_1(Z(S))^{\#}$. By Z^*-theorem there exists an element $x(\neq z)$ of S *such* that x is conjugate to z in G. Then there exists an element $k\in G$ and subgroup H of S such that $z^k=x$ and $k\in N_G(H)$, z, $x\in H$. Since $C_{Z(Q)}(S)$ is weakly closed in S, $N_G(H)=C_G(H)N_{N_G(H)}(C_{Z(Q)}(S))$ by the Frattini argument. Then we may assume $k\in N_G(C_{Z(Q)}(S))\subseteq N_G(Q)$. Hence $z=z^k=x$, a contradiction. Hence $C_{Z(Q)}(S_0)=1$. By (2.5) $\Omega_1(Z(Q))=\langle a\rangle\oplus\langle a^{\alpha}\rangle\oplus\cdots\oplus\langle a^{\alpha^{p-1}}\rangle$, where $\langle a^{\alpha^i}\rangle$ is a Wedderburn component, $0\leqslant i\leqslant p-1$. Let $v\in S_0^{\#}$. If $a^v=a^{-1}$, $(a^{\alpha^i})^v=a^{\alpha^i}$ for $i=1,\cdots,p-1$, then $a^{vv^{\alpha}}=a^{-1}$ and $(a^{\alpha})^{vv^{\alpha}}=a^{-\alpha}$. We set $b=a^{-1}a^{\alpha}$, then $b^w=b^{-1}$ and $b\in[Z(Q),\alpha]$. By the Frattini argument $N_G(\langle b\rangle)=C_G(b)N_{N_G(\langle b\rangle)}(P)$. Hence $N_G(P)$ is even order, this implies $N_G(S)=N_G(P)$, a contradiction. Hence $C_G(x)$ is a $\{2,3\}'$-group for each $x\in P^{\#}$.

LEMMA 4.9. *$C_G(t)$ is solvable for every 2-element and 3-element t of G. In particular $O(C_G(x))=1$ for every involution x of G.*

PROOF. Let R be a α-invariant S_7-subgroup of G. Assume that $R\neq 1$ and $d(Z(R))\leqslant 2$, then $p=2$ or 3. Then G is odd order or $3'$-group, a contradiction. Hence we may assume that $R=1$ or $d(Z(R))\geqslant 3$. Assume $d(Z(R))\geqslant 3$. Then we can repeat the proof of Lemma 4.6, 4.7 and 4.8 verbatim with R in place of P to obtain that $C_G(y)$ is a $\{2, 3\}'$-group for each $y\in R^{\#}$. Hence $C_G(t)$ is a $\{7, 13\}'$-group for every 2-element and 3-element t of G. In particular $C_G(t)$ is solvable by Lemma 4.3. Assume $SCN_3(2)=\phi$. Then $|\Omega_1(Z(S))|\leqslant 4$. Hence p $(=|\alpha|)=3$, a contradiction. Hence we may assume $SCN_3(2)\neq\phi$. By (2.14) $O(C_G(x))=1$ for every involution x of G. Assume $R=1$.

Then $C_G(t)$ is a $\{7, 13\}'$-group for every 2-element and 3-element t of G is a $7'$-group. Hence Lemma 4.9 is proved.

LEMMA 4.10. *$O_{3'}(C_G(x))$ is odd order for every element x of $Q^{\#}$.*

PROOF. Suppose false. Then there exists an element x of $Q^{\#}$ such that $O_{3'}(C_G(x))$ is even order. Since $Z(Q)$ is non-cyclic and the centralizer of every non-trivial 3-element is solvable, we may assume that $x\in Z(Q)$. By (2.10) $W=\langle O_{3'}(C_G(x))|x\in Z(Q)^{\#}\rangle$ is a solvable $3'$-group of G. Then W is α-invariant and even order. Let S_1 be a $\langle\alpha\rangle Q$-invariant S_2-subgroup of W. Let K be a maximal α-invariant subgroup of G which contains $S_1 Q$. Suppose that K is 3-nilpotent, then $Q\subseteq N_G(S)$, a contradiction. Hence K is 3-closed. It follows $[S_1, Q]\subseteq S_1\cap Q=1$. Let L be a maximal α-invariant subgroup which contains $C_G(S_1)$. Then L is 3-closed. Hence $Z(S)\subseteq N_G(Q)$. If $C_{Z(S)}(Q)\neq 1$, then $S\subseteq N_G(Q)$. If $\Omega_1(Z(S))$ is weakly closed in S, then G is a JR-group, $L_2(q)$, $q\equiv 3, 5\ (8)$, $L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$, which is a contradiction. Hence $\Omega_1(Z(S))$ is not weakly closed in S with respect to G. Hence there exists an element $h\in G$ such that $h\in N_G(H)$ and $\Omega_1(Z(S))^h\neq\Omega_1(Z(S))$, $H=\langle\Omega_1(Z(S))^k|k\in\langle h\rangle\rangle\subseteq S$. If $[H, Q]=1$, then $N_G(H)=C_G(H)\, N_{N_G(H)}(Q)$. Thus we may assume that $h\in N_G(Q)$, this follows $\Omega_1(Z(S))^h=\Omega_1(Z(S))$, a contradiction. Hence we may assume $[\Omega_1(Z(S))^h, Q]\neq 1$. Since $\Omega_1(Z(S))$ is non-cyclic, $Q=\langle C_Q(x)|x\in\Omega_1(Z(S))^{h\#}\rangle$. Since $[\Omega_1(Z(S))^h, Q]\neq 1$, there exist elements x, $y\in\Omega_1(Z(S))^h$ and $a\in Q$ such that $[a, x]=1$ and $[a, y]\neq 1$. Then $y\in O_2(C_G(x))$ since $C_G(x)$ is solvable and $O(C_G(x))=1$, $y\in Z(S)^h$, S^h is a S_2-subgroup of $C_G(x)$. Hence $[a, y]\subseteq O_2(C_G(x))\cap Q=1$, a contradiction. Suppose $C_{Z(S)}(Q)=1$, then we have a contradiction by a similar argument. Hence $O_{3'}(C_G(x))$ is odd order for each each $x\in Q^{\#}$.

LEMMA 4.11. *G does not exist.*

PROOF. Since $N_S(Q)$ acts irreducibly on $\Omega_1(Z(Q))$, there exist elements $u\in N_S(Q)$ and a, $b\in\Omega_1(Z(Q))$ such that u centralizes $\langle a\rangle\times\langle b\rangle$ and u is an involution. Then $\langle a\rangle\times\langle b\rangle$ acts faithfully on $O_2(C_G(u))$ since $C_G(\mathrm{u})$ is solvable and $O(C_G(u))=1$. Hence we may assume that there exists an element $x\in$

$O_2(C_G(u))$ such that $[a, x]=1$ and $[b, x]\neq 1$ since $O_2(C_G(u))=\langle C_{O_2(C_G(u))}(d)| d\in\langle a\rangle\times\langle b\rangle^{\#}\rangle$. Since $b\in O_{3',3}(C_G(a))$ and $O_{3',3}(C_G(a))$ is odd order, $[b, x]\subseteq O_{3',3}(C_G(a))\cap O_2(C_G(u))=1$, a contradiction.

2. The case $SCN_3(\mathbf{P})=\phi$

Suppose $P=1$. Then $C_G(t)$ is a $\{7, 13\}'$-group for every 2-element and 3-element t of G since G is a $13'$-group. Hence Lemma 4.9 is satisfied. By Lemma 4.9 and 4.10, we have a contradiction. Hence $P\neq 1$. Suppose $Z(P)$ is a cyclic group, then $p=7$, in particular $L_2(7)$ is not involuved in G. Hence we may assume that $L_3(3)$ is involved in G. Since $SCN_3(P)=\phi$, $d_n(P)\leq 2$, which yields $\Omega_1(P)\subseteq Z(P)$.

LEMMA 4.12. *$g\in N_G(\langle x\rangle)$ for each $x\in\Omega_1(P)^{\#}$.*

PROOF. $\Omega_1(P)$ is normalized by $\langle\alpha\rangle\times\langle g\rangle$. By (2.5) the number of Wedderburn components of $\Omega_1(P)$ with respect to $\langle g\rangle$ is one since $C_{\Omega_1(P)}(\alpha)=1$. Then $\Omega_1(P)=P_1\oplus P_2$, where P_i is a $\langle g\rangle$-isomorphic cyclic subgroup of $\Omega_1(P)$ for $i=1, 2$, since g normalizes a cyclic subgroup of $\Omega_1(P)$. Hence g normalizes every cyclic subgroup of $\Omega_1(P)$.

LEMMA 4.13. $C_Q(S)=1$.

PROOF. Suppose false. We set $Q^*=C_Q(S)$, then $Q^*\neq 1$. In the first we prove that $C_G(x)$ is odd order for each $x\in P^{\#}$. Suppose false. Then there exists an element $x\in P^{\#}$ such that $C_G(x)$ is even order. P normalizes a $V\in S_2$-subgroup of $C_G(x)$ since $C_G(x)$ is 13-nilpotent. Let M be a maximal α-invariant subgroup which contains $N_G(Q^*)$. Suppose M is 3-nilpotent, then $N_G(Q)=N_G(S)$ is nilpotent, a contradiction. Hence $S\subseteq N_G(Q)$. If S is abelian, then G is JR-type or $L_2(q)$, $q\equiv 3, 5\ (8)$, $L_2(2^n)$, a contradiction. This follows that $C_S(Q)\neq 1$ since $S'\subseteq C_S(Q)$. We set $\Omega_1(F)=\langle x\rangle\times\langle y\rangle$, then y acts fixed point free on a Hall $\{2, 3\}$-subgroup W of $C_G(x)$ which contains V. Because suppose false, then $C_G(\Omega_1(P))$ is even order or $3||C_G(\Omega_1(P))|$. If $C_G(\Omega_1(P))$ is even order, then we see that $N_G(S)=N_G(P)$, a contradiction. If $3||C_G(\Omega_1(P))|$, then we have a contradiction by a similar argument of Lemma 4.8. Hence W is nilpotent. Since $O_{13'}(C_G(x))$ is solvable, $V\subseteq O_{\{2,3\}}(C_G(x))$. Since $W\cap O_{\{2,3\}}(C_G(x))$ is nilpotent, $V=O_2(C_G(x))$. Now we prove that $C_G(V)$ is 13-nilpotent. Suppose false. Since a S_{13}-subgroup of $C_G(V)$ is cyclic, we may assume that $N_{C_G(V)}(\langle x\rangle)/C_{C_G(V)}(x)$ is not a 13-group. Since $N_G(\langle x\rangle)=\langle g\rangle PO_{13'}(C_G(x))$, every S_3-subgroup of $N_G(\langle x\rangle)$ is written by $\langle g^k\rangle U$ for some $k\in N_G(\langle x\rangle)$ and $U\in S_3$-subgroup of $O_{13'}(C_G(x))$. Then $\langle g^k\rangle U\subset C_G(V)=U$ or $\langle g^k\rangle U$ since $[U, V]=1$. Suppose that $[g^k, V]=1$, then $[g, V]=1$ since $k\in N_G(\langle x\rangle)$ and $V\lhd N_G(\langle x\rangle)$. Since $\langle g\rangle\langle y\rangle$ is a Frobenius group, $[y, V]=1$, a contradiction. Hence every S_3-subgroup of $N_G(\langle x\rangle)C_G(V)$ is contained in $O_{13'}(C_G(x))$. Then $N_{C_G(V)}(\langle x\rangle)/C_{C_G(V)}(x)$ is a 13-group, a contradiction. Hence $C_G(V)$ is 13-nilpotent. By taking a conjugation of V, we may assume that $V\subseteq S$. Then $Q^*\subseteq C_G(V)$ and $h\in C_G(V)$, where h is a non-trivial 13-element. Let Q_0 be a S_3-subgroup of $C_G(V)$ which contains Q^*. We may assume $h\in N_G(Q_0)$ since $C_G(V)$ is 13-nilpotent. Now $C_G(Q_0)$ is a 13′-group since $C_G(Q_0)\subseteq C_G(Q^*)$ and $C_G(Q^*)$ is a α-invariant 13′-group. By taking a conjugation of Q_0, we may assume that $Q_0\subseteq Q$ and $C_Q(Q_0)$ is a S_3-subgroup of $C_G(Q_0)$. We set $Q_1=C_Q(Q_0)$, then $Z(Q)\subseteq Q_1$. Since $g\in Z(Q)$ $C_S(Q)$ is a S_2-subgroup of $C_G(Z(Q))$. Hence $C_S(Q)$ is a S_2-subgroup of $C_G(Q_1)$. Now $C_G(Q_1)$ is a 13′-group since $C_G(Q_1)\subseteq C_G(Z(Q))$. Hence by the Frattini argument we may assume that $h\in N_G(C_S(Q))$. Since $C_S(Q)\neq 1$, we see that $N_G(S)=N_G(P)$ is nilpotent, a contradiction. Hence we have $C_G(x)$ is odd order for each $x\in P^{\#}$. In particular $C_G(t)$ is solvable for every involution t of G. By (2.14) we see that $O(C_G(t))=1$ for every involution t. But now we have a contradiction by a similar argument of Lemma 4.9. Hence $C_Q(S)=1$.

LEMMA 4.14. $\Omega_1(Z(Q))\subseteq Z(Q)$.

PROOF. We set $\Phi_0(Q)=Q$ and $\Phi_1(Q)=\Phi(Q)$, $\Phi_{i+1}(Q)=\Phi(\Phi_i(Q))$, $\Phi_{n+1}(Q)=1$. Let $S_0=N_S(Q)$. Now we prove that $\langle\alpha\rangle S_0$ acts irreducibly on $\Phi_i(Q)/\Phi_{i+1}(Q)$, $0\leq i\leq n$ Suppose false. Since $C_Q(\alpha)$ is cyclic, we have $C_Q(S_0)\neq 1$. By Lemma 4.13 $S\neq S_0$. Let M be a maximal α-invariant subgroup of G which contains $N_G(S_0)$, then M is 3-nilpotent, hence $N_G(S)=N_G(Q)$, a contradiction. Hence $\langle\alpha\rangle S_0$ acts irreducibly on $\Phi_i(Q)/\Phi_{i+1}(Q)$, $0\leq i\leq n$. Next we consider the structure of $\overline{\Phi_i(Q)}=\Phi_i(Q)/\Phi_{i+2}(Q)$, $0\leq i\leq n-1$. Then class $\overline{\Phi_i(Q)}\leq 2$ and $\Omega_1(\overline{\Phi_i(Q)})=\overline{\Phi_{i-1}(Q)}$ or $\overline{\Phi_i(Q)}$. Now the exponent of $\Omega_1(\overline{\Phi_i(Q)})=3$ since class $\overline{\Phi_i(Q)}\leq 2$. Suppose that $\Omega_1(\overline{\Phi_i(Q)})=\overline{\Phi_i(Q)}$, then $|C_{\overline{\Phi_i(Q)}}(\alpha)|=3$. Since $C_{S_0}(\overline{\Phi_{i+1}(Q)})=1$, we have $C_{\overline{\Phi_{i+1}(Q)}}(\alpha)\neq 1$. Hence $C_{\overline{\Phi_i(Q)}}(\alpha)\subseteq\overline{\Phi_{i+1}(Q)}$. But now $C_{\Phi_i(Q)/\Phi_{i+1}(Q)}(\alpha)=1$, a contradiction. Hence we see that $\Omega_1(\overline{\Phi_i(Q)})=\overline{\Phi_{i+1}(Q)}$. Let $a\in Q$ and $|a|=3$. Then there exists a number j, $0\leq j\leq n$, such that $a\in\Phi_j(Q)-\Phi_{j+1}(Q)$. Suppose that $j<n$, then $a\in\Phi_j(Q)/\Phi_{j+2}(Q)$. Since $|a|=3$, we see that $a\in\Omega_1(\Phi_j(Q))=\Phi_{j+1}(Q)$. Hence $a\in\Phi_{j+1}(Q)$, a contradiction. Hence $a\in\Phi_n(Q)\subseteq Z(Q)$, this implies $\Omega_1(Q)\subseteq Z(Q)$.

LEMMA 4.15. *$C_G(x)$ is a 3′-group for each $x\in P^{\#}$. In particular the centralizer of every non-trivial 3-element is solvable.*

PROOF. Suppose false. Then there exists an element $x\in\Omega_1(P)$ such that $3||C_G(x)|$. We set $L=O_{13'}(C_G(x))$, then $N_G(\langle x\rangle)=\langle g\rangle PL$. Let A be a S_3-subgroup of $N_G(\langle x\rangle)$ which contains the element g. Then $\langle g\rangle P$ acts on $O_{3',3}(L)/O_{3'}(L)$. But now $O_{3',3}(L)=O_{3'}(L)(A\cap O_{3',3}(L))$. Since $|g|=3$, we have $[g, A]=1$ by Lemma 4.14. Hence g centralizes $O_{3',3}(L)/O_{3'}(L)$. Since $\langle g\rangle P$ is a Frobenius group, this follows that $[P, O_{3',3}(L)]\subseteq O_{3'}(L)$. Hence $3||C_G(P)|$. But now we have a contradiction by a similar argument of Lemma 4.8. Hence $C_G(x)$ is a $3'$-group for each $x\in P^{\#}$. By Lemma 4.3 the centralizer of every non-trivial 3-element is solvable.

LEMMA 4.16. *$C_G(x)$ is odd order for each $x\in P^{\#}$. In particular $C_G(t)$ is solvable and $O(C_G(t))=1$ for every involution t of G.*

PROOF. Suppose false. Then there exists an element x of $P^{\#}$ such that an S_2-subgroup V of $C_G(x)$ is non-trivial. Then by Lemma 4.13 $V\lhd C_G(x)$. We set $\Omega_1(P)=\langle x\rangle\times\langle y\rangle$, then y acts fixed point free on V. By Lemma 4.13 $C_G(V)$ is 13-nilpotent. By taking a conjugation of V we may assume that $V\subseteq S$ and $C_S(V)$ is a S_2-subgroup of $C_G(V)$. Let $S^*=C_S(V)$, then $Z(V)\subseteq S^*$. By taking a conjugation of x, we may assume that $x\in N_G(S^*)$. Assume that $N_G(S^*)$ is solvable, then x normalizes a S_2-subgroup K_1 of $N_G(S^*)$. Futhermore assume that $N_G(K_1)$ is solvable, then x normalizes a S_2-subgroup K_2 of $N_G(K_1)$. By a similar argument we see that $13||N_G(S)|$, then $N_G(S)=N_G(P)$, a contradiction. Hence there exists a 2-group K which contains S^* and such that $N_G(K)$ is non-solvable. Hence $N_G(K)$ involves $L_3(3)$, in particular a S_3-subgroup of $N_G(K)$ is non-cyclic. By taking a conjugation of K we may assume that $\langle a\rangle\times\langle b\rangle\subseteq Q\subset N_G(K)$. Let $c\in\langle a\rangle\times\langle b\rangle^{\#}$, then $C_G(c)\subseteq O_{3'}(C_G(c))\,N_G(Q)$ since $C_G(c)$ is solvable and $\Omega_1(Q)\subseteq Z(Q)$. By the Signalizer functor theorem $\langle O_{3'}(C_G(d))|d\in\Omega_1(Q)^{\#}\rangle=L$ is a α-invariant solvable $3'$-group. Suppose that $L\neq 1$. Let M be a maximal α-invariant subgroup of G which contains QL. Suppose that M is 3-nilpotent. If L is even order, then $N_G(S)=N_G(Q)$, a contradiction. If L is odd order, then we yield a contradiction by a similar argument of Lemma 4.8. Hence M is 3-closed and so $L\subseteq N_G(Q)$. Hence $C_G(c)\subseteq N_G(Q)$. In particular $K=\langle C_K(c)|c\in\langle a\rangle\times\langle b\rangle^{\#}\rangle\subseteq N_G(Q)$. Let $W=\Omega_1(Z(V))$, then we may assume that $W\subseteq N_S(Q)$. On the other hand $C_W(g_1)\cap C_W(g_1^{y_1})\subseteq C_W(y_1)=1$ for some conjugate elements g_1, y_1 of g, y. Hence $C_W(g_1)\oplus C_W(g_1^{y_1})\subseteq W$. We set $|W|=2^m$, then $2^m\geq 2^{12}$ since y_1 acts fixed point free on W. Let $|C_W(g_1)|=2^n$, then $2^{2n}\leq 2^m$. Assume that $n\gneqq m-6$, then $m\geq 2n\gneqq 2(m-6)$, this follows $m\gneqq 12$, a contradiction. Hence $n\leq m-6$. We set $W_0=W\cap C_S(Q)$, then $|W;W_0|\leq 2^6$, hence $|W_0|\geq 2^{m-6}$. Assume that $W=W_0$. Then $y_1\in N_G(W)\subseteq C_G(W)\,N_G(Q)$. Hence $13||N_G(Q)|$, a contradiction. Hence $W\supsetneqq W_0$. Let $v\in W-W_0$ and $X=\langle v\rangle\times W_0$. Then $|C_W(g_1)|=2^n\leq 2^{m-6}\lneqq 2^{m-5}\leq|X|$. But now $C_G(X)$ is 13-nilpotent by a similar argument of Lemma 4.13. Then $C_Q(v)=C_Q(X)\neq 1$ since $L_3(3)$ is involved in G and so Q is non-abelian. Let Q^* be a S_3-subgroup of $C_G(x)$. Since $C_G(X)$ is 13-nilpotent, $x_1 \in N_G(Q^*)$ for some x_1 which is conjugate to x. Let Q_0 be a S_3-subgroup of G which contains Q^*. Since $N_G(Q^*)$ is 3-constrained by Lemma 4.15 and $\Omega_1(Q_0) \subseteq Z(Q_0)$, we see $N_G(Q^*) \rhd \Omega_1(Q_0) O_{3'}(N_G(Q^*))$. Suppose that $x_1 \in O_{3'}(N_G(Q^*))$, then $[x_1, Q^*] \subseteq Q^* \subset O_{3'}(N_G(Q^*)) = 1$, which is a contradiction by Lemma 4.15. Hence we may assume that $x_1 \in N_G(\Omega_1(Q_0)) = N_G(Q_0)$. Hence $13 | |N_G(Q)|$, then $N_G(S) = N_G(P)$ is nilpotent, a contradiction. Hence $C_G(x)$ is odd order for each $x \in P^{\#}$.

Now we see that $O_{3'}(C_G(y))$ is odd order for each $y \in Q^{\#}$ by a similar argument of Lemma 4.10. And by a similar argument of Lemma 4.11 we have a final contradiction. Hence the Theorem is proved.

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