Hokkaido Mathematical Journal Vol. 8 (1979) p. 294-299

The F. and M. Riesz theorem and group structures

By Shozo KOSHI and Hiroshi YAMAGUCHI*

(Received April 10, 1979)

1. Introduction

In [4] and [5], one of the authors of the present note considered the relation between the F. and M. Riesz theorem and the structures of locally compact abelian (LCA) groups. In this note, we shall refine these results in a more general setting.

Let G be a LCA group with the algebraically ordered dual $\hat{G}=\Gamma$. If there exists a semi-group P of Γ such that

(AO 1) $$P\cup(-P)=\Gamma$$

(AO 2) $$P\cap(-P)=\{0\},$$

then Γ is called algebraically ordered, and P is called a semi-group with (AO)-conditions.

Let $M_{P}^{a}(G)$ be the set of all analytic measures with respect to P. Here a measure μ on G is analytic *w.r.* to P if and only if the Fourier transformation $\hat{\mu}$ is vanishing outside P. $L^{1}(G)$ denotes the space of all integrable functions with respect to the Haar measure on G.

The main result is: If $0\neq M_{P}^{a}(G)\subset L^{1}(G)$, then G must be isomorphic to one of the following

$$R\times D \quad \text{or} \quad T\times D$$

where R is the reals, T the torus and D some devisible discrete abelian group.

We express our hearty thanks to Professor J. Inuoue who has given important ideas in order to prove the theorems.

2. Main theorems

Let G be a LCA group with the algebraically ordered dual $\Gamma=\hat{G}$ so that there exists a semi-group P of Γ with (AO)-conditions. Then $\Lambda=\bar{P}\cap(\overline{-P})$ is a closed subgroup of Γ. It is easy to see that Λ is the boundary of P and $-P$ respectively.

---

* H. Yamaguchi is the same person as H. Otaki.

LEMMA 1. *Let F be a compact subgroup of Γ. Then Λ contains F.*

PROOF. $F\cap P$ is a semigroup with (AO)-conditions in F. Since F is compact, $F\cap P$ is dense in F. (cf. Lemma 1 of [5]). Hence, $F\subset\bar{P}$ and similarly $F\subset(\overline{-P})$.

LEMMA 2. *Let $\gamma\in\Gamma\backslash\Lambda$. Then, there exists an open set $V\ni\gamma$ with $V+\Lambda\subset P\backslash\{0\}$ or $V+\Lambda\subset-P\backslash\{0\}$.*

PROOF. It $\gamma\in P\backslash\Lambda$, then there exists an open set $V\ni\gamma$ such that $V\subset P\backslash\Lambda$. In this case, we have $V+\Lambda\subset P\backslash\{0\}$. If $\gamma\in-P\backslash\Lambda$, then there exists an open set $V\ni\gamma$ with $V+\Lambda\subset-P\backslash\{0\}$. q. e. d.

LEMMA 3. *Let $\tilde{P}=\{[\gamma]\,;\,[\gamma]=\gamma+\Lambda\subset P\}\cup[0]$. Then, $\tilde{P}$ is a closed semigroup with (AO)-conditions in Γ/Λ. Moreover $\pi(P)=\tilde{P}$ and $\pi(-P)=-\tilde{P}$ where π is a natural homomorphism $\Gamma\rightarrow\Gamma/\Lambda$.*
This is an easy consequence of Lemma 2.

LEMMA 4. *If $\{0\}\neq M_P^a(G)\subset L^1(G)$, then Γ/Λ is isomorphic to R or Z.*

PROOF. If Γ/Λ is not isomorphic to R or Z, then there exists a measure μ on $(\widehat{\Gamma/\Lambda})=\Lambda^{\perp}$ (annihilator of Λ) with $\mu\in M_{\tilde{P}}^a(\Lambda^{\perp})$ and $\hat{\mu}\notin C_0(\Gamma/\Lambda)$ (cf. [4]). Since μ is considered as a measure on G, and the Fourier transform $\hat{\mu}$ is vanishing outside P and $\tilde{\mu}\notin C_0(\Gamma)$ by Lemma 3, we have a contradicition.

LEMMA 5. *Under the assumption of Lemma 4, $\Gamma\cong\Lambda\times R$ or $\Gamma\cong\Lambda\times Z$.*

PROOF. By structure theorem ([3], (24.30) Theorem), $\Gamma\cong R^k\times F$ (k: integer) where F contains an open compact subgroup. If $\Gamma/\Lambda\cong R$, then $R\cong\Gamma/\Lambda\supset(R^k+\Lambda)/\Lambda\cong R^k/(\Lambda\cap R^k)$. Since $S=R^k/\Lambda\cap R^k$ is isomorphic to a closed subgroup of R and is not discrete, we have (i) $S\cong R$ or (ii) $S\cong 0$.

Assume (i) is occured, then $R^k\cong R\times(\Lambda\cap R^k)$. Hence, $\Gamma\cong R\times(\Lambda\cap R^k)\times F$. Since $\Lambda\cap[R\times\{0\}\times\{0\}]=\{0\}$, and $\Gamma/\Lambda\cong R$, we have $\Gamma\cong R\times\Lambda$.

In the case (ii), $\Lambda\supset R^k$ and so $R\cong\Gamma/\Lambda\cong(\Gamma/R^k)/(\Lambda/R^k)\cong F/\Lambda/R^k)$. But, this is a contradiction, since F has an open compact subgroup.

If $\Gamma/\Lambda\cong Z$, then it is clear that $\Gamma\cong\Lambda\times Z$.

We state now the main theorem.

THEOREM 1. *Let G a LCA group with the algebraically ordered dual $\Gamma=\hat{G}$, and let P be a semi-group with (AO)-conditions of Γ. If $\{0\}\neq M_P^a(G)\subset L^1(G)$, then G is isomorphic to one of both $R\times D$ or $T\times D$ where D is a divisible discrete group.*

PROOF. By Lemma 5, $\Gamma\cong\Lambda\times H$ where $H\cong R$ or Z. We have only to prove that Λ is compact. Remark now that $G=\hat{\Gamma}\cong H^{\perp}\times\Lambda^{\perp}$, $\hat{H}^{\perp}=\Lambda$ and $(\hat{\Lambda}^{\perp})=H$. Assume that Λ is not compact, then there exists a $\nu\in M(H^{\perp})$ (measure on $H^{\perp}$) with $\hat{\nu}\notin C_0(\Lambda)$. By Lemma 2, there exists a $\gamma\in H$ and an open set $V \ni \gamma$ with $V + \Lambda \subset P \setminus \{0\}$. Hence, there exists a non zero $\rho \in L^1(\Lambda^\perp)$ with supp $\hat{\rho} \subset V \cap H$. Putting $\mu = \nu \times \rho$, we have

$$\operatorname{supp} \hat{\mu} \subset \Lambda + V \subset P \quad \text{and} \quad \hat{\mu} \notin C_0(\Gamma).$$

Hence $\mu \in M_P^a(G) \setminus L^1(G)$. This is a contradiction.

REMARK 1. Theorem 1 in ([5]) is an immediate consequence of Theorem 1 of this paper.

REMARK 2. Let F be a compact abelian torsion-free group.

(1) If P is a semigroup with (AO)-conditions of $R \times E$, and if P is not dense in $R \times F$, then P is either

$$\{(x, f) \in R \times F;\ x > 0, \quad \text{or } x = 0 \text{ and } f \geqq_P 0\}$$

or

$$\{(x, f) \in R \times F;\ x < 0, \quad \text{or } x = 0 \text{ and } f \geqq_P 0\}.$$

Here '$>$' denotes the usual order of R and '$\geqq_P$' denotes the order of F induced by the semigroup P.

(2) Let P be a semigroup with (AO)-conditions of $Z \times F$. If P is not dense in $Z \times F$, then P is either

$$\{(n, f) \in Z \times F;\ n > 0, \quad \text{or } n = 0 \text{ and } f \geqq_P 0\}$$

or

$$\{(n, f) \in Z \times F;\ n < 0, \quad \text{or } n = 0 \text{ and } f \geqq_P 0\}.$$

THEOREM 2. *Suppose G is one of both $T \times D$ and $R \times D$, where D is a discrete abelian group such that $\hat{D}$ is torsion-free. Assume that P is a semigroup of $\hat{G}$ with (AO)-conditions such that it is not dense in $\hat{G}$. Then, $M_P^a(G) \subset L^1(G)$.*

Indeed, when G is either T or R this is the F. and M. Riesz theorem. The proof is given in Proposition A of [5] for $G = T \times D$. The latter case can be proved in the same way as the case $G = T \times D$.

3. Denseness of a semigroup with (AO)-conditions

An abelian group G is algebraically ordered if and only if it is torsion-free, since every group G can be considered as a discrete group (c. f. 8. 1. 2 of [6]). Therefore, these two terms are synonymous.

PROPOSITION 1. *Let R be the additive group of the real numbers.*

There exists a dense subset of which is a semigroup with (AO)-conditions.

PROOF. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be a Hamel basis of R with respect to the rational number field. We can assume that every e_λ is positive. We introduce a linear order in Λ. Every $x\in R$ is written into the following:

$$x=\sum_{k=1}^{n}\alpha_k e_{\lambda_k}\,;\ \lambda_1>\lambda_2>\cdots>\lambda_n\,,$$

where α_k is a rational number for $k=1, 2, \cdots, n$. Let $P=\left\{x\,;\ x=\sum_{k=1}^{n}\alpha_k e_{\lambda_k}\,;\ \lambda_1>\lambda_2>\cdots>\lambda_n \text{ and } \alpha_1>0\right\}\cup\{0\}$.

Then it is easy to see that P is a semigroup with (AO)-conditions in R. An order induced by P is known as the lexicographic order. But P contains not only positive numbers, but also negative numbers. So, it is easy see that P is dense in R by the usual topology.

PROPOSITION 2. *Let G_1 and G_2 be LCA groups with semigroups P_1 and P_2 with (AO)-conditions. If P_1 is dense in G_1, then there exists in $G=G_1\times G_2$ a dense semigroup P with (AO)-conditions. If P_1 is not dense in G_1, then there exists a semigroup P with (AO)-conditions which is not dense in $G=G_1\times G_2$.*

PROOF. We construct a linear order by lexicographic method in $G=G_1\times G_2$ so that $g_1+g_2\geqq g_1'+g_2'$ means that $g_1-g_1'\in P_1/\{0\}$ or $\{g_1=g_1',\ g_2\geqq g_2'\}$. Then the semigroup induced by this linear order satisfies above conditions.

THEOREM 3. *Let Γ be a nondiscrete locally compact abelian torsion-free group. Then there exists a dense subset which is a semigroup with (AO)-conditions.*

PROOF. By structure theorem ([3], (24. 30) Theorem), $\Gamma\cong R^n\times F$, where n is a nonnegative integer and F is a LCA group with a compact open subgroup F_0. By Propositions 1 and 2, we can easily construct a semigroup with the required condition if $n\geqq 1$. Therefore we will prove the case $n=0$, *i. e*, $\Gamma\cong F$. The proof of this case calls for Zorn's lemma. Since Γ is not discrete, it follows that $F_0\neq\{0\}$.

Let $\mathscr{F}$ denote the set of the following pairs (Λ, P): Λ is an open subgroup which includes F_0 as a subgroup. P is a dense subset of Λ which is a semigroup with (AO)-conditions. $\mathscr{F}$ is not empty because F_0 has such a semigroup. For (Λ_1, P_1), $(\Lambda_2, P_2)\in\mathscr{F}$, define $(\Lambda_1, P_1)\leqq(\Lambda_2, P_2)$ if and only if $\Lambda_1\subset\Lambda_2$ and $P_1\subset P_2$. By this "$\leqq$", $\mathscr{F}$ is partially ordered.

Let $\{\Lambda_\alpha, P_\alpha)\}_{\alpha\in I}$ be a totally ordered subset of $\mathscr{F}$. Let Λ_0 and P_0 denote $\bigcup_{\alpha\in I}\Lambda_\alpha$ and $\bigcup_{\alpha\in I}P_\alpha$ respectively. Then Λ_0 is an open subgroup of Γ and $F_0\subset$

Λ_0. It is easy to check that P_0 is a semigroup with (AO)-conditions and P_0 is dense in Λ_0. Thus, $(\Lambda_0, P_0)\in\mathscr{F}$ and $(\Lambda_\alpha, P_\alpha)<(\Lambda_0, P_0)$ for every $\alpha\in I$. By Zorn's lemma, there exists a maximal element $(\Lambda_*, P_*)\in\mathscr{F}$. It is sufficient to show $\Lambda_*=\Gamma$ in order to prove the theorem. We suppose $\Lambda_*\subsetneqq\Gamma$ and derive a contradiction. First, consider the case that Γ/Λ_* is not a torsion group. There exists an element $\gamma_0\in\Gamma$ (but $\gamma_0\notin\Lambda_*$) such that $n\gamma_0+\Lambda_*\neq\Lambda_*$ for every natural number n. Thus, the open subgroup $[\gamma_0, \Lambda_*]$ generated by γ_0 and Λ_* is isomorphic to $Z\times\Lambda_*$. Let $P=\{(n, \gamma)\in[\gamma_0, \Lambda_*]; \gamma\in P_*\backslash\{0\}, n\in Z\}\cup\{(n, 0)\in[\gamma_0, \Lambda_*]; n\geqq 0\}$. Then P is a semigroup with (AO)-conditions which is dense in $[\gamma_0, \Lambda_*]$. That is, $([\gamma_0, \Lambda_*], P)$ belongs to $\mathscr{F}$ and $(\Lambda_*, P_*)\lneqq([\gamma_0, \Lambda_*], P)$. This contradicts the maximality of (Λ_*, P_*). If Γ/Λ_* is a torison group, define a semigroup P as follows:

$$P=\{\gamma\in\Gamma;\ n\gamma\in P_* \text{ for some positive integer } n \text{ satisfying } n\gamma+\Lambda_*=\Lambda_*\}.$$

Then, since Γ is torsion-free, P is a semigroup with (AO)-conditions. Since $P\supset P_*$ and P_* is dense in Λ_*, $\bar{P}\cap(\overline{-P})\supset\Lambda_*$. If we suppose that P is not dense in Γ, there exists $\gamma\in\Gamma$ with $\gamma+\Lambda_*\subset P$ by Lemma 2. Since there exists a positive number n with $n\gamma\in\Lambda_*$, it follows that $P_*=\Lambda_*\cap P\supset n(\gamma+\Lambda_*)=\Lambda_*$. This is impossible. Hence, P is dense in Γ. This implies that $(\Gamma, P)\in\mathscr{F}$ and $(\Lambda_*, P_*)\lneqq(\Gamma, P)$, contradicting the maximality of (Λ_*, P_*). Q. E. D.

PROPOSITION 3. *Suppose Γ is a locally compact abelian torsion-free group and $\Gamma\cong R^n\times F$, where n is a nonnegative integer and F is a* LCA *group which has a compact open subgroup F_0. Then there is a semigroup P with (AO)-conditions of Γ which is not dense in Γ in $n\geqq 1$.*

In case $n=0$, such a semigroup exists if and only if Γ/F_0 is not a torsion group.

PROOF. If $n\geqq 1$, it is easy to construct a such semigroup P by proposition 1. Thus we consider the case $n=0$. If Γ/F_0 is a torsion group, then a semigroup S with (AO)-conditions of Γ is always dense in Γ by the fact that every semigroup with (AO)-conditions in a compact group is always dense, because for every $\gamma\in\Gamma$ there exists an integer n such that $\{\gamma+F_0\}\cup\{z\gamma+F_0\}\cup\cdots\cup\{n\gamma+F_0\}$ is a compact open subgroup of Γ. Hence, Γ/F_0 is not a torsion group if there exists a semigroup P with (AO)-conditions not dense in Γ. Now, we prove the converse.

Let $\mathscr{F}=\{\gamma+F_0;\ O(\gamma+F_0)<\infty \text{ in } \Gamma/F_0\}$ where $O(\gamma+F_0)$ denotes the order of the coset in Γ/F_0, and let $F_{0,\mathscr{F}}=\cup\{\gamma+F_0;\ \gamma+F_0\in\mathscr{F}\}$. By the hypothesis, $F_{0,\mathscr{F}}\neq\Gamma$. Furthermore, $\Gamma/F_{0,\mathscr{F}}$ is torsion-free, as we can prove in the following way. Suppose that there exist $\gamma_0\notin F_{0,\mathscr{F}}$ and an integer $n>1$ such that $n\gamma_0+F_{0,\mathscr{F}}=F_{0,\mathscr{F}}$. Then $n\gamma_0+F_0=\gamma+F_0$ for some $\gamma+F_0\in\mathscr{F}$, and $mn\gamma_0+F_0=F_0$ where $m=O(\gamma+F_0)$ in Γ/F_0. This implies $\gamma_0+F_0\in\mathscr{F}$. This is a contradiction.

Now, we can take a semigroup P_1 with (AO)-conditions of $F_{0,\mathscr{F}}$, since $F_{0,\mathscr{F}}$ is a torsion free group.

Since $\Gamma/F_{0,\mathscr{F}}$ is torsion-free, there exists a semigroup P_2 with (AO)-conditions of $\Gamma/F_{0,\mathscr{F}}$.

Let $P=\{\gamma\in\Gamma;\ [\gamma]\in P_2\backslash\{[0]\}\}\cup P_1$, where $[\gamma]$ denotes the coset of γ in $\Gamma/F_{0,\mathscr{F}}$. Then P is a semigroup with (AO)-conditions which is not dense in Γ. Q. E. D.

References

[1] DOSS, R.: On measures with small transforms. Pacific Journal of Math. 26, 257-263 (1968).

[2] HELSON, H. and LOWDENSLAGER, D.: Prediction theory and Fourier series in several variables. Acta Math. 99, 165-202 (1958).

[3] HEWITT, E. and ROSS, K. A.: Abstract Harmonic Analysis Vol 1. Berlin-Heidelberg-New York: Springer 1963.

[4] OTAKI, H.: Remarks on characterization of locally compact abelian groups. Hokkaido Math. J. 6, 52-55 (1977).

[5] OTAKI, H.: A ralation between the F. and M. Riesz theorem and the structure of LCA groups. Hokkaido Math. J. 6, 306-312 (1977).

[6] RUDIN, W.: Fourier Analysis on Groups. New York-London: Interscience 1962.

Department of Mathematics
Hokkaido University
Sapporo 060, Japan