Hokkaido Mathematical Journal Vol. 8 (1979) p. 283-287

Notes on extremizations

By Yukio NAGASAKA

(Received April 26, 1979)

1. Let G be a subregion of a hyperbolic Riemann surface R with an analytic relative boundary ∂G, compact or noncompact. We denote by $HP_0(G)$ the class of nonnegative continuous functions on R which are harmonic on G and vanish on $R-G$. Let $u\in HP_0(G)$. Then $\{H_u^{R_n}\}_n$ is an increasing sequence, where $\{R_n\}$ is an exhaustion of R and $H_u^{R_n}$ is a harmonic function of R_n with boundary values u on ∂R_n. Then $Eu=E_G u=\lim_{n\to\infty} H_u^{R_n}$ is harmonic on R or identically $+\infty$. If Eu is harmonic on R, Eu is called the extremization of u relative (R, G) in the Kuramochi terminology. If u is bounded on G or the Dirichlet integral $D_G(u)$ of u on G is finite, then u has the extremization. But $u\in HP_0(G)$ has not always the extremization. In the present paper, we give a sufficient condition for u to have the extremization and give an example of G such that any $u(\neq 0)$ of $HP_0(G)$ has not the extremization.

2. Let $u\in HP_0(G)$ and $z_0\in G$. We denote by $G_n(z, z_0)$ and $G(z, z_0)$ the Green's function on R_n and R with pole at z_0 respectively. By Green's formula,

$$\int_\alpha u(z)\frac{\partial}{\partial n}G_n(z, z_0)\,ds=\int_\alpha G_n(z, z_0)\frac{\partial u}{\partial n}ds\,,$$

where $\alpha=\partial(R_n\cap G-(|z-z_0|\leqq\varepsilon))$ for small $\varepsilon>0$,

$$-2\pi u(z_0)+\int_{\partial R_n\cap G} u(z)\frac{\partial}{\partial n}G_n(z, z_0)\,ds=\int_{\partial G\cap R_n} G_n(z, z_0)\frac{\partial u}{\partial n}ds\,.$$

Then

$$H_u^{R_n}(z_0)=\frac{1}{2\pi}\int_{\partial R_n\cap G} u(z)\frac{\partial}{\partial n}G_n(z, z_0)\,ds=u(z_0)+\frac{1}{2\pi}\int_{\partial G\cap R_n} G_n(z, z_0)\frac{\partial u}{\partial n}ds\,.$$

Since $G_n(z, z_0)\uparrow G(z, z_0)$ on ∂G, we have

$$Eu(z_0)=u(z_0)+\frac{1}{2\pi}\int_{\partial G} G(z, z_0)\frac{\partial u}{\partial n}ds\,, \tag{1}$$

where the normal n is taken inward with respect to G. From (1) we obtain the next Theorem 1. We use the notations

$$G_a = G_a^u = \{z ;\ u(z) > a\}, \qquad L_a = L_a^u = \partial G_a^u$$

and

$$G_{ab} = G_{ab}^u = \{z ;\ a < u(z) < b\} \qquad \text{for every } 0 \leqq a < b .$$

THEOREM 1 (*Z. Kuramochi* [1]) *If* $\int_{L_a} \frac{\partial u}{\partial n} ds < +\infty$ *for some* $a \geqq 0$, *then* $E_G u$ *is harmonic.*

PROOF. Let $u_1 = u - a$ on G_a and take $z_0 \in G_a$. We apply (1) with $u = u_1$, $G = G_a$. Then we have

$$(2) \qquad E_{G_a} u_1(z_0) = u_1(z_0) + \frac{1}{2\pi} \int_{L_a} G(z, z_0) \frac{\partial u_1}{\partial n} ds .$$

Since $\sup_{z \in L_a} G(z, z_0) < +\infty$ and $\int_{L_a} \frac{\partial u_1}{\partial n} ds = \int_{L_a} \frac{\partial u}{\partial n} ds < +\infty$, it follows from (2) that $E_{G_a} u_1(z_0) < +\infty$. Hence $E_{G_a} u_1$ is harmonic. Since $u_1 \leqq E_{G_a} u_1$ on G_a, $u \leqq a + E_{G_a} u_1$ on G. This shows that $E_G u$ is harmonic.

In the next theorem, we use terms of the Royden compactification R^* of R. For a subset A of R we denote by $\bar{A}^*$ the closure of A with respect to R^*.

THEOREM 2. *Let* $u \in HP_0(G)$. *If* $\overline{L_a^u}^* \cap \overline{L_b^u}^* = \phi$ *for some* a *and* b $(0 \leqq a < b)$, *then* $E_G u$ *is harmonic.*

PROOF. By $\bar{L}_a^* \cap \bar{L}_b^* = \phi$, there exists a bounded continuous Tonelli function f on R with finite Dirichlet integral over R such that $f | L_a = 0$ and $f | L_b = 1$ (cf. p. 156 in L. Sario and M. Nakai [2]). Let ω_n be the harmonic function in $G_{ab} \cap R_n$ which has the boundary values 0 on $\overline{L_a \cap R_n}$ and 1 on $\overline{L_b \cap R_n}$ and whose normal derivative vanishes on the rest of the boundary. From the existence of the above f we see that ω_n converges to a function $\omega \in HD(G_{ab})$ locally uniformly and in Dirichlet norm. Then ω has the boundary values 0 on L_a and 1 on L_b. By Green's formula,

$$D(\omega_n) = -\int_{\partial(G_{ab} \cap R_n)} \omega_n \frac{\partial \omega_n}{\partial n} ds = \int_{L_b \cap R_n} \frac{\partial \omega_n}{\partial n} ds = \int_{L_a \cap R_n} \frac{\partial \omega_n}{\partial n} ds .$$

Since $\frac{\partial \omega_n}{\partial n} \geqq 0$ on $L_a \cap R_n$,

$$\varliminf_{n \to \infty} \int_{L_a \cap R_n} \frac{\partial \omega_n}{\partial n} ds \geqq \int_{L_a} \frac{\partial \omega}{\partial n} ds$$

by Fatou's lemma. Since $D(\omega)\geqq D(\omega_n)$, this shows

$$0\leqq\int_{L_a}\frac{\partial\omega}{\partial n}ds\leqq D(\omega)<+\infty\,. \tag{3}$$

Set $E_n=(G_a-G_b)\cap(R-R_n)$ and $F_n=\bar{G}_b\cap(R-R_n)$. Let $v=u-a$ on G_a. For a closed set F of G_a we denote by v_F the regularized reduced function of v relative to F in G_a. Consider v_{E_n} and v_{F_n}. Since $\{v_{E_n}\}$ and $\{v_{F_n}\}$ are decreasing sequences, $v_1=\lim_{n\to\infty} v_{E_n}$ and $v_2=\lim_{n\to\infty} v_{F_n}$ are harmonic on G_a. Since $v_1\leqq v\leqq b-a$ on G_a, v_1 is bounded on G_a and so $E_{G_a}v_1$ is harmonic on R. On the other hand, since $v_2\leqq v_{\bar{G}_b}\leqq(b-a)\,\omega$ on G_{ab} and $v_2=(b-a)\,\omega=0$ on L_a, we see

$$0\leqq\int_{L_a}\frac{\partial v_2}{\partial n}ds\leqq(b-a)\int_{L_a}\frac{\partial\omega}{\partial n}ds<+\infty$$

by (3). By Theorem 1, this shows that $E_{G_a}v_2$ is harmonic. Since $v\leqq v_{E_n}+v_{F_n}$ for any n, $v\leqq v_1+v_2$ and so $v\leqq E_{G_a}v_1+E_{G_a}v_2$ on G_a. Hence we see that $E_{G_a}v$ is harmonic. Since $v\leqq E_{G_a}v$ on G_a, $u\leqq a+E_{G_a}v$ on G. This shows that $E_G u$ is harmonic on R.

THEOREM 3. *Let $G\subset\{z\,;\,G(z,z_0)>\delta\}$ for some $\delta>0$. Then $u\in HP_0(G)$ has extremization if and only if $\int_{L_a}\frac{\partial u}{\partial n}ds=\int_{L_0}\frac{\partial u}{\partial n}ds<+\infty$ for any $a>0$.*

PROOF. "if" part follows from Theorem 1.

Next suppose that $E_G u$ is harmonic. Since $u-a\leqq E_G u$ on G_a for every $a\geqq 0$, $E_{G_a}(u-a)\leqq E_G u$ and so $E_{G_a}(u-a)$ is harmonic for every $a\geqq 0$. Let $0\leqq a\leqq M<+\infty$ and take $z_1\in G_M$. Then

$$E_{G_a}(u-a)\,(z_1)=(u-a)\,(z_1)+\frac{1}{2\pi}\int_{L_a}G(z,z_1)\frac{\partial u}{\partial n}ds$$

by (2). Since $G_a\subset G\subset\{z\in R\,;\,G(z,z_1)>\delta'\}$ for some $\delta'>0$, we have

$$\int_{L_a}\frac{\partial u}{\partial n}ds\leqq\frac{2\pi}{\delta'}(E_G u)\,(z_1)<+\infty\qquad\text{for }0\leqq a\leqq M.$$

Hence by Lemma 5 in [1],

$$D_{G_{0M}}(u)=\int_0^M\left(\int_{L_a}\frac{\partial u}{\partial n}ds\right)da\leqq\frac{2\pi M}{\delta'}(E_G u)\,(z_1)<+\infty\,.$$

This shows that $D_G(\min(u,M))<+\infty$ for every $M>0$. Take any $a>0$. Here we note that the double $\hat{G}_{0a}$ of G_{0a} about $L_0\cup L_a$ is parabolic. Since

$D_{G_{0a}}(u) < +\infty$, there exists a exhaustion G_n of $\hat{G}_{0a}$ such that

$$\lim_{n\to\infty}\int_{G_{0a}\cap\partial G_n}\left|\frac{\partial u}{\partial n}\right| ds = 0$$

This shows that

$$\int_{L_a}\frac{\partial u}{\partial n} ds = \int_{L_0}\frac{\partial u}{\partial n} ds < +\infty .$$

THEOREM 4. *Let* $G \subset \{z \in R;\ G(z, z_0) > \delta\}$ *for some* $\delta > 0$. *Then* $E_G u$ *is harmonic if and only if there is a constant* $\alpha > 0$ *such that*

$$D\big(\min(u, M)\big) = \alpha M \qquad \textit{for every } M > 0 .$$

PROOF. Let $D(\min(u, M)) < +\infty$ for every M. Take a and b $(0 \leqq a < b)$. Since $\min(u, b)$ is a bounded continuous Tonelli function on R with finite Dirichlet integral on R such that $\min(u, b)|L_a^u = a$ and $\min(u, b)|L_b^u = b$. This shows $\bar{L}_a^* \cap \bar{L}_b^* = \phi$. Hence we have $E_G u$ is harmonic by Theorem 2.

Suppose on the other hand that $E_G u$ is harmonic. Then, by Theorem 3, there is a non-negative constant α such that $\int_{L_a}\frac{\partial u}{\partial n} ds = \alpha$ for every $a \geqq 0$. Hence we have

$$D\big(\min(u, M)\big) = \int_0^M\left(\int_{L_a}\frac{\partial u}{\partial n} ds\right) da = \alpha M .$$

This completes the proof.

Here we use the following Kuramochi's result [1] : Let R be a hyperbolic Riemann surface and suppose $\lim_{z\to\infty} G(z, z_0) = 0$, where $G(z, z_0)$ is the Green's function of R and ∞ is the Alexandroff's ideal boundary point. If a positive harmonic function u on R satisfies $\varlimsup_{M\to\infty}\frac{D(\min(u, M))}{M} < +\infty$, then u is quasibounded on R. Using this result we obtain the next theorem.

THEOREM 5. *Let* $G \subset \{z \in R;\ G(z, z_0) > \delta\}$ *for some* $\delta > 0$ *and suppose*

$$\lim_{G \ni z \to \infty} G'(z, z_0) = 0 ,$$

where $G'(z, z_0)$ *is the Green's function on* G. *Then any function* $u (\not\equiv 0)$ *of* $HP_0(G)$ *has not the extremization.*

PROOF. Suppose that a function u of $HP_0(G)$ has the extremization. Then $\varlimsup_{M\to\infty}\frac{D(\min(u, M))}{M} \leqq \alpha < +\infty$ by Theorem 4. Hence, by the above result, we see that u is a quasibounded on G. Since G is an SO_{HB} region, this implies $u\equiv 0$ on G.

References

[1] Z. KURAMOCHI: On quasi-Dirichlet bounded harmonic functions, Hokkaido Math. J., 8 (1979), 1-22.

[2] L. SARIO and M. NAKAI: Classification theory of Riemann surfaces, Springer, 1970.

Department of Mathematics
Hokkaido University