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On infinitesimal holomorphically projective transformations in compact Kaehlerian manifolds

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§ 1. Introduction.

One of the authers has proved the following theorems (K. Yamauchi [6], [8]).

THEOREM A. *In a compact orientable Riemannian manifold with non-positive constant scalar curvature, an infinitesimal projective transformation is necessarily an infinitesimal isometry.*

THEOREM B. *Let M be a compact, orientable and simply connected n-dimensional $(n\geqq 3)$ Riemannian manifold with constant scalar curvature R. If M admits a non-isometric infinitesimal projective transformation, then M is isometric to a sphere of radius $\sqrt{n(n-1)/R}$.*

It is natural to consider the Kaehlerian analogues corresponding to the above theorems. In this paper, we shall investigate the infinitesimal holomorphically projective transformations in compact Kaehlerian manifolds with constant scalar curvature and prove the following theorems.

THEOREM 1. *In a compact Kaehlerian manifold with non-positive constant scalar curvature, an infinitesimal holomorphically projective transformation is necessarily an infinitesimal isometry.*

THEOREM 2. *Let M be a compact and simply connected n-dimensional $(n=2m\geqq 4)$ Kaehlerian manifold with constant scalar curvature R. If M admits a non-isometric infinitesimal holomorphically projective transformation, then M is holomorphically isometric to a complex m-dimensional projective space with the Fubini-Study metric of constant holomorphic sectional curvature $R/m(m+1)$.*

T. Kashiwada announced Theorem 1. However the proof given in [2] turned to be incomplete.

In this paper, we assume that the Riemannian manifolds under consideration are connected, differentiable and of dimension $\geqq 3$.

§ 2. Preliminaries.

Let M be a Kaehlerian manifold of real dimension $n(n=2m\geqq 4)$. Then the Riemannian metric g_{ji} and the complex structure J_i^h satisfy the following equations:

$$(2.1)\qquad J_i^a J_a^h = -\delta_i^h, \qquad g_{ba} J_j^b J_i^a = g_{ji}, \\ \nabla_k J_i^h = 0, \qquad \nabla_k g_{ji} = 0,$$

where ∇_k denotes the operator of covariant differentiation with respect to g_{ji}.

Let $R_{kji}{}^h$ be the Riemannian curvature tensor and put $R_{ji}:=R_{aji}{}^a$ (Ricci tensor), $R:=g^{ba}R_{ba}$ (scalar curvature) and $H_{ji}:=J_j^a R_{ai}$. Then we can easily verify that these tensors satisfy the following identities:

$$(2.2)\qquad R_{kji}{}^a J_a^h = R_{kja}{}^h J_i^a, \qquad R_{kjih} = R_{kjba} J_i{}^b J_h{}^a, \\ R_{ji} = R_{ba} J_j^b J_i^a, \qquad H_{ji} + H_{ij} = 0, \\ H_{ji} = H_{ba} J_j^b J_i^a = -(1/2)\, J^{ba} R_{baji} = J^{ba} R_{bjia}.$$

An infinitesimal isometry or a Killing vector field X^h is defined by

$$(2.3)\qquad \mathscr{L}_X g_{ji} \equiv \nabla_j X_i + \nabla_i X_j = 0,$$

where $\mathscr{L}_X$ denotes the operator of Lie differentiation with respect to X^h. In a compact orientable Riemannian manifold, a necessary and sufficient condition for a vector field X^h to be an infinitesimal isometry is

$$(2.4)\qquad \nabla_a X^a = 0$$

and

$$(2.5)\qquad \nabla^a \nabla_a X^h + R_a^h X^a = 0.$$

An infinitesimal affine transformation X^h is defined by

$$(2.6)\qquad \mathscr{L}_X \left\{ {h \atop ji} \right\} \equiv \nabla_j \nabla_i X^h + R_{aji}{}^h X^a = 0,$$

where $\left\{ {h \atop ji} \right\}$ is the Christoffel's symbol.
In a compact orientable Riemannian manifold, an infiitesimal affine transformation is necessarily an infinitesimal isometry.

An infinitesimal homorphically projective transformation or, for simplicity, an infinitesimal HP-transformation X^h is defined by

$$(2.7)\qquad \mathscr{L}_X \left\{ {h \atop ji} \right\} = F_j \delta_i^h + F_i \delta_j^h - F_a J_j^a J_i^h - F_a J_i^a J_j^h,$$

where F_i is a certain vector.
In this case, we shall call F_i the associated vector of the transformation. If F_i vanishes, then the infinitesimal HP-transformation reduces to an affin one.
Contracting (2. 7) with respect to h and i, we get

(2. 8) $$\nabla_j \nabla_a X^a = (n+2) F_j ,$$

which shows that the associated vector is gradient.

A vector field X^h is called contravariant analytic or, for simplicity, analytic, if it satisfies

(2. 9) $$\mathcal{L}_X J_i^h \equiv -J_i^a \nabla_a X^h + J_a^h \nabla_i X^a = 0 .$$

Transvecting (2. 7) with g^{ji}, we have (2. 5). In a compact Kaehlerian manifold, (2. 5) is equivalent to (2. 9), whence an infinitesimal HP-transformation is analytic.

For a vector field X^h and tensor field Y_i^h, the following identities are well known:

(2. 10) $$\nabla_k \mathcal{L}_X \begin{Bmatrix} h \\ ji \end{Bmatrix} - \nabla_j \mathcal{L}_X \begin{Bmatrix} h \\ ki \end{Bmatrix} = \mathcal{L}_X R_{kji}{}^h ,$$

(2. 11) $$\mathcal{L}_X \nabla_j Y_i^h - \nabla_j \mathcal{L}_X Y_i^h = Y_i^a \mathcal{L}_X \begin{Bmatrix} h \\ ja \end{Bmatrix} - Y_a^h \mathcal{L}_X \begin{Bmatrix} a \\ ji \end{Bmatrix} .$$

Using these identities for the infinitesimal analytic HP-transformation X^h with the associated vector F_i, we obtain the following identities:

(2. 12) $$\nabla^a \nabla_a F^h + R_a^h F^a = 0 ,$$

(2. 13) $$2R_a^h F^a = -\nabla^h (\Delta f) ,$$

where $f := \frac{1}{n+2} \nabla_a X^a$ and $\Delta f := \nabla^a \nabla_a f$,

(2. 14) $$J_j^a \nabla_i F_a + J_i^a \nabla_a F_j = 0 ,$$

(2. 15) $$\mathcal{L}_X R_{ji} = -(n+2) \nabla_j F_i ,$$

and

(2. 16) $$\mathcal{L}_X R_{kji}{}^h = -\delta_k^h \nabla_j F_i + \delta_j^h \nabla_k F_i + J_k^h J_i^a \nabla_j F_a - J_j^h J_i^a \nabla_k F_a - 2 J_i^h J_j^a \nabla_k F_a .$$

§ 3. Proofs of theorems.

LEMMA 1. *If a compact Kaehlerian manifold M with constant scalar curvature R admits an infinitesimal HP-transformation X^h, then there exists the following formula:*

(3. 1) $$\Delta f=-\frac{2R}{n}f,$$

where $f:=\frac{1}{n+2}\nabla_a X^a$.

PROOF. From (2. 7), (2. 14), Ricci identity and Bianchi identity, we have

(3. 2) $$\begin{aligned}0&=\nabla^b(\nabla_b\nabla_j X_i+R_{abji}X^a-F_b g_{ji}-F_j g_{bi}+F_a J_b{}^a J_{ji}+F_a J_j{}^a J_{bi})\\&=(R_{abji}-2R_{bjia})\nabla^b X^a-R_{ai}\nabla_j X^a+R_j{}^a\nabla_a X_i-(\nabla_a R_{ji})X^a\\&\quad-(\Delta f)g_{ji}-\nabla_i F_j+\nabla^b F_a J_j{}^a J_{bi}.\end{aligned}$$

Operating ∇^j to the above equation, we obtain

(3. 3) $$\begin{aligned}0&=-2\nabla_i R_{ba}\nabla^b X^a\\&=\nabla_i R_{ba}\mathcal{L}_X g^{ba}\\&=n\nabla_i(\Delta f)+2RF_i,\end{aligned}$$

whence $n\Delta f+2Rf$ is constant.
Then we have

$$\Delta f=-\frac{2R}{n}f$$

because of

(3. 4) $$\int_M \Delta f d\sigma=\int_M f d\sigma=0,$$

where $d\sigma$ denotes the volume element of M. Q. E. D.

PROOF OF THEOREM 1.

Since

(3. 5) $$\Delta\rho^2=2\rho\Delta\rho+2(\nabla^a\rho)(\nabla_a\rho)$$

for any function ρ on M, it follows

(3. 6) $$\int_M(\nabla^a\rho)(\nabla_a\rho)\,d\sigma=-\int_M\rho\Delta\rho d\sigma.$$

From (3. 6) and Lemma 1, we have

(3. 7) $$\int_M F^a F_a d\sigma=\frac{2R}{n}\int_M f^2 d\sigma.$$

So, if R is non-positive, then F_i vanishes. This means X^h is an infinitesimal affine transformation and consequently an infinitesimal isometry by the compactness of M. Q. E. D.

M. Obata [3] announced and S. Tanno [5] proved the following

LEMMA 2. *Let M be a complete and simply connected Kaehlerian manifold. In order for M to admit a non-constant function ρ satisfying*

$$(3.8)\qquad \nabla_k\nabla_j\nabla_i\rho+(c/4)(2\nabla_k\rho g_{ji}+\nabla_j\rho g_{ki}+\nabla_i\rho g_{kj}-\nabla_a\rho J_j{}^a J_{ki} -\nabla_a\rho J_i{}^a J_{kj})=0$$

for some positive constant c, it is necessary and sufficient that M is holomorphically isometric to a complex projective space with Fubini-Study metric of constant holomorphic sectional curvature c.

LEMMA 3. *Let M be a compact Kaehlerian manifold with constant scalar curvature R. If M admits an infinitesimal HP-transformation X^h,* then $Y^h := \frac{2R}{n(n+2)}X^h+F^h$ *is an infinitesimal isometry and consequently F^h is an infinitesimal HP-transformation.*

PROOF. Using Lemma 1, we have

$$(3.9)\qquad \nabla_a Y^a=\frac{2R}{n(n+2)}\nabla_a X^a+\Delta f = \frac{2R}{n}f-\frac{2R}{n}f=0 .$$

On the other hand, we have

$$(3.10)\qquad \nabla^a\nabla_a Y^h+R_a{}^h Y^a=0 ,$$

because X^h and F^h are analytic.
Thus Y^h is an infinitesimal isometry and it is clear that F^h is an infinitesimal HP-transformation. Q. E. D.

PROOF OF THEOREM 2.

Using Lemma 3 and Lemma 1, we have

$$(3.11)\qquad \nabla_k\nabla_j F_i=-R_{akji}F^a-\frac{2R}{n(n+2)}(F_k g_{ji}+F_j g_{ki}-F_a J_k{}^a J_{ji} -F_a J_j{}^a J_{ki}) .$$

Since F_i is gradient, using (3.11), we have

$$(3.12)\qquad 0=\nabla_k\nabla_j F_i-\nabla_k\nabla_i F_j = -2R_{akji}F^a-\frac{2R}{n(n+2)}(F_j g_{ki}-F_i g_{kj}-2F_a J_k{}^a J_{ji} -F_a J_j{}^a J_{ki}+F_a J_i{}^a J_{kj}) .$$

Substituting this result into (3.11), we obtain

$$(3.13)\qquad \nabla_k \nabla_j F_i + \frac{R}{n(n+2)}(2F_k g_{ji} + F_j g_{ki} + F_i g_{kj} - F_a J_j{}^a J_{ki} - F_a J_i{}^a J_{kj}) = 0 .$$

Since X^h is non-isometric, R is positive. Therefore, from Lemma 2, Theorem 2 was proved. Q. E. D.

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