Hokkaido Mathematical Journal Vol. 8 (1979) p. 253-259

The factorization in the commutant of a unitary operator

By Katsutoshi TAKAHASHI

(Received April 19, 1979)

1. Introduction.

In this paper we generalize the results concerning the factorization of positive (i. e. positive semidefintie) operator valued functions on the unit circle to the abstract context. Let $\mathscr{L}$ be a complex Hilbert space, U a unitary operator on $\mathscr{L}$ and $\mathscr{H}$ a closed subspace of $\mathscr{L}$ which is invariant under U. Let $\{U\}'$ denote the commutant of U and $\mathscr{A}$ the algebra consisting of all bounded operators A in $\{U\}'$ such that $A\mathscr{H}\subseteq\mathscr{H}$. We ask the following question; which positive operator T in $\{U\}'$ is factorable in the sense that $T=A^{*}A$ for some A in $\mathscr{A}$?

Let us recall a classical example. Let $\mathscr{E}$ be a separable Hiblert space, $L_{\mathscr{E}}^{2}$ the Hilbert space of all Lebesgue measurable $\mathscr{E}$-valued functions on the unit circle having square-integrable norm, and U_{0} the bilateral shift on $L_{\mathscr{E}}^{2}$, i. e. $(U_{0}f)(e^{i\theta})=e^{i\theta}f(e^{i\theta})$. Also let $L_{\mathscr{B}(\mathscr{E})}^{\infty}$ denote the algebra of all Legesgue measurable, essentially bounded functions from the unit circle to the algebra $\mathscr{B}(\mathscr{E})$ of bounded operators on $\mathscr{E}$, and M_{F} the multiplication operator on $L_{\mathscr{E}}^{2}$ by F in $L_{\mathscr{B}(\mathscr{E})}^{\infty}$, i. e. $(M_{F}f)(e^{i\theta})=F(e^{i\theta})f(e^{i\theta})$. It is known that the map $F\rightarrow M_{F}$ is a $*$-isomorphism from the algebra $L_{\mathscr{B}(\mathscr{E})}^{\infty}$ with involution $F^{*}(e^{i\theta})=(F(e^{i\theta}))^{*}$ onto the commutant $\{U_{0}\}'$ of U_{0}. (See, for example, [6, P48 and P50]). Let $H_{\mathscr{E}}^{2}$ and $H_{\mathscr{B}(\mathscr{E})}^{\infty}$ be the Hardy subspaces of $L_{\mathscr{E}}^{2}$ and $L_{\mathscr{B}(\mathscr{E})}^{\infty}$ respectively. It is easy to see that A lies in $H_{\mathscr{B}(\mathscr{E})}^{\infty}$ if and only if M_{A} maps $H_{\mathscr{E}}^{2}$ into itself. Thus the above question is essentially the factorization problem for positive operator valued functions if $\mathscr{L}=L_{\mathscr{E}}^{2}$, $\mathscr{H}=H_{\mathscr{E}}^{2}$ and $U=U_{0}$.

The above question was considered by Page and Gellar, in [5] and [2]. In [5], Page studies the invertibility of an operator $PA|\mathscr{H}$, where A lies in $\{U\}'$ and P is the orthogonal projection of $\mathscr{L}$ onto $\mathscr{H}$, and showed that every invertible positive operator in $\{U\}'$ is factorable. Subsequently Gellar and Page [2] generalized this result, but only in an unsatisfactory way.

In the present paper we first prove a theorem which gives necessary and sufficient conditions for factorability. This contains the theorem of

Gellar and Page, and Lowdenslager's characterization [3, P117, Lemma] for factorability of operator valued functions. Then we generalize Deviratz' factorization theorem for operator valued functions having invertible values a. e. ([3] and [8]), and the operator generalization ([7] and [8]) of the Fejer-Riesz theorem on the factorization of trigonometric polynomials.

The author wishes to thank Prof. T. Ando and Prof. T. Nakazi for many helpful conversations,

2. Factorization theorem.

LEMMA 1. *Let $T\in\{U\}'$ and $A\in\mathscr{A}$. Then $T^*T=A^*A$ if and only if $T=VA$ where V is a partial isometry in $\{U\}'$ with initial space $(A\mathscr{L})^-$.*

PROOF. Let $T^*T=A^*A$. Then the operator V defined by $V(Af)=Tf$ for all $f\in\mathscr{L}$ and $V|(A\mathscr{L})^\perp=0$ is a partial isometry with initial space $(A\mathscr{L})^-$. The operator V commutes with U because $(A\mathscr{L})^-$ is a reducing subspace of U. The converse is obvious.

By Lemma 1 our question is equivalent to the following; Which operator $T\in\{U\}'$ can be factored in the form $T=VA$, where $A\in\mathscr{A}$ and V is a partial isometry in $\{U\}'$ with initial space $(A\mathscr{L})^-$?

LEMMA 2. *Let $T\in\{U\}'$ and $\mathscr{M}$ a reducing subspace for U. Then there exists a partial isometry $V\in\{U\}'$ with initial space $(T\mathscr{M})^-$ and final space contained in $\mathscr{M}$. If further $T|\mathscr{M}$ is one-to-one, then the final space of V is equal to $\mathscr{M}$.*

PROOF. Let P be the orthogonal projection of $\mathscr{L}$ onto $\mathscr{M}$. Let $TP=WQ$ be the polar decomposition of TP, so W is a partial isometry with initial space $(\mathrm{Ker}\ TP)^\perp$, and Q is positive. Since TP is in the von Neumann algebra $\{U\}'$, W lies in $\{U\}'$. Setting $V=W^*$, we complete the proof of Lemma.

When $\mathscr{K}$ is a reducing subspace of U, the answer to our question is the following;

COROLLARY 1. *If $\mathscr{K}$ is a reducing subspace of U, then every operator $T\in\{U\}'$ can be factored $T=VA$, where $A\in\mathscr{A}$ and V is a partial isometry in $\{U\}'$ with initial space $(A\mathscr{L})^-$.*

PROOF. By Lemma 2 we obtain a partial isometry $W_1\in\{U\}'$ such that $(\mathrm{Ker}\ W_1)^\perp=(T\mathscr{K})^-$ and $\mathrm{Im}\ W_1\subseteq\mathscr{K}$. (In denoting the range.) Let P be the orthogonal projection onto $(T\mathscr{L})^-\ominus(T\mathscr{K})^-$. Since T commutes with U^* as well as U, the subspace $(T\mathscr{L})^-\ominus(T\mathscr{K})^-$ is U-reducing and $P\in\{U\}'$. We apply Lemma 2 to $PT\in\{U\}'$ and a U-reducing subspace $\mathscr{L}\ominus\mathscr{K}$ to obtain a partial isometry $W_2\in\{U\}'$ such that $(\mathrm{Ker}\ W_2)^\perp=(PT(\mathscr{L}\ominus\mathscr{K}))^-=$

$(T\mathscr{L})^- \ominus (T\mathscr{H})^-$ and Im $W_2 \subseteq \mathscr{L} \ominus \mathscr{H}$. We set $V = W_1^* + W_2^*$ and $A = V^*T$. Since the initial spaces of W_1 and W_2 are mutually orthogonal and so are their final spaces, V^* is a partial isometry whose initial space is equal to $(\text{Ker } W_1)^\perp \oplus (\text{Ker } W_2)^\perp = (T\mathscr{L})^-$. Also $A\mathscr{H} = W_1 T\mathscr{H} \subseteq \mathscr{H}$. Clearly V and A are in $\{U\}'$. This completes the proof.

We call an operator A outer if A lies in $\mathscr{A}$ and A satisfies $(A\mathscr{H})^\perp \cap \mathscr{H} = (A\mathscr{L})^\perp \cap \mathscr{H}$. Let $\mathscr{L}$, $\mathscr{H}$ and U be L^2_{ℓ}, H^2_{ℓ} and U_0 respectively. Then it is easy to see that if A is an outer function in $H^\infty_{\mathscr{B}(\ell)}$ ([3], [8]), then the multiplication operator M_A is outer in the above sense.

In [2], Gellar and Page proved the following theorem; Let $T \in \{U\}'$. If there exists an invertible operator $X \in \{U\}'$ such that $XT \in \mathscr{A}$, then $T = VA$ where A is outer and V is a partial isometry in $\{U\}'$ with initial space $(A\mathscr{L})^-$.

We weaken the condition of Gellar and Page to obtain a necessary and sufficient condition for factorability.

THEOREM 1. *Let $T \in \{U\}'$. The following statements are equivalent.*

(i) *$T = VA$ where $A \in \mathscr{A}$ and V is a partial isometry in $\{U\}'$ with initial space $(A\mathscr{L})^-$.*

(ii) *There exists an operator $X \in \{U\}'$ such that $XT \in \mathscr{A}$ and $X|(T\mathscr{H})^-$ is one-to-one.*

(iii) *There exists an one-to-one operator Y from $\bigcap_{n=0}^{\infty} U^n(T\mathscr{H})^-$ into $\bigcap_{n=0}^{\infty} U^n\mathscr{H}$ such that $YU = UY$ on $\bigcap_{n=0}^{\infty} U^n(T\mathscr{H})^-$.*

(iv) *$T = VA$ where A is outer and V is a partial isometry in $\{U\}'$ with initial space $(A\mathscr{L})^-$.*

PROOF. (iv) implies (i); This is trivial.

(i) implies, (ii); Take V^* for X in (ii).

(ii) implies (iii); For X in (ii), $X|\bigcap_{n=0}^{\infty} U^n(T\mathscr{H})^-$ is one-to-one, and

$$X\left(\bigcap_{n=0}^{\infty} U^n(T\mathscr{H})^-\right) = \bigcap_{n=0}^{\infty} XU^n(T\mathscr{H})^- = \bigcap_{n=0}^{\infty} U^n X(T\mathscr{H})^- \subseteq \bigcap_{n=0}^{\infty} U^n\mathscr{H} .$$

Hence $X|\bigcap_{n=0}^{\infty} U^n(T\mathscr{H})^-$ meets the requirement on Y in (iii).

(iii) implies (iv); Since $\mathscr{H}$ and $(T\mathscr{H})^-$ are invariant under U, $U|\mathscr{H}$ and $U|(T\mathscr{H})^-$ are isometries on $\mathscr{H}$ and $(T\mathscr{H})^-$ respectively. From the Wold decompositions of isometries $U|\mathscr{H}$ and $U|(T\mathscr{H})^-$, we have the following decomposition;

$$\mathscr{H} = \left(\sum_{n=0}^{\infty} \oplus U^n \ell\right) \oplus \mathscr{K} ,$$

where $\mathscr{L}=\mathscr{M}\ominus U\mathscr{M}$, $\mathscr{K}=\bigcap_{n=0}^{\infty}U^n\mathscr{M}$, and $U|\mathscr{K}$ is unitary;
and

$$(T\mathscr{M})^- = \left(\sum_{n=0}^{\infty}\oplus U^n\mathscr{L}_1\right)\oplus\mathscr{K}_1,$$

where $\mathscr{L}_1=(T\mathscr{M})^-\ominus U(T\mathscr{M})^-$, $\mathscr{K}_1=\bigcap_{n=0}^{\infty}U^n(T\mathscr{M})^-$, and $U|\mathscr{K}_1$ is unitary. Let $\mathscr{M}_{-\infty}$ denote the smallest reducing subspace for U that contains $\mathscr{M}$;

$$\mathscr{M}_{-\infty}=\left(\sum_{n=-\infty}^{\infty}\oplus U^n\mathscr{L}\right)\oplus\mathscr{K}.$$

Then

$$\mathscr{L}=(\mathscr{L}\ominus\mathscr{M}_{-\infty})\oplus\left(\sum_{n=-\infty}^{\infty}\oplus U^n\mathscr{L}\right)\oplus\mathscr{K},$$

and

$$(T\mathscr{L})^- = \left((T\mathscr{L})^-\ominus(T\mathscr{M}_{-\infty})^-\right)\oplus\left(\sum_{n=-\infty}^{\infty}\oplus U^n\mathscr{L}_1\right)\oplus\mathscr{K}_1.$$

Let Q be the orthogonal projection of $\mathscr{L}$ onto $(T\mathscr{L})^-\ominus(T\mathscr{M}_{-\infty})^-$. Since $(T\mathscr{L})^-\ominus(T\mathscr{M}_{-\infty})^-$ is a reducing subspace of U, $Q\in\{U\}'$. We apply Lemma 2 to $QT\in\{U\}'$ and a U-reducing subspace $\mathscr{L}\ominus\mathscr{M}_{-\infty}$ to obtain a partial isometry $W_1\in\{U\}'$ such that $(\mathrm{Ker}\ W_1)^\perp=(QT(\mathscr{L}\ominus\mathscr{M}_{-\infty}))^-=(T\mathscr{L})^-\ominus(T\mathscr{M}_{-\infty})^-$ and $\mathrm{Im}\ W_1\subseteq\mathscr{L}\ominus\mathscr{M}_{-\infty}$.

From observations similar to the ones used in the proof of [2, Theorem 2], we know that $\dim\mathscr{L}_1\leq\dim\mathscr{L}$. Therefore there exists an isometry W_2 mapping $\mathscr{L}_1$ into $\mathscr{L}$. We extend W_2 to a partial isometry on $\mathscr{L}$ by defining $W_2(U^nf)=U^n(W_2f)$ for each $f\in\mathscr{L}_1$ and $n=0, \pm1, \pm2, \cdots$ and $W_2=0$ on $\mathscr{L}\ominus\left(\sum_{n=-\infty}^{\infty}\oplus U_n\mathscr{L}_1\right)$. Clearly $(\mathrm{Ker}\ W_2)^\perp=\sum_{n=-\infty}^{\infty}\oplus U^n\mathscr{L}_1$, $\mathrm{Im}\ W_2\subseteq\sum_{n=-\infty}^{\infty}\oplus U^n\mathscr{L}$, and $W_2\in\{U\}'$.

Let us extend Y in (iii) to $\mathscr{L}$ by defining $Y=0$ on $\mathscr{K}_1^\perp$. Then Y is in $\{U\}'$. Applying Lemma 2 to Y and $\mathscr{K}_1$, we obtain a partial isometry $W_3\in\{U\}'$ with initial space contained in $\mathscr{K}$ and final space $\mathscr{K}_1$ because $\mathrm{Im}\ Y\subseteq\mathscr{K}$ and $Y|\mathscr{K}_1$ is one-to-one.

We now set $V=W_1^*+W_2^*+W_3$ and $A=V^*T$. The clearly V^* is a partial isometry in $\{U\}'$ with initial space $(T\mathscr{L})^-$, and so $T=VA$. Taking account of the initial spaces and final spaces of W_1, W_2 and W_3, it is easily checked that A is outer. Therefore (iii) implies (iv).

By Lemma 1 we obtain the following theorem equivalent to Theorem 1.

THEOREM 1′. *Let T be a positive operator in $\{U\}'$. The following* *statements are equivalent.*

(i) *T is factorable.*

(ii) *There exists an operator $X\in\{U\}'$ such that $XT^{1/2}\in\mathscr{A}$ and $X|(T^{1/2}\mathscr{H})^-$ is one-to-one.*

(iii) *There exists an one-to-one operator Y from $\bigcap_{n=0}^{\infty}U_n(T^{1/2}\mathscr{H})^-$ into $\bigcap_{n=0}^{\infty}U^n\mathscr{H}$ such that $YU=UY$ on $\bigcap_{n=0}^{\infty}U^n(T^{1/2}\mathscr{H})^-$.*

(iv) *$T=A^*A$ where A is outer.*

The following lemma shows that we have only to consider the case where the smallest reducing subspace $\mathscr{H}_{-\infty}$ for U containing $\mathscr{H}$ is equal to $\mathscr{L}$.

LEMMA 3. *Let T be a positive operator in $\{U\}'$ and $P_{-\infty}$ the orthogonal projection of $\mathscr{L}$ onto $\mathscr{H}_{-\infty}$. Then T is factorable if and only if $P_{-\infty}TP_{-\infty}$ is.*

PROOF. Let $T=A^*A$ for some $A\in\mathscr{A}$. Then $P_{-\infty}TP_{-\infty}=(AP_{-\infty})^*(AP_{-\infty})$, and clearly $AP_{-\infty}\in\mathscr{A}$. Hence $P_{-\infty}TP_{-\infty}$ is factorable.

Conversely, suppose that $P_{-\infty}TP_{-\infty}$ is factorable, so there is a partial isometry W_1 with initial space $(T^{1/2}P_{-\infty}\mathscr{L})^-$ such that $W_1T^{1/2}P_{-\infty}\in\mathscr{A}$, by Lemma 1. As in the proof of Theorem 1, we use Lemma 2 to obtain a partial isometry $W_2\in\{U\}'$ with initial space $(T^{1/2}\mathscr{L})\ominus(T^{1/2}P_{-\infty}\mathscr{L})^-$ and final space contained in $\mathscr{L}\ominus P_{-\infty}\mathscr{L}$. We define W by $W=W_1+W_2$. Then W is a partial isometry in $\{U\}'$ with initial space $(T^{1/2}\mathscr{L})^-$ such that $WT^{1/2}\in\mathscr{A}$, and so T is factorable by Lemma 1.

REMARK. Let V be an isometry on a Hibert space $\mathscr{H}$. Moore, Rosenblum and Rovnyak proved a theorem [4, Theorem 4] which characterized the product A^*A where A commutes with V. It turns out that our Theorem 1′ is quivalent to [4, Theorem 4] under Lemma 3 and the following fact (see [1, Theorem 2] and its proof.): Let V be an isometry on a Hilbert space $\mathscr{H}$ and U the minimal unitary extension of V on a Hilbert space $\mathscr{K}$. Let P be the orthogonal projection of $\mathscr{K}$ onto $\mathscr{H}$. Then an operator T on $\mathscr{H}$ satisfies $V^*TV=T$ if and only if there exists an operator $\tilde{T}\in\{U\}'$ such that $T=P\tilde{T}|\mathscr{H}$. In this case, moreover, (i) T is positive if and only if $\tilde{T}$ is positive, and (ii) $T=A^*A$ for some $A\in\{V\}'$ if and only if $\tilde{T}=A^*A$ for some $\tilde{A}\in\{U\}'$ such that $\tilde{A}\mathscr{H}\subseteq\mathscr{H}$.

3. Applications.

Let F be a positive operator valued function in $L^{\infty}_{\mathscr{B}(\mathscr{E})}$ whose values are invertible a.e.. The Devinatz theorem (see e.g. [2, p119]) asserts that if

$\log \|F(e^{i\theta})^{-1}\|^{-1}$ is integrable, then F is factorable, that is, $F(e^{i\theta})=A^*(e^{i\theta}) A(e^{i\theta})$ a. e. for some $A\in H^\infty_{\mathscr{B}(\mathscr{E})}$.

From the fact that $F_1(e^{i\theta})\geq F_2(e^{i\theta})$ a. e. if and only if $M_{F_1}\geq M_{F_2}$ for $F_1, F_2\in L^\infty_{\mathscr{B}(\mathscr{E})}$, it follows for a positive operator valued function $F\in L^\infty_{\mathscr{B}(\mathscr{E})}$ that F has invertible values a. e. if and only if $M_F\geq M_{wI}$ where w is a bounded positive (non-zero) scalar function and I is the identity operator on $\mathscr{E}$. And clearly M_{wI} is one-to-one operator in the double commutant $\{U_0\}''$ of U_0.

Returning to the general case, let us consider the factorability of an operator $T\in\{U\}'$ for which there exists an one-to-one positive operator $D\in\{U\}''$ such that $D\leq T$.

THEOREM 2. *Let T be a positive operator in $\{U\}'$ and D an one-to-one positive operator in $\{U\}''$ such that $D\leq T$. Assume that there exists an one-to-one factorable operator $T_1\in\{U\}'$ such that $T_1\leq T$. Then T is factorable.*

PROOF. Since T_1 is factorable, $T_1=A_1^* A_1$ for some $A_1\in\mathscr{A}$. For each $f\in\mathscr{L}$, we have $\|A_1f\|=\|T_1^{1/2}f\|\leq\|T^{1/2}f\|$ because $T_1\leq T$, and so we can define a bounded operator X by $X(T^{1/2}f)=A_1f$ for $f\in\mathscr{L}$ and $X|(T^{1/2}\mathscr{L})^\perp=0$. Then $XT^{1/2}\in\mathscr{A}$ and X commutes with U because $T^{1/2}$ and $A_1\in\{U\}'$. By Theorem 1′ it is now enough to show that $X|T^{1/2}\mathscr{H})^-$ is one-to-one. If $Xg=0$ for some $g\in(T^{1/2}\mathscr{H})^-$, then there is a sequence $\{f_n\}$ in $\mathscr{H}$ such that $T^{1/2}f_n\to g$ and $A_1f_n\to 0$. Since $T_1=A_1^* A_1$, $T_1^{1/2}f_n\to 0$. Since $T\geq D$, there is a vector $h\in\mathscr{L}$ such that $D^{1/2}f_n\to h$. Then $T_1^{1/2}h=\lim_{n\to\infty} T_1^{1/2}D^{1/2}f_n=\lim_{n\to\infty} D^{1/2}T_1^{1/2}f_n$ (because $D^{1/2}\in\{U\}''$)$=0$, so $h=0$ because T_1 is one-to-one. Hence we have $D^{1/2}g=\lim_{n\to\infty} D^{1/2}T^{1/2}f_n=\lim_{n\to\infty} T^{1/2}D^{1/2}f_n=0$, and $g=0$ because D is one-to-one. Therefore $X|(T^{1/2}\mathscr{H})^-$ is one-to-one. This completes the proof.

The following corollary is an abstract generalization of the Devinatz theorem.

COROLLARY 2. *Let T be a positive operator in $\{U\}'$. If there exists an one-to-one factorable operator D in $\{U\}''$ such that $D\leq T$, then T is factorable.*

Our last theorem contains the operator generalization ([7] and [8]) of the Fejer-Riesz theorem which asserts that every positive trigonometric polynomial w is of the form $w=|f|^2$, where f is a analytic trigonometric polynomial of degree equal to the one of w.

THEOREM 3. *Let T be a positive operator in $\{U\}'$. Assume that there exists an operator $X\in\{U\}'$ such that $XT\in\mathscr{A}$ and $X|(T\mathscr{H})^-$ is one-to-one. Then T is factorable and its outer factor A satisfies $XA^*\in\mathscr{A}$.*

PROOF. We can assume, without loss of generality, that $T\leq I$. By assumption and Theorem 1', $T^2=A_1^* A_1$ for some $A_1\in\mathscr{A}$. Since $T\geq T^2=A_1^* A_1$ (because $T\leq I$), we have an operator $X_1\in\{U\}'$ such that $X_1 T^{1/2}=A_1$. From $T^2=A_1^* A_1$ and $(T^{1/2}\mathscr{L})^-=(\mathrm{Ker}\, T^{1/2})^\perp$, it follows that $X_1|(T^{1/2}\mathscr{L})^-$ is one-to-one. Therefore T satisfies the condition (ii) of Theorem 1', so T is factorable.

Let $T=A^*A$ where A is outer. Then we have $XA^*(Af)=XTf\in\mathscr{K}$ for all $f\in\mathscr{K}$. If $f\in\mathscr{K}\cap(A\mathscr{K})^\perp$ then $f\in(A\mathscr{L})^\perp$, so $XA^*f=0$. Hence $\mathscr{K}$ is invariant for XA^* and $XA^*\in\mathscr{A}$.

Now the operator generalization of the Fejer-Riesz theorem follows immediately. In fact, let F be a positive operator valued trigometric polynomial of degree N. Then the multiplication operator M_F satisfies the assumption in Theorem 3 with $X=M_{e^{iN\theta}I}$ (the multiplication operator by $e^{iN\theta}I$).

References

[1] R. G. DOUGLAS: On the operator equation $S^*XT=X$ and related topics, Acta Sci. Math. (Szeged) 30 (1969), 19–32.

[2] R. GELLAR and L. PAGE: Inner-outer factorizations of operators, J. Math. Anal. Appl. 61 (1977), 151–158.

[8] H. HELSON: Lecture on invariant subspaces, Acadimic Press, New York, 1964.

[4] B. MOORE, M. ROSENBLUM and J. ROVNYAK: Toeplitz operators associated with isometries, Proc. Amer. Math. Soc. 49 (1975), 189–194.

[5] L. PAGE: Applications of Sz.-Nagy and Foias lifting theorem, Indiana Univ. Math. J. 20 (1970), 135–145.

[6] H. RADJAVI and P. ROSENTHAL: Invariant subspaces, Springer-Verlag, 1973.

[7] M. ROSENBLUM: Vectorial Toeplitz operators and the Fejer-Riesz theorem, J. Math. Anal. Appl. 23 (1968), 139–147.

[8] M. ROSENBLUM and J. ROVNYAK: The factorization problem for nonnegative operator valued functions, Bull. Amer. Math. Soc. 77 (1971), 287–318.

Division of Applied Mathematics
Research Institute of Applied Electricity
Hokkaido University
Sapporo, Japan