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Notes on eigenvalues of Laplacian acting on p-forms

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1. Introduction.

By generalizing the method of Payne-Pólya-Weinberger ([4]), for a compact domain on a minimal hypersurface in the Euclidean space, S. Y. Cheng ([2]) proved an inequality between successive eigenvalues of the Laplacian acting on C^{∞}-functions on this domain. On the other hand, M. Maeda ([3]) has got a similar result for a compact minimal submanifold with or without boundary in the unit sphere. The purpose of the present note is to present a similar inequality between successive eigenvalues of the Laplacian acting on p-forms (resp. 1-forms) on a compact and oriented minimal hypersurface (resp. minimal submanifold of any codimension) without boundary in the unit sphere.

Let M be an $m(\geqslant 2)$-dimensional compact and oriented Riemannian manifold without boundary with the Riemannian metric g. For each $p=0, 1, \cdots, m$, $A^{p}(M)$ denotes the space of all differential p-forms on M. For ω, $\eta\in A^{p}(M)$, we can define a C^{∞}-function $(\omega|\eta)$ on M as follows: $(\omega|\eta)$ is locally given by

$$(\omega|\eta)=\sum_{\substack{j_{1}\cdots j_{p}\\ k_{1}\cdots k_{p}=1}}^{m}\omega_{j_{1}\cdots j_{p}}\eta_{k_{1}\cdots k_{p}}g^{j_{1}k_{1}}\cdots g^{j_{p}k_{p}}$$

where we put $g_{jk}=g\left(\frac{\partial}{\partial z^{j}}, \frac{\partial}{\partial z^{k}}\right)$ for a local coordinate system $\{z^{j}; j=1, \cdots, m\}$ of M and (g^{jk}) is the inverse matrix of (g_{jk}). The inner product $\langle\ ,\ \rangle$ on $A^{p}(M)$ is defined by $\langle\omega, \eta\rangle:=\int_{M}(\omega|\eta)\,dV_{M}$. Here dV_{M} denotes the volume form of M. We put $||\omega||:=\sqrt{\langle\omega, \omega\rangle}$. Let $d: A^{p}(M)\rightarrow A^{p+1}(M)$ be the operator of exterior differentiation and $\delta: A^{p+1}(M)\rightarrow A^{p}(M)$ be the adjoint operator of d with respect to $\langle\ ,\ \rangle$. Then the Laplace-Beltrami operator (Laplacian for short) Δ acting on p-forms is defined by $\Delta:=d\delta+\delta d: A^{p}(M)\rightarrow A^{p}(M)$. The Laplacian is elliptic and a self-adjoint operator. Thus all eigenvalues of the Laplacian form a discrete infinite sequence:

$$0=\lambda_{0}^{p}\leqslant\lambda_{1}^{p}\leqslant\cdots\leqslant\lambda_{n}^{p}\leqslant\cdots;\ \lambda_{n}^{p}\longrightarrow\infty$$

where each λ_{n}^{p} is repeated as many time as its multiplicity. Let $\{\varphi_{r}; r=$

$1, 2, \cdots\}$ be a complete orthonormal base in $A^p(M)$ consisting of eigenforms of the Laplacian. Then we have the so-called minimum principle.

Minimum principle.

$$\lambda_{n+1}^p = \inf\left\{\langle \Delta\omega, \omega\rangle/\|\omega\|^2 ;\ \omega\in A^p(M),\ \omega\neq 0,\ \langle\omega, \varphi_r\rangle = 0 \right.$$
$$\left. \text{for all } r=1, \cdots, n\right\}.$$

In Section 2, we shall prove the following theorem.

THEOREM 1. *Let M be an $m(\geqslant 2)$-dimensional compact and oriented Riemannian manifold without boundary minimally immersed in the unit $(m+1)$-sphere. Then for each numbers $n\geqslant 1$ and $1\leqslant p\leqslant m$, we have the inequality*:

$$(1)\qquad \lambda_{n+1}^p < \lambda_n^p + \frac{2(\Omega+1)}{nm}\left\{\sum_{r=1}^{n}\lambda_r^p + pn\left(p-m+S(M)^2\right)\right\} + \left(\frac{1}{\Omega}+1\right)m\,.$$

Here $S(M)$ denotes the maximum of the length of the second fundamental form of the immersion and Ω is a free parameter such that $\Omega\geqslant 1$.

In Section 3, we shall prove the following theorem.

THEOREM 2. *Let M be an $m(\geqslant 2)$-dimensional compact and oriented Riemannian manifold without boundary minimally immersed in the unit sphere (of any codimension). Then for each number $n\geqslant 1$, we have the inequality*:

$$(2)\qquad \lambda_{n+1}^1 < \lambda_n^1 + \frac{2(\Omega+1)}{nm}\left\{\sum_{t=1}^{n}\lambda_t^1 - nr(M)\right\} + \left(\frac{1}{\Omega}+1\right)m\,.$$

Here $r(M)$ denotes the minimum of the Ricci curvature of M and Ω is a free parameter such that $\Omega\geqslant 1$.

2. Estimate of λ_{n+1}^p.

Let M be an $m(\geqslant 2)$-dimensional compact and oriented Riemannian manifold without boundary minimally immersed in the unit $(m+1)$-sphere $S^{m+1}\subset E^{m+2}$. Let $\{x^\alpha ;\ \alpha=1, \cdots, m+2\}$ denote an orthonormal coordinate system in E^{m+2}. We also denote by x^α the restriction of x^α to M. The next is well known (cf. T. Takahashi [6]).

LEMMA 2.1. $\Delta x^\alpha = m x^\alpha$ $(1\leqslant\alpha\leqslant m+2)$.

Fix arbitrary numbers $1\leqslant p\leqslant m$ and $n\geqslant 1$. We use the following convention on the range of indices unless otherwise stated: $i, j, k, l, j_1, \cdots, j_p = 1, \cdots, m$; $\mu, \nu = 1, \cdots, p$; $r, t = 1, \cdots, n$ and $\alpha = 1, \cdots, m+2$.

LEMMA 2.2. *For any $\omega\in A^p(M)$, we have the inequality*

$$(3)\qquad \langle \nabla\omega, \nabla\omega\rangle \leqslant \langle \Delta\omega, \omega\rangle + p\left(p-m+S(M)^2\right)\cdot\|\omega\|^2 .$$

Here ∇ denotes the Levi-Civita connection defined by the Riemannian metric g of M.

PROOF. It is known that $\Delta\omega$ is locally expressed in terms of ∇, the Riemannian curvature tensor R of M and the Ricci tensor ρ of M (cf. G. de Rham [5], p. 131). That is, if $p\geqslant 2$, then we have

$$(4)\qquad (\Delta\omega)_{j_1\cdots j_p} = -\sum_i \nabla^i\nabla_i\omega_{j_1\cdots j_p} - \sum_{k,\nu}(-1)^\nu \rho_{j_\nu\cdot}{}^k\cdot\omega_{kj_1\cdots\hat{j}_\nu\cdots j_p} + 2\sum_{i,k}\sum_{\mu<\nu}(-1)^{\mu+\nu} R^{k\ i}_{j_\nu\cdot j_\mu}\cdot\omega_{ikj_1\cdots\hat{j}_\mu\cdots\hat{j}_\nu\cdots j_p}$$

and if $p=1$, then we have

$$(5)\qquad (\Delta\omega)_j = -\sum_i \nabla^i\nabla_i\omega_j + \sum_k \rho_j{}^k\cdot\omega_k .$$

Here $\wedge$ over j_ν indicates that it is omitted. Fix an arbitrary point $z_0\in M$. Since the codimension of M in S^{m+1} is equal to one, we can choose a normal coordinate system $\{z^j\}$ of M about z_0 such that the second fundamental form (S_{jk}) at z_0 is given by a diagonal matrix, i. e., $S_{jk}=\delta_{jk}\xi_k$. Here ξ_k denote the principal curvatures at z_0. Then, at z_0, the equation of Gauss implies

$$(6)\qquad R^{ki}_{j\cdot l} = \delta_{jl}\cdot\delta_{ik} - \delta_{kl}\cdot\delta_{ij} + \delta_{jl}\cdot\delta_{ik}\cdot\xi_l\cdot\xi_k - \delta_{kl}\cdot\delta_{ij}\cdot\xi_l\cdot\xi_j .$$

By means of $\sum_i \xi_i = 0$, we have, at z_0,

$$(7)\qquad \rho_{j\cdot}{}^k = -\sum_i R^{ki}_{i\cdot j} = (m-1-\xi_j\cdot\xi_k)\cdot\delta_{jk} .$$

First we assume that $p\geqslant 2$. From (4), (6) and (7), we have

$$-\sum_i \nabla^i\nabla_i\omega_{j_1\cdots j_p} = (\Delta\omega)_{j_1\cdots j_p} + p(p-m)\cdot\omega_{j_1\cdots j_p} + (\sum_\nu \xi_{j_\nu})^2\cdot\omega_{j_1\cdots j_p} .$$

Since $\omega^{j_1\cdots j_p} = \omega_{j_1\cdots j_p}$ at z_0, we get

$$(-\sum_i \nabla^i\nabla_i\omega|\omega) = (\Delta\omega|\omega) + p(p-m)\cdot(\omega|\omega) + \sum_{j_1\cdots j_p}(\sum_\nu \xi_{j_\nu})^2\cdot(\omega_{j_1\cdots j_p})^2 .$$

On the other hand, by Schwartz inequality, we have

$$\begin{aligned}\sum_{j_1\cdots j_p}(\sum_\nu \xi_{j_\nu})^2\cdot(\omega_{j_1\cdots j_p})^2 &= p\,!\sum_{j_1<\cdots<j_p}(\sum_\nu \xi_{j_\nu})^2\cdot(\omega_{j_1\cdots j_p})^2 \\ &\leqslant p\,!\sum_{j_1<\cdots<j_p} p(\sum_\nu \xi_{j_\nu}^2)\cdot(\omega_{j_1\cdots j_p})^2 \leqslant p(\sum_i \xi_i^2)\cdot p\,!\sum_{j_1<\cdots<j_p}(\omega_{j_1\cdots j_p})^2 \\ &\leqslant p\cdot S(M)^2\cdot(\omega|\omega) ,\end{aligned}$$

because $\sum_i \xi_i^2$ is equal to the value of the square of the length of the second fundamental form at z_0. Hence we have

$$(-\sum_i \nabla^i \nabla_i \omega|\omega) \leqslant (\Delta\omega|\omega) + p\left(p-m+S(M)^2\right)(\omega|\omega) .$$

By integrating both sides of the above inequality, we get by Stokes formula

$$\begin{aligned}\langle \nabla\omega, \nabla\omega\rangle &= \langle -\sum_i \nabla^i \nabla_i \omega, \omega\rangle \\ &\leqslant \langle \Delta\omega, \omega\rangle + p\left(p-m+S(M)^2\right)\cdot\|\omega\|^2 .\end{aligned}$$

Next in the case $p=1$, by means of (5) and (7), we have a similar argument as above to obtain

$$\langle \nabla\omega, \nabla\omega\rangle \leqslant \langle \Delta\omega, \omega\rangle + \left(1-m+S(M)^2\right)\cdot\|\omega\|^2 . \qquad \text{q. e. d.}$$

REMARK 2. 1. The Riemannian curvature tensor R is defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]\, Z - \nabla_{[X,Y]} Z$$

and we put

$$R\left(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^l}\right)\frac{\partial}{\partial z^j} = \sum_i R^i_{jkl}\frac{\partial}{\partial z^i} .$$

The Ricci tensor ρ is defined by $\rho(Y, Z) = Trace\,(X \to R(X, Y)\, Z)$. Thus we have

$$\rho_{jk} = \rho\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right) = \sum_i R^i_{jik} .$$

REMARK 2. 2. If M is not totally geodesic, then by means of the inequality of J. Simons we have $S(M)^2 \geqslant m$ (cf. B. Y. Chen [1], p. 94).

Let $\{\varphi_h;\ h=1, 2, \cdots\}$ be a complete orthonormal base in $A^p(M)$ consisting of eigenforms of the Laplacian. Put $a^\alpha_{rt} := \langle x^\alpha\varphi_r, \varphi_t\rangle$ $(=a^\alpha_{tr})$ and $U^\alpha_r := x^\alpha\varphi_r - \sum_t a^\alpha_{rt}\varphi_t$.

LEMMA 2. 3. $\sum_{\alpha,r}\langle \nabla_{grad(x^\alpha)}\varphi_r, U^\alpha_r\rangle = -(m/2)\sum_{\alpha,r,t}(a^\alpha_{rt})^2$.

PROOF. By means of $\sum_\alpha (x^\alpha)^2 = 1$, we have

(i) $$\sum_\alpha \langle \nabla_{grad(x^\alpha)}\varphi_r, x^\alpha\varphi_r\rangle = \langle \nabla_{\sum_\alpha x^\alpha\cdot grad(x^\alpha)}\varphi_r, \varphi_r\rangle = 0 .$$

On the other hand, we have

$$\begin{aligned}\sum_r \langle \nabla_{grad(x^\alpha)}\varphi_r, \sum_t a^\alpha_{rt}\varphi_t\rangle &= \sum_{r,t} a^\alpha_{rt}\langle \nabla_{grad(x^\alpha)}\varphi_r, \varphi_t\rangle \\ &= \sum_{r,t} a^\alpha_{rt}\int_M grad(x^\alpha)\cdot(\varphi_r|\varphi_t)\, dV_M - \sum_{r,t} a^\alpha_{rt}\langle \varphi_r, \nabla_{grad(x^\alpha)}\varphi_t\rangle \\ &= \sum_{r,t} a^\alpha_{rt}\left\langle dx^\alpha, d(\varphi_r|\varphi_t)\right\rangle - \sum_{r,t} a^\alpha_{rt}\langle \nabla_{grad(x^\alpha)}\varphi_r, \varphi_t\rangle\end{aligned}$$

because of $a_{rt}^{\alpha}=a_{rt}^{\alpha}$. Thus we have

(ii) $$\sum_{r}\langle\nabla_{grad(x^{\alpha})}\varphi_{r}, \sum_{t}a_{rt}^{\alpha}\varphi_{t}\rangle=(1/2)\sum_{r,t}a_{rt}^{\alpha}\left\langle\Delta x^{\alpha}, (\varphi_{r}|\varphi_{t})\right\rangle$$
$$=(m/2)\sum_{r,t}a_{rt}^{\alpha}\left\langle x^{\alpha}, (\varphi_{r}|\varphi_{t})\right\rangle=(m/2)\sum_{r,t}(a_{rt}^{\alpha})^{2}.$$

From (i) and (ii) we get the assertion of our lemma. q. e. d.

LEMMA 2. 4. $\sum_{\alpha,r}||\nabla_{grad(x^{\alpha})}\varphi_{r}||^{2}\leqslant\sum_{r}\lambda_{r}^{p}+pn(p-m+S(M)^{2})$.

PROOF. Fix an arbitrary point $z_{0}\in M$. Let $\{z^{j}\}$ be a normal coordinate system of M about z_{0}. Put $e_{j}=\left.\frac{\partial}{\partial z^{j}}\right|_{z_{0}}$, then we have, at z_{0},

$$\sum_{\alpha}(\nabla_{grad(x^{\alpha})}\varphi_{r}|\nabla_{grad(x^{\alpha})}\varphi_{r})$$
$$=\sum_{\alpha,j,k}g\left(grad(x^{\alpha}), e_{j}\right)\cdot g\left(grad(x^{\alpha}), e_{k}\right)\cdot(\nabla_{j}\varphi_{r}|\nabla_{k}\varphi_{r})$$
$$=\sum_{j,k}\sum_{\alpha}(e_{j}x^{\alpha})\cdot(e_{k}x^{\alpha})\cdot(\nabla_{j}\varphi_{r}|\nabla_{k}\varphi_{r})=\sum_{j}(\nabla_{j}\varphi_{r}|\nabla_{j}\varphi_{r})=(\nabla\varphi_{r}|\nabla\varphi_{r})$$

because of $\sum_{\alpha}(e_{j}x^{\alpha})\cdot(e_{k}x^{\alpha})=g_{jk}(z_{0})=\delta_{jk}$. Thus, by lemma 2.2, we get

$$\sum_{\alpha,r}||\nabla_{grad(x^{\alpha})}\varphi_{r}||^{2}=\sum_{r}\langle\nabla\varphi_{r}, \nabla\varphi_{r}\rangle$$
$$\leqslant\sum_{r}\left\{\langle\Delta\varphi_{r}, \varphi_{r}\rangle+p\left(p-m+S(M)^{2}\right)\cdot||\varphi_{r}||^{2}\right\}$$
$$=\sum_{r}\lambda_{r}^{p}+pn\left(p-m+S(M)^{2}\right).$$ q. e. d.

We see easily that $\langle U_{r}^{\alpha}, \varphi_{t}\rangle=0$ for any α, r, t. Thus by means of the minimum principle, we get

(8) $$\lambda_{n+1}^{p}\sum_{\alpha,r}||U_{r}^{\alpha}||^{2}\leqslant\sum_{\alpha,r}\langle\Delta U_{r}^{\alpha}, U_{r}^{\alpha}\rangle.$$

If $U_{r}^{\alpha}=0$ for any α and r, then $x^{\alpha}\varphi_{r}=\sum_{t}a_{rt}^{\alpha}\varphi_{t}$. So $1=\sum_{\alpha}\langle x^{\alpha}\varphi_{r}, x^{\alpha}\varphi_{r}\rangle=\sum_{\alpha,t}(a_{rt}^{\alpha})^{2}$ for all r. Hence $n=\sum_{\alpha,r,t}(a_{rt}^{\alpha})^{2}$. This contradicts lemma 2. 3. Therefore we have $\sum_{\alpha,r}||U_{r}^{\alpha}||^{2}>0$. On the other hand, we have $\sum_{\alpha,r}||U_{r}^{\alpha}||^{2}=n-\sum_{\alpha,r,t}(a_{rt}^{\alpha})^{2}$. Put $A:=\sum_{\alpha,r,t}(a_{rt}^{\alpha})^{2}$, then from (8) we get

(9) $$\lambda_{n+1}^{p}\leqslant\left(\sum_{\alpha,r}\langle\Delta U_{r}^{\alpha}, U_{r}^{\alpha}\rangle\right)\Big/(n-A).$$

It is known that for $f\in A^{0}(M)$ and $\omega\in A^{p}(M)$, we have $\Delta(f\omega)=(\Delta f)\omega+f\Delta\omega-2\nabla_{grad(f)}\omega$ (cf. G. de Rham [5], p. 129). Therefore we get

$$\Delta U_{r}^{\alpha}=(\Delta x^{\alpha})\varphi_{r}+x^{\alpha}\Delta\varphi_{r}-2\nabla_{grad(x^{\alpha})}\varphi_{r}-\sum_{t}a_{rt}^{\alpha}\Delta\varphi_{t}$$
$$=(m+\lambda_{r}^{p})x^{\alpha}\varphi_{r}-2\nabla_{grad(x^{\alpha})}\varphi_{r}-\sum_{t}a_{rt}^{\alpha}\lambda_{t}^{p}\varphi_{t}$$

and

$$\langle \Delta U_r^\alpha, U_r^\alpha\rangle = (m+\lambda_r^p)\langle x^\alpha \varphi_r, U_r^\alpha\rangle - 2\langle \nabla_{grad(x^\alpha)}\varphi_r, U_r^\alpha\rangle$$
$$\leqslant (m+\lambda_n^p)\cdot ||U_r^\alpha||^2 - 2\langle \nabla_{grad(x^\alpha)}\varphi_r, U_r^\alpha\rangle .$$

Hence, by means of lemma 2. 3, we have

$$\sum_{\alpha,r}\langle \Delta U_r^\alpha, U_r^\alpha\rangle \leqslant (m+\lambda_n^p)(n-A)+mA .$$

From (9), we get

(10) $$\lambda_{n+1}^p - \lambda_n^p - m \leqslant mA/(n-A) = -m+nm/(n-A) .$$

(11) $$\lambda_{n+1}^p - \lambda_n^p \leqslant nm/(n-A) .$$

By means of Schwartz inequality, lemma 2.3 and lemma 2.4, we have

(12) $$m^2A^2/4 = \Big(\sum_{\alpha,r}\langle \nabla_{grad(x^\alpha)}\varphi_r, U_r^\alpha\rangle\Big)^2$$
$$\leqslant \Big(\sum_{\alpha,r}||\nabla_{grad(x^\alpha)}\varphi_r||^2\Big)\cdot\Big(\sum_{\alpha,r}||U_r^\alpha||^2\Big)$$
$$\leqslant \Big\{\sum_r \lambda_r^p + pn\big(p-m+S(M)^2\big)\Big\}(n-A) .$$

If $A=0$, then by (10) we have $\lambda_{n+1}^p - \lambda_n^p - m \leqslant 0$.
This estimate is sharper than the case $A\neq 0$ because of $A\geqslant 0$. So hereafter we consider the case $A\neq 0$. Form (10) and (12), we get

(13) $$\lambda_{n+1}^p - \lambda_n^p - m \leqslant 4\Big\{\sum_r \lambda_r^p + pn\big(p-m+S(M)^2\Big\}\Big/mA .$$

Put $X := \lambda_{n+1}^p - \lambda_n^p - m$ *and* $K := \sum_r \lambda_r^p + pn(p-m+S(M)^2)$, *then* (11) and (13) mean

(11′) $$X+m \leqslant nm/(n-A) .$$

(13′) $$X \leqslant 4K/mA .$$

From (11′) and (13′), we have

$$nmX^2 - 4KX - 4Km \leqslant 0 .$$

Thus we get

$$X \leqslant 2(K+\sqrt{K^2+Knm^2})/nm$$
$$< 2(K+\Omega K+nm^2/2\Omega)/nm$$
$$= 2(\Omega+1)\,K/nm + m/\Omega .$$

Here Ω is a free parameter such that $\Omega\geqslant 1$. Therefore we obtain the assertion of our theorem 1.

3. Another estimate of λ^1_{n+1}

Let M be an $m(\geqslant 2)$-dimensional compact and oriented Riemannian manifold without boundary minimally immersed in the unit sphere (of any codimension). By means of (5), we have

$$\langle \nabla\omega, \nabla\omega \rangle = \langle -\sum \nabla^i \nabla_i \omega, \omega \rangle \leqslant \langle \Delta\omega, \omega \rangle - \int_M \rho(\omega^\#, \omega^\#)\, dV_M$$

for $\omega \in A^1(M)$. Here $\omega^\#$ denotes the vector field on M such that $g(\omega^\#, X) = \omega(X)$ for any vector field X on M. Thus we get

$$(14) \qquad \langle \nabla\omega, \nabla\omega \rangle \leqslant \langle \Delta\omega, \omega \rangle - r(M) \cdot \|\omega\|^2 .$$

Using (14) instead of (3), we have a similar argument as Section 2 to obtain the assertion of our theorem 2.

REMARK 3. 1. Let M be a Riemannian manifold as in theorem 1. Let $\{z^j\}$ be a normal coordinate system of M about a point $z_0 \in M$ as in proof of lemma 2. 2. Then by means of (7), we have, at z_0,

$$\rho(\omega^\#, \omega^\#) = \sum_j (m-1-\xi_j^2) \cdot \omega_j^2 \qquad \big(\omega \in A^1(M)\big) .$$

Hence for any $\omega \in A^1(M)$, we have

$$\big(m-1-S(M)^2\big) \cdot (\omega|\omega) \leqslant \rho(\omega^\#, \omega^\#) \leqslant (m-1) \cdot (\omega|\omega) .$$

In particular, we get $m-1-S(M)^2 \leqslant r(M)$. Thus for $p=1$, the inequality (1) follows from the inequality (2).

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