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Extension of involutions on spheres

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Introduction

Let Z_{2q} be a cyclic group of order $2q$ generated by T'. Suppose that a free involution T is given on the sphere S^n. If there exists a free Z_{2q}-action on S^n such that the restriction of the Z_{2q}-action to the Z_2-action coincides with T on S^n, i.e., $T'|Z_2=T'^q=T$, then we call that the involution T on S^n extends to a free Z_{2q}-action. In this paper, we show that:

THEOREM. *Let q be any integer and $n\geqq 1$. Then, every picewise linear (resp. topological) free involution on S^{2n+1} extends to a picewise linear (resp. topological) free Z_{2q}-action on S^{2n+1}.*

The theorem follows from a similar method to the proof of the following proposition.

PROPOSITION 3.1. *Let T be a free involution on a homotopy sphere Σ^{2n+1} such that the normal invariant $\eta(\Sigma^{2n+1}/T)\in Im\{p^* : [L^{2n+1}(2q), G/H]\to[p^{2n+1}, G/H]\}$ and $(q, |\Theta_{2n+1}(\partial\pi)|)=1$, where $p: p^{2n+1}\to L^{2n+1}(2q)$ is the projection and $H=O$, PL or TOP and $n\geqq 2$. Then, T extends to a free Z_{2q}-action on Σ^{2n+1}.*

§1 and §2 will be devoted to the preliminaries of the above proposition. In §3, we shall prove it and the above theorem.

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1. Definition of transfer

Let X^{2n-1} be a $(2n-1)$-dimensional closed oriented manifold with fundamental group π. Denote by $\mathscr{S}^{\varepsilon}_{H}(X)$ the set of ε-homotopy structures on X, where $H=O$ or PL and $\varepsilon=h$ or s. An ε-homotopy equivalence $f: M\to X$ determines a normal map

$$\begin{array}{ccc} \nu_M & \xrightarrow{\ b\ } & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{\ f\ } & X, \end{array}$$

$$\tau\Big(\theta(F_y H)\Big)=\theta\Big((F_y)_1 H_1\Big)=\theta\Big((F_y)_1\Big)$$
$$=\tau\Big(\theta(F_y)\Big),$$

we have

$$\tau(x+y)=\tau(x)+\tau(y).$$

REMARK 1.2. The inclusion $i: \pi_1 \subset \pi$ induces a homomorphism $i_*: L_{2n}^{\varepsilon}(\pi_1)\to L_{2n}^{\varepsilon}(\pi)$. Then we have the following as a property of the transfer (See [2, p. 54]).

PROPOSITION 1.3. *Let $C(\pi)$ be the center of π. If $\pi_1\subset C(\pi)$, then*

$$\tau i^*(x)=[\pi;\pi_1]\,x: \quad L_{2n}^{\varepsilon}(\pi_1)\xrightarrow{i_*} L_{2n}^{\varepsilon}(\pi)\xrightarrow{\tau} L_{2n}^{\varepsilon}(\pi_1),$$

where $[\pi;\pi_1]$ is the index of π_1 in π.

The trivial map $p: \pi\to 1$ induces the homorphism $p_*: L_*^{\varepsilon}(\pi)\to L_*(1)$ which is onto, and so we have

$$L_*^{\varepsilon}(\pi)=L_*^{\varepsilon}(\widetilde{\pi})\oplus L_*(1),$$

where $L_*^{\varepsilon}(\widetilde{\pi})=Ker\,[p_*: L_*^{\varepsilon}(\pi)\to L_*(1)]$ is the reduced Wall group.

Our goal in this section is the following lemma.

LEMMA 1.4. $\tau: L_0^{\varepsilon}(Z_{2q})\to L_0(Z_2)$ is onto modulo $L_0(1)$. Here $L_0(1)\subset L_0(Z_2)$.

Proof. $L_0(Z_2)$ is isomorphic to $8Z\oplus 8Z$. The correspondence is given by

$$x=\theta(F,W)\longmapsto\Big(I(W),I(\widetilde{W})\Big),$$

where $F: W\to P^{4k-1}\times I$ is a normal map, P^{4k-1} the standard projective $(4k-1)$-space, and $I(W)$ (resp. $I(\widetilde{W})$) is the index of W (resp. $\widetilde{W}$), $\widetilde{W}$ the universal cover of W.

Let T be a generator of Z_2. The multi-signature invariant $\rho(\mathrm{T},\mathrm{x})$ for $x\in L_0(Z_2)$ is given by

(1) $$\rho(T,x)=Sign(T,\widetilde{W})=2I(W)-I(\widetilde{W}).$$

It follows that

(2) $$\rho(T,-): L_0(\widetilde{Z_2})\longrightarrow 8Z$$

is an isomorphism, and Ker $\rho=L_0(1)$ which is isomorphic to $\{(m,2m)\}_{m\in Z}\subset L_0(Z_2)$. For the Atiyah-Singer invariants $\sigma(T,\partial_{\pm}\widetilde{W})$ of $\partial_{\pm}\widetilde{W}$, we have

(3) $$\rho(T,x)=\sigma(T,\partial_-\widetilde{W})-\sigma(T,\partial_+\widetilde{W}).$$

by taking $\xi=g^*\nu_M$, where ν_M is the normal bundle of M and g is an ε-homotopy inverse of f. Take $x\in L_{2n}^{\varepsilon}(\pi)$. By the realization theorem of Wall [9], there is a triad $(W, \partial_+ W, \partial_- W)$ and a map F of this to the triad $(X\times I,\ X\times 0,\ X\times 1)$ satisfying that

(1) There is a bundle map B covering F of the normal bundle of W which extends the bundle map b.

(2) $(\partial_- W, F|\partial_- W)=(M,f)$.

(3) $F|\partial_+ W$ is an ε-homotopy equivalence.

(4) $\theta(F, W)=x\in L_{2n}^{\varepsilon}(\pi)$.

Let π_1 be a subgroup of π and let $\tilde{X}$ be the universal cover of X. Put $X_1=\tilde{X}/\pi_1$. For the projection $p: X_1\to X$, we consider the following pull-back diagram

$$\begin{array}{ccc} W_1 & \xrightarrow{F_1} & X_1\times I \\ \downarrow p & & \downarrow p \\ W & \xrightarrow{F} & X\times I. \end{array}$$

The pair (F_1, W_1) has the same properties corresponding with (1), (2) and (3). We set

$$\tau(x)=\theta(F_1)\in L_{2n}^{\varepsilon}(\pi_1).$$

It is easy to see that the definition of τ is independent of choices of the cobordisms which represent x. In particular, we may start from $(X, id)\in \mathcal{S}_H^{\varepsilon}(X)$ in place of (M,f).

LEMMA 1.1. *$\tau: L_{2n}^{\varepsilon}(\pi)\to L_{2n}^{\varepsilon}(\pi_1)$ is a well-defined homomorphism.*

PROOF. We can show that τ is a homomorphism similarly as the proof of the theorem [7, p. 50]. Let $x,\ y\in L_{2n}^{\varepsilon}(\pi)$, and let $F_x: W_x\to X\times I$ be the cobordism between $id: X\to X$ and $f_x: M_x\to X$ such that $\theta(F_x)=x$. We represent y by the cobordism $F_y: W_y\to X\times I$ similarly. We consider cobordisms

(i) $F_{-x}: W_{-x}\to X\times I$ between f_x and id such that $\theta(F_{-x})=-x$.

(ii) $F_{yx}: W_{yx}\to M_y\times I$ between $id: M_y\to M_y$ and $f_{yx}: M_{yx}\to M_y$ such that $\theta(F_{yx})=x$.

Combining these with $id: W_y\to W_y$, we have a map

$$H': W_{-x}\cup W_y\cup W_{yx}\longrightarrow W_y.$$

It follows that $\theta(H')=\theta(F_{-x})+\theta(id)+\theta(F_{yx})=0$. We have an ε-homotopy equivalence $H: W_y'\to W_y$. We take $F_x\cup F_y H: W_x\cup W_y'\to X\times I$ as the normal map $F_{x+y}: W_{x+y}\to X\times I$ corresponding to $x+y$. Since

By [8], there exists a homotopy equivalence f_i of a homotopy complex projective 3-space HCP^3 into the complex projective 3-space CP^3 which is transverse regular to CP^2 and such that the restricted normal map

$$\bar{f}_i:\ N^4=f_i^{-1}(CP^2)\longrightarrow CP^2 \qquad \text{satisfies}$$

$$(4) \qquad \theta(\bar{f}_i)=8i \quad \text{for any} \quad i\in Z.$$

Consider the S^1-fibration $p:\ L^7(2q)\to CP^3$, where $L^7(2q)$ being the 7-dimensional standard lens space. Pulling back this fibration by f_i, we have a homotopy lens space L_i^7 and an ε-homotopy equivalence

$$(5) \qquad g_i:\ L_i^7\to L^7(2q),$$

which is transverse to $L^5(2q)$, $L_i^5=g_i^{-1}(L^5(2q))$, and $\bar{g}_i=g_i|L_i^5:\ L_i^5\to L^5(2q)$ is the restricted normal map. Since the surgery obstruction $\theta(\bar{g}_i)=0$ in $L_5^{\varepsilon}(Z_{2q})$, $\bar{g}_i$ is normally cobordant to an ε-homotopy equivalence $g_i':\ L_i^{5\prime}\to L^5(2q)$. By the normal cobordism extension property (See [7, p. 45]), we may extend the normal cobordism between $\bar{g}_i:\ L_i^5\to L^5(2q)$ and $g_i':\ L_i^{5\prime}\to L^5(2q)$ to a cobordism between $g_i:\ L_i^7\to L^7(2q)$ and $h_i:\ L_i^{7\prime}\to L^7(2q)$ such that $h_i^{-1}(L^5(2q))=L_i^{5\prime}$, and $h_i|L_i^{5\prime}=g_i'$. Let $N(L^5(2q))$ be a tubular neighbourhood of $L^5(2q)$ in $L^7(2q)$. Then the surgery obstruction of h_i, $\theta(h_i)$ is equal to the surgery obstruction of the restriction map

$$h_i:\ L_i^{7\prime}-\text{int}\ N(L_i^{5\prime})\longrightarrow L^7(2q)-\text{int}\ N\big(L^5(2q)\big)\cong D^6\times S^1$$

which is a homotopy equivalence on the boundary, i. e.,

$$\theta(h_i)=\theta\big(h_i\big|L_i^{7\prime}-\text{int}\ N(L_i^{5\prime})\big)\in L_7(Z)\cong L_6(1)=Z_2.$$

Here $N(L_i^{5\prime})$ is a tubular neighbourhood of $L_i^{5\prime}$ in $L_i^{7\prime}$. Since $\theta(h_i)=\theta(g_i)=0$, there exists a normal cobordism rel. boundary between $h_i|(L_i^{7\prime}-\text{int}\ N(L_i^{5\prime}))$ and a homotopy equivalence $h_i':\ E\to D^6\times S^1$. Put $M_i^7=E\cup N(L_i^{5\prime})$. There is an ε-homotopy equivalence

$$k:\ M_i^7\longrightarrow L^7(2q)$$

defined to be h_i' on E and h_i on $N(L_i^{5\prime})$. Combining these cobordisms, there is a normal cobordism

$$F:\ V^8\longrightarrow L^7(2q)\times I$$

between $g_i:\ L_i^7\to L^7(2q)$ and $k:\ M_i^7\to L^7(2q)$. It follows that $\theta(F, V)\in L_8^{\varepsilon}(Z_{2q})$. We have

$$(6) \qquad \tau\theta(F, V)=\theta(F_1, V_1)\in L_8(z_2),$$

where $F_1: V_1 \to P^7 \times I$ is a normal map, and the universal cover $\partial \tilde{V}_1 = \tilde{L}_i^7 \cup \tilde{M}_i^7$.

Since $L_i^7 \to HCP^3$ is the S^1-fibration, so $\sigma(T, \tilde{L}_i^7)$ is the value $\sigma(-1, \widetilde{HCP^3})$ of the Atiyah-Singer invariant $\sigma(t, \widetilde{HCP^3})$, $t \in S^1$ at $t=-1$. Hence from (4) we have

$$(7) \qquad \sigma(T, \tilde{L}_i^7) = \sigma(-1, \widetilde{HCP^3}) = 8i\,.$$

On the other hand, the Atiyah-Singer invariant $\sigma(T, Q^{4k-1})$ is equal to the Browder-Livesay invariant $\sigma(Q^{4k-1})$ for a homotopy projective $(4k-1)$-space Q^{4k-1} (See [3]). If we note that the Browder-Livesay invariant $\sigma(Q^{4k-1})$ is the desuspension invariant of Q^{4k-1}, we see that the q-fold covering manifold $\bar{M}_i^7$ is a homotopy projective space which desuspends (in fact, $(T, \tilde{M}_i^7)$ is a double suspension). Hence we have

$$(8) \qquad \sigma(T, \tilde{M}_i^7) = 0\,.$$

By (3), (6), (7) and (8), it follows that

$$\tau\big(\theta(F, V)\big) = 8i\,.$$

Therefore, by (2), $\tau: L_0^\varepsilon(Z_{2q}) \to L_0(Z_2)$ is onto modulo $L_0(1)$. This completes the proof of the lemma.

2. Surgery exact sequence

Let $\mathcal{S}_H^\varepsilon(X)$ be the set of ε-homotopy structures on X and let $[X, G/H]$ be the set of normal cobordisms classes of normal maps into X. We consider the surgery exact sequences for $X = P^{2n+1}$ and $L^{2n+1}(2q)$ (See [9]). The projection $p: P^{2n+1} \to L^{2n+1}(2q)$ induces a map

$$p!: \mathcal{S}_H^\varepsilon\big(L^{2n+1}(2q)\big) \longrightarrow \mathcal{S}_H^\varepsilon(P^{2n+1})$$

by taking q-fold covering. Similarly, p induces a map

$$p^*: \big[L^{2n+1}(2q), G/H\big] \longrightarrow [P^{2n+1}, G/H]\,.$$

Then we have the following commutative diagram of exact sequences for $n \geqq 2$.

$$(2.1) \qquad \begin{array}{ccccccc} L_{2n+2}(Z_2) & \xrightarrow{\omega} & \mathcal{S}_H(P^{2n+1}) & \xrightarrow{\eta} & [P^{2n+1}, G/H] & \xrightarrow{\theta} & L_{2n+1}(Z_2) \\ \uparrow \tau & & \uparrow p! & & \uparrow p^* & & \\ L_{2n+2}^\varepsilon(Z_{2q}) & \xrightarrow{\omega} & \mathcal{S}_H^\varepsilon(L^{2n+1}(2q)) & \xrightarrow{\eta} & [L^{2n+1}(2q), G/H] & \xrightarrow{\theta} & L_{2n+1}^\varepsilon(Z_{2q}) \end{array}$$

LEMMA 2.2. *p^* : $[L^{2n+1}(2q), G/H] \to [P^{2n+1}, G/H]$ is onto for $H=PL$, TOP.*

PROOF. The projection $p: L^{2n+1}(s) \to CP^n$ induces a map $p^*: [CP^n, G/H] \to [L^{2n+1}(s), G/H]$ for each integer $s \in Z$. The lemma follows from the fact that p^* is onto (See [9, LEMMA 14 A. 2, p 186]).

The natural projection $d: Z_{2q} \to Z_2$ induces the isomorphism

$$d: L_3^s(Z_{2q}) \cong Z_2 \longrightarrow L_3(Z_2) \cong Z_2 . \tag{2.3}$$

We have the following lemma.

LEMMA 2.4. *There is a following commutative diagram for $H=O$, PL, or TOP and $k \geq 1$.*

$$\begin{array}{ccc} [P^{4k+3}, G/H] & \xrightarrow{\theta} & L_3(Z_2) \\ \uparrow p^* & & \uparrow d \\ [L^{4k+3}(2q), G/H] & \xrightarrow{\theta} & L_3^s(Z_{2q}) \end{array}$$

REMARKS. The lemma of *PL* case is seen in [9, THEOREM 14. 4] and the smooth case is seen in [5, THEOREM 3.7]. Throughout the cases $H=O$, *PL*, or *TOP*, the proof of this lemma is to determine the nontrivial obstruction for the fundamental group of a cyclic group of even order in place of a cyclic group of odd order in [1, THEOREM 1′].

PROOF. Take a normal map $f: L^{4k+3} \to L^{4k+3}(2q)$. As in the proof of the lemma 1.4, there is a normal map $g: L_1^{4k+3} \to L^{4k+3}(2q)$ which is normally cobordant to f such that $g: g^{-1}(L^{4k+1}(2q)) \to L^{4k+1}(2q)$ is an ε-homotopy euivalence. Let $N(g^{-1}(L^{4k+1}(2q))$ be a tubular neighbourhood of $g^{-1}L^{4k+1}(2q))$ in L_1^{4k+3}. It follows that

$$\theta(f) = \theta\left(g \middle| L_1^{4k+3} - \text{int } N\left(g^{-1}\left(L^{4k+1}(2q)\right)\right)\right),$$

where $g: L_1^{4k+3} - \text{int } N(g^{-1}(L^{4k+1}(2q))) \to D^{4k+2} \times S^1$ is a normal map which is a homotopy equivalence on the boundary. Make g transverse to $D^{4k+2} \times t \subset D^{4k+2} \times S^1$ such that $g^{-1}(S^{4k+1} \times t)$ is a homotopy sphere and $g: g^{-1}(S^{4k+1} \times t) \to S^{4k+1}$ is a homotopy equivalence. Let $d'': L_3^s(Z_{2q}) \to L_2(1)$ be the homomorphism defined by the composition of $L_3^s(Z_{2q}) \to L_3(Z)$ and $L_3(Z) \underset{\cong}{\longrightarrow} L_2(1)$. We have

$$d''\theta(f) = \theta\left(g \middle| g^{-1}(D^{4k+2} \times t)\right) \in L_2(1) \cong Z_2 .$$

The q-fold covering map of f induces a normal map $p^*(f): Q \to P^{4k+3}$. Here Q is the q-fold cover of L^{4k+3}. Then it follows that the surgery obstruction

$\theta(p^*(f))$ is equal to $\theta(g|g^{-1}(D^{4k+2}\times t))$, i.e., $d'\theta(p^*(f))=\theta(g|g^{-1}(D^{4k+2}\times t))$, where $d' : L_3(Z_2)\to L_2(1)$ is the isomorphism. From the commutative diagram

$$\begin{array}{ccc} L_3(Z_2) & \xrightarrow{\ d'\ } & L_2(1) \\ \uparrow d & \nearrow d'' & \\ L_3^s(Z_{2q}) & & \end{array},$$

we have $d\theta(f)=\theta(p^*(f))$. This proves the lemma.

3. Proof of Theorem

First we prove the following.

PROPOSITION 3.1. *Let $H=O$, PL, or TOP and $n\geqq 2$. Let T be a free involution on a homotopy sphere Σ^{2n+1} such that $\eta(\Sigma^{2n+1}/T)\in Im\,[p^* : [L^{2n+1}(2q), G/H]\to[P^{2n+1}, G/H]]$ and $(q, |\Theta_{2n+1}(\partial\pi)|)=1$, where $\Theta_{2n+1}(\partial\pi)$ is the group of homotopy $(2n+1)$-spheres which bound parallelizable manifolds, and $|\Theta_{2n+1}(\partial\pi)|$ is the order of $\Theta_{2n+1}(\partial\pi)$. Then, T extends to a free Z_{2q}-action on Σ.*

REMARK. For example, $q=3$ satisfies $(q, |\Theta_{2n+1}(\partial\pi)|)=1$ for any $n\geqq 2$.

PROOF. Case 1. $n\equiv 0(2)$. Let T be a free involution on Σ^{4k+1} such that $\eta(\Sigma^{4k+1}/T)\in \mathrm{Im}\,p^*$. Since $\theta: [L^{4k+1}(2q), G/H]\to L_{4k+1}^s(Z_{2q})$ is zero, there exists an element $L_1^{4k+1}\in\mathcal{S}_H^s(L^{4k+1}(2q))$ such that

$$\eta\big(p!(L_1^{4k+1})\big)=\eta(\Sigma^{4k+1}/T)\,.$$

Since the action ω of $L_2(Z_2)\cong L_2(1)$ is to add the Kervaire manifold, we have

$$\Sigma^{4k+1}/T\cong p!(L_1^{4k+1})$$

or

$$\Sigma^{4k+1}/T\cong p!(L_1^{4k+1})\,\#\,\Sigma_K^{4k+1}\,,$$

where Σ_K^{4k+1} is the Kervaire sphere.
If $p!(L_1^{4k+1})\#\Sigma_K^{4k+1}\cong\Sigma^{4k+1}/T$, we take $L_2^{4k+1}=L_1^{4k+1}\#\Sigma_K^{4k+1}\in\mathcal{S}_H^s(L^{4k+1}(2q))$. Since $(q, |\Theta_{4k+1}(\partial\pi)|)=1$ and the order of $\Theta_{4k+1}(\partial\pi)$ is at most 2, we have

$$\begin{aligned} p!(L_2^{4k+1}) &\cong p!(L_1^{4k+1})\,\#\,q\Sigma_K^{4k+1} \\ &\cong p!(L_1^{4k+1})\,\#\,\Sigma_K^{4k+1}\cong\Sigma^{4k+1}/T\,. \end{aligned}$$

Hence T extends to a free Z_{2q}-action.

Case 2. $n\equiv 1(2)$. Let T be a free involution on Σ^{4k+3} such that $\eta(\Sigma^{4k+3}/T)\in\mathrm{Im}\,p^*$. By Lemma 2.4, there exists an element $L_1^{4k+3}\in\mathcal{S}_H^s(L^{4k+3}$

$(2q))$ such that $\eta(p!(L_1^{4k+3}))=\eta(\Sigma^{4k+3}/T)$. From (2. 1), we have $\Sigma^{4k+3}/T=\omega(x, p!(L_1^{4k+3}))$ for some $x\in L_0(Z_2)$. By Lemma 1. 4, there exists $y\in L_0^\varepsilon(Z_{2q})$ such that $x-\tau(y)=x_0$ for some $x_0\in L_0(1)\subset L_0(Z_2)$. Put $L_2=\omega(y, L_1^{4k+3})\in\mathcal{S}_H^\varepsilon(L^{4k+3}(2q))$. We have from the commutativity of (2. 1) that

$$\Sigma^{4k+3}/T=\omega\left(x_0, p!(L_2)\right).$$

Since the action ω of $L_0(1)$ is to add the Milnor manifolds, it follows that

$$\Sigma^{4k+3}/T=p!(L_2)\#m\Sigma_1^{4k+3}$$

for some $m\in Z$, where Σ_1^{4k+3} is a generator of $\Theta_{4k+3}(\partial\pi)$. By the condition $(q, |\Theta_{4k+3}(\partial\pi)|)=1$, there is an integer n such that $nq\equiv 1 \mod|\Theta_{4k+3}(\partial\pi)|$. We take $L_3=L_2\# nm\Sigma_1^{4k+3}\in\mathcal{S}_H^\varepsilon(L^{4k+3}(2q))$. Then we have

$$\begin{aligned} p!(L_3) &\cong p!(L_2)\# qnm\Sigma_1^{4k+3} \\ &\cong p!(L_2)\# m\Sigma_1^{4k+3}\cong\Sigma^{4k+3}/T. \end{aligned}$$

Hence T extends to a free Z_{2q}-action. This proves the proposition.

PROOF OF THEOREM IN INTRODUCTION.

By [6], any free involution on S^3 is conjugate to the antipodal map. Therefore, T extends to a free Z_{2q}-action on S^3. Let T be a free involution on S^{2n+1} for $n\geqq 2$. It follows from Lemma 2.2 that $\eta(S^{2n+1}/T)\in\mathrm{Im}p^*$. Similarly to Proposition 3. 1, $S^{2n+1}/T\cong\omega(x, p!(M))$ for some $M\in\mathcal{S}_H^\varepsilon(L^{2n+1}(2q))$ and $x\in L_{2n+2}(Z_2)$. Since the action ω of $L_{2n+2}(1)$ on $\mathcal{S}_H(P^{2n+1})$ is trivial for $H=PL$, TOP, we have $S^{2n+1}/T\cong p!(M_1)$ for some $M_1\in\mathcal{S}_H^\varepsilon(L^{2n+1}(2q))$. Hence T extends to a free Z_{2q}-action.

COROLLARY. (See [4]) *There exist non-triangulable (simple) homotopy lens spaces* $\bar{L}^{2n+1}(2q)$ *for* $n\geqq 2$ *and* $q\geqq 1$.

PROOF. From the computations of $[P^{2n+1}, G/H]$ for $H=PL$, TOP, there is an exact sequence

$$\mathcal{S}_{PL}(P^{2n+1})\xrightarrow{\ \Phi\ }\mathcal{S}_{TOP}(P^{2n+1})\xrightarrow{\ \Psi\ }Z_2\longrightarrow 0,$$

where Φ is the obvious map, and Ψ is the obstruction map (See [7]).

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