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Variation of harmonic mapping caused by a deformation of Riemannian metric

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1. Introduction

Most of the studies of geodesics or harmonic mappings are concerned with the numbers of them. In contrast, we study in this paper a local variation of harmonic mapping caused by a deformation of riemannian metric.

Let (M, g) be a compact n-dimensional riemannian manifold and $\overline{M}$ another compact manifold. Let $\bar{g}_0$ be a riemannian metric on $\overline{M}$ and let $\phi_0: M\rightarrow\overline{M}$ be a harmonic imbedding. We prove the following results: *If there is essentially no non-zero Jacobi field, then a deformation of* $\bar{g}_0$ *causes a simple variation of* ϕ_0. (Corollary 4. 3). *But, if there exist essentially non-zero Jacobi fields, then for some deformation of* $\bar{g}_0$, *harmonic mapping vanishes or branches off* (Theorem 4. 7).

This paper is organized as follows: after preliminaries in **2**, we show in **3** that the space $\mathscr{H}^{r,s}$ of harmonic imbeddings becomes a Hilbert manifold (Theorem 3. 4). Though the isometry group I of (M, g) $(n\neq 2)$ acts on $\mathscr{H}^{r,s}$, we construct another manifold $\mathscr{N}^{r,s}$ instead of $\mathscr{H}^{r,s}/I$ (Proposition 3. 10) and explain their relations between $\mathscr{H}^{r,s}/I$ and $\mathscr{N}^{r,s}$ (Lemma 3. 14, Lemma 3. 16). Combining these informations we get the main results in **4** stated above.

2. Preliminaries

In this section, we give notations, fundamental definitions and fundamental lemmas. Let (M, g) be a compact n-dimensional C^{∞}-riemannian manifold and $\overline{M}$ a compact $\bar{n}$-dimensional C^{∞}-manifold, where $\bar{n}\geqq n$. We *denote by* $H^{s}(F)$ the set of all H^{s}-cross sections of a fiber bundle F over a compact C^{∞}-manifold, where H^{s} means an object which has derivatives defined almost everywhere up to oder s and such that each partial derivative is square integrable. $H^{s}(F)$ becomes a Hilbert manifold, in particular, a Hilbert space if F is a vector bundle. (See Palais [**6**, § 4, § 9, § 10, § 11].)

We *denote by* $S^{2}\overline{M}$ the vector bundle of symmetric bilinear forms on $\overline{M}$. Then, if $r>\bar{n}/2$, *the set* $\mathscr{M}^{r}$ of all H^{r}-metrics on $\overline{M}$ is an open set of

$H^r(S^2\bar{M})$, and so becomes a Hilbert manifold. Similarly, we see that *the set* $\mathscr{P}^s$ of H^s-imbedding of M into $\bar{M}$ becomes an open manifold of $H^s(\bar{M}\times M)$ if $s>n/2+1$, where $\bar{M}\times M$ is a trivial fiber bundle over M.

Now, we fix $\bar{g}\in\mathscr{M}^r$ and $\phi\in\mathscr{P}^s$. We *denote by* $T_\phi\bar{M}$ the vector bundle over M induced by ϕ. That is, the fibre of $T_\phi\bar{M}$ at $x\in M$ is the tangent space $T_{\phi(x)}\bar{M}$ of $\bar{M}$ at $\phi(x)$. Then, there is *a cannonical inner product* $\langle\ ,\ \rangle$ on $H^s(T_\phi\bar{M})$;

$$\langle\ ,\ \rangle=\int_M \bar{g}(\ ,\ )\,v_g, \tag{2.1}$$

where v_g is the volume element of (M, g). Moreover, there is the covariant derivative $\bar{\nabla}$ along ϕ on the vector bundle $T_\phi\bar{M}$. Let $\{x^i\}$, $\{y^i\}$, and $\bar{\Gamma}^i{}_{jk}$ be a local coordinate of M, a local coordinate of $\bar{M}$ and the Cristoffel's symbol of $\bar{g}$, respectively. If a vector field ξ along ϕ and a vector field X on M are given by

$$\xi(x)=\xi^i(x)\cdot\frac{\partial}{\partial y^i}\bigg|_{\phi(x)}$$

and

$$X(x)=X^a(x)\cdot\frac{\partial}{\partial x^a}\bigg|_x,$$

respectively then we define *the covariant derivative* $\bar{\nabla}_X\xi$ *along* ϕ as

$$\begin{aligned}(\bar{\nabla}_X\xi)(x)=&\left(X^a\cdot\frac{\partial\xi^i}{\partial x^a}\right)(x)\cdot\frac{\partial}{\partial y^i}\bigg|_{\phi(x)}\\&+\bar{\Gamma}^i{}_{jk}\big(\phi(x)\big)\cdot\xi^j(x)\cdot\left(X^a\cdot\frac{\partial\phi^k}{\partial x^a}\right)(x)\cdot\frac{\partial}{\partial y^i}\bigg|_{\phi(x)}.\end{aligned} \tag{2.2}$$

DEFINITION 2.1. *The fundamental from* α *of* ϕ *with respect to* $\bar{g}$ is a cross section of the tensor bundle $T_\phi\bar{M}\otimes S^2M$ over M which is given by the following equation. Let X and Y be tangent vectors of M at x, and denote a local extension of Y by the same latter Y. Then,

$$\alpha(X, Y)=\bar{\nabla}_X(\phi^*Y)-\phi^*(\nabla_X Y), \tag{2.3}$$

where ∇ is the riemannian connection of (M, g). We see easily that $\alpha(X, Y)$ does not depend on extensions of the vector Y and that $\alpha(Y, X)=\alpha(X, Y)$.

DEFINITION 2.2. *The tension field* τ *of* ϕ *with respect to* $\bar{g}$ is the trace of the fundamental form α with respect to g. That is,

$$\begin{aligned}\tau^i(x)=&\,g^{ab}(x)\cdot\frac{\partial^2\phi^i}{\partial x^a\partial x^b}(x)-g^{ab}(x)\cdot\Gamma^c{}_{ab}(x)\cdot\frac{\partial\phi^i}{\partial x^c}(x)\\&+g^{ab}(x)\cdot\bar{\Gamma}^i{}_{jk}\big(\phi(x)\big)\cdot\frac{\partial\phi^j}{\partial x^a}(x)\cdot\frac{\partial\phi^k}{\partial x^b}(x),\end{aligned}$$

where $\Gamma^c{}_{ab}$ is the Cristoffel's symbol of g. (See Eells and Sampson [**3**, p. 116 (5)].)

DEFINITION 2. 3. If the tension field τ of ψ with respect to $\bar{g}$ vanishes, then ψ is said to be *harmonic with respect to* $\bar{g}$.

We *denote by* $\bar{\mathscr{H}}^{r,s}$ the subset of $\mathscr{M}^r \times \mathscr{P}^s$ of all pairs $(\bar{g}, \psi)$ such that ψ is harmonic with respect to $\bar{g}$. We *denote by* $\mathscr{M}$, $\mathscr{P}$ and $\mathscr{H}$ the subset of $\mathscr{M}^r$, $\mathscr{P}^s$ and $\bar{\mathscr{H}}^{r,s}$ of all C^∞-objects, respectively.

Finally, we show some fundamental propositions concerning the differentiability of the composition of H^s-mappings. Owing to Palais [**6**, § 4], $H^s(F)$ has a system of coordinate neighbourhoods such that each neighbourhood is diffeomorphic to an open set of a closed subspace of $\Sigma H^s(D^p, \boldsymbol{R})$, where p is the dimension of the base manifold of F and $H^s(D^p, \boldsymbol{R})$ is the vector space of all $\boldsymbol{R}$-valued H^s-functions on a closed p-dimensional disc D^p. Therefore the differentiability is reduced to that following lemmas.

LEMMA 2. 4 (Palais [**6**, 11.3 Theorem]). *Assume that* $s>q/2+1$. *If* $\xi\in C^\infty(D^p, \boldsymbol{R})$, $\eta\in H^s(D^q, D^p)$ *and* $\eta(D^q)\subset \mathrm{int}\,(D^p)$ *then* $\xi\circ\eta\in H^s(D^q, \boldsymbol{R})$. *Moreover, there is a neighbourhood* W *of* η *in* $H^s(D^q, D^p)$ *such that the composition*: $W\to H^s(D^q, \boldsymbol{R})$ *is* C^∞.

LEMMA 2. 5. *Assume that* $s>q/2+1$ *and* $r>s+p/2$. *If* $\xi\in H^r(D^p, \boldsymbol{R})$, $\eta\in H^s(D^q, D^p)$ *and* $\eta(D^q)\subset \mathrm{int}\,(D^p)$ *then* $\xi\circ\eta\in H^s(D^q, \boldsymbol{R})$. *Moreover there is a neighbourhood* W *of* (ξ, η) *in* $H^r(D^p, \boldsymbol{R})\times H^s(D^q, D^p)$ *such that the composition*: $W\to H^s(D^q, \boldsymbol{R})$ *is* $C^{r-s-[p/2]-1}$.

PROOF. Immediate from Omori [**5**, 1. 4 Corollary] and Sobolev's lemma.

Q. E. D.

3. A Manifold structure of the set $\mathscr{H}^{r,s}$

First we give some propositions to see that there is an open subset of $\bar{\mathscr{H}}^{r,s}$ which becomes a Hilbert manifold. We fix an element $(\bar{g}_0, \psi_0)$ of $\mathscr{H}$ and a sufficiently small neighbourhood in $\bar{\mathscr{H}}^{r,s}$. Assume that $s>n/2+3$ and $r>s+\bar{n}/2-1$. Note that the Cristoffel's symbol $\bar{\Gamma}^i{}_{jk}$ of $\bar{g}$ is represented as a rational function of $\bar{g}_{ij}$ and $\partial\bar{g}_{ij}/\partial y^k$. Therefore, by the formula (2. 4), τ^i is represented as a rational function of $\bar{g}_{ij}$, $\partial\bar{g}_{ij}/\partial y^k$, ψ^i, $\partial\psi^i/\partial x^a$, $\partial^2\psi^i/\partial x^a\partial x^b$ and their compositions. Hence Lemma 2. 4 and Lemma 2. 5 imply that the map: $(\bar{g}, \psi)\to\tau$ is $C^{r-s-[\bar{n}/2]}$ (in the sense of local expression) as a map from H^r-metric and H^s-imbedding to H^{s-2}-vector field.

Let x be a point in M and let z be a point in $\bar{M}$ which is sufficiently near to $\psi_0(x)$. For a vector $\xi\in T_z\bar{M}$, we obtain a vector at $\psi_0(x)$ by the parallel transport along the minimal geodesic connecting z and $\psi_0(x)$ with respect to $\bar{g}_0$. We denote this vector at $\psi_0(x)$ by $p(x, \xi)$ or simply $p\xi$. Note that the map: $(x, \xi)\to p(x, \xi)$ is C^∞. Thus Lemma 2.4 implies

LEMMA 3.1. *Assume that $s>n/2+3$ and $r>s+\bar{n}/2-1$. Then there is a neighbourhood W of $(\bar{g}_0, \psi_0)$ in $\mathcal{M}^r\times\mathcal{P}^s$ such that the map: $(\bar{g}, \psi)\to p\tau$ is $C^{r-s-[\bar{n}/2]}$ as a map: $W\to H^{s-2}(T_{\psi_0}\bar{M})$.*

We shall give the derivation of this map. Let $\bar{g}(t)$ be *a deformation of* $\bar{g}_0$, i.e., a 1-parameter family of riemannian metrics on $\bar{M}$ such that $\bar{g}(0)=\bar{g}_0$. Set $\bar{g}'(0)=h$. Then by Lichnerowicz [4, (17.2)], we see

$$\bar{g}_0\Big(\big[\alpha(X, Y)\big]', \xi\Big)=\bar{g}_0\Big(\bar{\nabla}'_X(\psi_{0*}Y), \xi\Big) \tag{3.1}$$
$$=\frac{1}{2}\Big\{(\bar{\nabla}_{\psi_{0*}X}h)(\psi_{0*}Y, \xi)+(\bar{\nabla}_{\psi_{0*}Y}h)(\psi_{0*}X, \xi)-(\bar{\nabla}_\xi h)(\psi_{0*}X, \psi_{0*}Y)\Big\}.$$

Let $\{X_a\}$ be a local orthonormal basis of (M, g). Then

$$\bar{g}_0\Big([p\tau]', \xi\Big)=\bar{g}_0(\tau', \xi) \tag{3.2}$$
$$=\sum_a\Big\{(\bar{\nabla}_{\psi_{0*}X_a}h)(\psi_{0*}X_a, \xi)-\frac{1}{2}(\bar{\nabla}_\xi h)(\psi_{0*}X_a, \psi_{0*}X_a)\Big\}.$$

We *denote by* $\gamma(h)$ the right hand side of this equation. Next, we consider *a variation of* ψ_0, i.e., a 1-parameter family $\psi(t)$ of imbeddings such that $\psi(0)=\psi_0$. Set $\psi'(0)=V$. In the following equation, $\bar{\nabla}$ means the covariant derivative along Φ: $M\times\boldsymbol{R}\to\bar{M}$, where Φ is defined by $\Phi(x, t)=\psi(t)(x)$. We omit ψ_* and Φ_* and set $\Phi_*(d/dt)=V$. Let X be a vector field on M, then

$$\bar{\nabla}_V X-\bar{\nabla}_X V=[V, X]=0. \tag{3.3}$$

Denote by $\bar{R}_0$ the curvature tensor of the metric $\bar{g}_0$. Now, we compute $V[p\tau]$.

$$V[p\tau]=\bar{\nabla}_V\tau=\mathrm{tr}\,(\bar{\nabla}_V\alpha),$$
$$\begin{aligned}\bar{\nabla}_V\Big(\alpha(X, Y)\Big)&=\bar{\nabla}_V(\bar{\nabla}_X Y-\nabla_X Y)\\&=\bar{R}(V, X)\,Y+\bar{\nabla}_X\bar{\nabla}_V Y-\bar{\nabla}_V\nabla_X X\\&=\bar{R}(V, X)\,Y+\bar{\nabla}_X\bar{\nabla}_Y V-\bar{\nabla}_{\nabla_X Y}V\\&=\bar{R}(V, X)\,Y+(\bar{\nabla}\bar{\nabla}V)(X, Y)+\bar{\nabla}_{\alpha(X,Y)}V.\end{aligned}$$

Thus we have, at $t=0$,

$$V[p\tau]=\sum_a\bar{R}_0(V, X_a)\,X_a+\sum_a(\bar{\nabla}\bar{\nabla}V)(X_a, X_a). \tag{3.4}$$

We denote by $\beta(V)$ the right side of this equation. Combining these formulae we get

LEMMA 3.2. *The derivative of the map*: $(\bar{g}, \phi)\to p\tau$ *at* $(\bar{g}_0, \phi_0)$ *is given by*

(3.5) $\qquad (h, V)\to\gamma(h)+\beta(V),$

where γ *is the first order differential operator defined by* (3.2) *and* β *the elliptic, self adjoint second order differential operator defined by* (3.4).

PROOF. It is sufficient to prove that β is self adjoint.

$$\begin{aligned}
&\bar{g}_0(\bar{\nabla}_X\bar{\nabla}_Y V-\bar{\nabla}_{\nabla_X Y} V, W)\\
&=X\big[\bar{g}_0(\bar{\nabla}_Y V, W)\big]-\bar{g}_0(\bar{\nabla}_Y V, \bar{\nabla}_X W)-(\nabla_X Y)\big[\bar{g}_0(V, W)\big]\\
&\qquad+\bar{g}_0(V, \bar{\nabla}_{\nabla_X Y} W)\\
&=X\big[Y\big[\bar{g}_0(V, W)\big]\big]-X\big[\bar{g}_0(V, \bar{\nabla}_Y W)\big]-\bar{g}_0(\bar{\nabla}_Y V, \bar{\nabla}_X W)\\
&\qquad-(\nabla_X Y)\big[\bar{g}_0(V, W)\big]+\bar{g}_0(V, \bar{\nabla}_{\nabla_X Y} W)\\
&=\big\{\nabla\nabla\big(\bar{g}_0(V, W)\big)\big\}(X, Y)-\big\{\nabla_X\big(\bar{g}_0(V, \bar{\nabla} W)\big)\big\}(Y)-\bar{g}_0(\bar{\nabla}_Y, V, \bar{\nabla}_X W)\,.
\end{aligned}$$

Set $X=Y=X_a$ and take summation over a. Q. E. D.

Denote by K the vector space of all Killing vector fields on (M, g) if $n\neq 2$, or the vector space of all conformal vector fields if $n=2$. That is, $Z\in K$ if and only if

(3.6) $\qquad \nabla_a Z_b+\nabla_b Z_a-\nabla^c Z_c\cdot g_{ab}=0\,.$

LEMMA 3.3. *Let* ϕ *be an imbedding of* M *into* $\bar{M}$ *and* τ *the tension field of* ϕ. *Then,*

(3.7) $\qquad \langle\tau, K\rangle=0\,.$

PROOF. Set $\phi^*\bar{g}=\tilde{g}$. For $Z\in K$ we see

$$\begin{aligned}
\bar{g}\big(\alpha(X, X), Z\big)&=\bar{g}(\bar{\nabla}_X X-\nabla_X X, Z)\\
&=X\big[\bar{g}(X, Z)\big]-\bar{g}(X, \bar{\nabla}_X Z)-\bar{g}(\nabla_X X, Z)\\
&=X\big[\tilde{g}(X, Z)\big]-\bar{g}(X, \bar{\nabla}_Z X)+\bar{g}\big(X, [Z, X]\big)-\tilde{g}(\nabla_X X, Z)\\
&=(\nabla_X\tilde{g})(X, Z)+\tilde{g}(X, \nabla_X Z)-\frac{1}{2}Z\big[\tilde{g}(X, X)\big]\\
&\qquad+\tilde{g}(X, \nabla_Z X)-\tilde{g}(X, \nabla_X Z)\\
&=(\nabla_X\tilde{g})(X, Z)+\tilde{g}(X, \nabla_Z Z)-\frac{1}{2}Z\big[\tilde{g}(X, X)\big]\\
&=(\nabla_X\tilde{g})(X, Z)-\frac{1}{2}(\nabla_Z\tilde{g})(X, X)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}\langle\tau, Z\rangle &= \int_M \left(\nabla_a \tilde{g}^a{}_b \cdot Z^b - \frac{1}{2} Z^b \nabla_b \tilde{g}^a{}_a\right) v_g \\ &= -\int_M \left(\nabla^a Z^b - \frac{1}{2} \nabla^c Z_c \cdot g^{ab}\right) \tilde{g}_{ab} \cdot v_g \\ &= 0 . \end{aligned}$$

Q. E. D.

THEOREM 3. 4. *Assume that* $s>n/2+3$ *and* $r>s+\bar{n}/2-1$. *Then,* $\bar{\mathscr{H}}^{r,s}$ *is closed in* $\mathscr{M}^r \times \mathscr{P}^s$ *and there is a* $C^{r-s-[\bar{n}/2]}$*– Hilbert submanifold of* $\mathscr{M}^r \times \mathscr{P}^s$ *which is open in* $\bar{\mathscr{H}}^{r,s}$ *and contains the set* $\mathscr{H}$. *We denote the manifold by* $\mathscr{H}^{r,s}$. *Then the tangent space of* $\mathscr{H}^{r,s}$ *at* $(\bar{g}_0, \phi_0) \in \mathscr{H}$ *is a subspace of* $H^r(S^2\bar{M}) \times H^s(T_{\phi_0}\bar{M})$ *of all pairs* (h, V) *such that* $\gamma(h)+\beta(V)=0$.

REMARK 3. 5. This theorem shows that $\mathscr{H}$ becomes an ILH-submanifold of $\mathscr{M} \times \mathscr{P}$. (For the term "ILH", see Omori [5, pp. 168–169].)

PROOF. Since the map: $(\bar{g}, \phi) \to \tau$ is continuous, $\bar{\mathscr{H}}^{r,s}$ is closed. Let $(\bar{g}_0, \phi_0) \in \mathscr{H}$ and take W given in Lemma 3. 1. Denote by $(p\tau)^{NK}$ the orthogonal part of $p\tau$ to K. We shall apply the implicit function theorem to the map: $(\bar{g}, \phi) \to (p\tau)^{NK}$ defined on W. Owing to Lemma 3. 1, this map is $C^{r-s-[\bar{n}/2]}$ as a map: $W \to [H^{s-2}(T_{\phi_0}\bar{M})]^{NK}$.

First, we show that if $(\bar{g}, \phi) \in W$ is sufficiently near to $(\bar{g}_0, \phi_0)$ and $(p\tau)^{NK}=0$ then $\tau=0$. We define a symmetric 2-form on M depending on $(\bar{g}, \phi)$ by

$$2S(X, Y) = \bar{g}(p^{-1}\phi_{0*} X, \phi_* Y) + \bar{g}(p^{-1}\phi_{0*} Y, \phi_* X) . \tag{3. 8}$$

Since S is positive definite for $\phi=\phi_0$, S is positive definite if ϕ is sufficiently near to ϕ_0 with respect to C^1-topology which is weaker than $H^{n/2+3}$-topology. Assume that $(p\tau)^{NK}=0$. Then, if we set $p\tau=X$, $X\in K$ and

$$S(X, X) = \bar{g}(p^{-1}\phi_{0*} X, \phi_* X) = \bar{g}(\tau, \phi_* X) .$$

Therefore, by Lemma 3. 3, we see

$$\int_M S(X, X)\, v_g = \langle\tau, X\rangle = 0 ,$$

which implies that $X=0$ and so $\tau=0$.

Next, we show that the derivative of the map: $(\bar{g}, \phi) \to (p\tau)^{NK}$ at $(\bar{g}_0, \phi_0)$ is surjective, which completes the proof. By Lemma 3. 2, we see that $\operatorname{Im}\beta$ is closed and has finite codimension. Therefore it is sufficient to prove that the orthogonal complement of $\operatorname{Im}\gamma$ coinsides with K. In fact, then, the image is closed and dense owing to Palais [7, Chapter VII Theorem 7]. Let η by any H^{r-1}-1-form along ϕ_0 which is orthogonal to M at each point of M. Since ϕ_0 is an imbedding, there is an H^r-function f on $\bar{M}$ such that $-1/2 \cdot \operatorname{tr}(\phi_{0*}\bar{g}_0) \cdot df = \eta$. If we note that f is constant on $\phi_0 M$, then we see

$$\bar{g}_0\Big(\gamma(f\cdot\bar{g}_0)\cdot\xi\Big)=\sum_a\Big\{(X_a f)\cdot\bar{g}_0(X_a,\xi)-\frac{1}{2}\bar{g}_0(X_a,X_a)\,\xi f\Big\}$$
$$=\bar{g}_0(\eta,\xi)\,,$$

where $\{X_a\}$ is an orthonormal basis of (M,g). Therefore $\operatorname{Im}\gamma$ contains such η. Hence, if Z is orthogonal to $\operatorname{Im}\gamma$ then Z is tangent to M at each point of M. Let h be a symmetric bilinear form on $\bar{M}$ and set $\psi_0{*}h=\tilde{h}$. Then

$$\begin{aligned}
&(\bar{\nabla}_X h)(X,Z)-\frac{1}{2}(\bar{\nabla}_Z h)(X,X)\\
&=X\big[\tilde{h}(X,Z)\big]-h(\bar{\nabla}_X X,Z)-h(X,\bar{\nabla}_X Z)\\
&\quad-\frac{1}{2}Z\big[\tilde{h}(X,X)\big]+h(\bar{\nabla}_Z X,X)\\
&=(\bar{\nabla}_X\tilde{h})(X,Z)+\tilde{h}(\nabla_X X,Z)+\tilde{h}(X,\nabla_X Z)-h(\bar{\nabla}_X X,Z)\\
&\qquad+h\big([Z,X],X\big)-\frac{1}{2}(\nabla_Z\tilde{h})(X,X)-\tilde{h}(\nabla_Z X,X)\\
&=(\nabla_X\tilde{h})(X,Z)-\frac{1}{2}(\nabla_Z\tilde{h})(X,X)-h\big(\alpha(X,X),Z\big)\,.
\end{aligned}$$

And

$$\begin{aligned}
0=\big\langle\gamma(h),Z\big\rangle&=\int_M\Big(\nabla^a\tilde{h}_{ab}\cdot Z^b-\frac{1}{2}Z^b\nabla_b\tilde{h}^a{}_a\Big)v_g\\
&=-\int_M\tilde{h}^{ab}\Big(\nabla_a Z_b-\frac{1}{2}\nabla^c Z_c\cdot g_{ab}\Big)v_g\,.
\end{aligned}$$

Since this equation holds for all $\tilde{h}$, the 2-tensor $\nabla_a Z_b-\frac{1}{2}\nabla^c Z_c\cdot g_{ab}$ is skew-symmetric, i.e.,

$$(\nabla_a Z_b+\nabla_b Z_a)-\nabla^c Z_c\cdot g_{ab}=0\,. \qquad \text{Q. E. D.}$$

Set $\ker\beta=J$. The above proof implies

COROLLARY 3.6.

$$\operatorname{Im}\beta=\big[H^{s-2}(T_{\psi_0}\bar{M})\big]^{NJ}\,, \tag{3.9}$$

$$\operatorname{Im}\gamma=\big[H^{r-1}(T_{\psi_0}\bar{M})\big]^{NK}\,, \tag{3.10}$$

where $[\ \ ]^{NJ}$ *and* $[\ \ ]^{NK}$ *mean the orthgonal compliments of* J *and* K, *respectively.*

PROOF. To show the formula (3.10), it is sufficient that the vector space of all elements of $\operatorname{Im}\gamma$ which is tangent to M contains a closed and finite codimensional subspace of $H^{r-1}(T_{\phi_0}\overline{M})$. (See the proof of Theorem 3.4) Let ξ be an H^{r+1}-vector field on M and denote by the same letter ξ an extension of $\phi_{0*}\xi$ to M. Set $h_{ij}=\bar{\nabla}_i\xi_j+\bar{\nabla}_j\xi_i$. For a vector field Y on M, we have

$$\begin{aligned}
\bar{g}_0\big(\gamma(h), Y\big) &= \sum_a (\bar{\nabla}_{X_a}h)(X_a, Y)-\frac{1}{2}\sum_a(\bar{\nabla}_Y h)(X_a, X_a)\\
&=\sum_a\Big\{X_a\big[h(X_a, Y)\big]-h(\bar{\nabla}_{X_a}X_a, Y)-h(X_a, \bar{\nabla}_{X_a}Y\Big\}\\
&\quad -\frac{1}{2}\sum_a\Big\{Y\big[h(X_a, X_a)\big]-2h(\bar{\nabla}_Y X_a, X_a)\Big\}\\
&=\sum_a\Big\{X_a\big[\bar{g}_0(\bar{\nabla}_{X_a}\xi, Y)+\bar{g}_0(X_a, \bar{\nabla}_Y\xi)\big]-\big[\bar{g}_0(\bar{\nabla}_{\bar{\nabla}_{X_a}X_a}\xi, Y)\\
&\quad +\bar{g}_0(\bar{\nabla}_{X_a}X_a, \bar{\nabla}_Y\xi)\big]-\big[\bar{g}_0(\bar{\nabla}_{X_a}\xi, \bar{\nabla}_{X_a}Y)+\bar{g}_0(X_a, \bar{\nabla}_{\bar{\nabla}_{X_a}Y}\xi)\big]\\
&\quad -Y\big[\bar{g}_0(\bar{\nabla}_{X_a}\xi, X_a)\big]+\big[\bar{g}_0(\bar{\nabla}_{\bar{\nabla}_Y X_a}\xi, X_a)+\bar{g}_0(\bar{\nabla}_Y X_a, \bar{\nabla}_{X_a}\xi)\big]\Big\}\\
&=\bar{g}_0\Big(\sum_a(\bar{\nabla}\bar{\nabla}\xi)(X_a, X_a), Y\Big)+\sum_a\bar{g}_0\Big(X_a, \bar{R}_0(X_a, Y)\xi\Big).
\end{aligned}$$

This equation shows that the differential operator: $\xi\rightarrow\gamma(h)$ is elliptic. Particularly, the image of this map is closed and has finite codimension in $H^{r-1}(T_{\phi_0}\overline{M})$. Q. E. D.

LEMMA 3.7. *The image of the projection*: $T_{(\bar{g}_0,\phi_0)}\mathscr{H}^{r,s}\rightarrow T_{\bar{g}_0}\mathscr{M}^r$ *coinsides wtih* $\gamma^{-1}[^{r-1}(T_{\phi_0}\overline{M})]^{NJ}$. *The image of the projection*: $T_{(\bar{g}_0,\phi_0)}\mathscr{H}^{r,s}\rightarrow T_{\phi_0}\mathscr{P}^s$ *coinsides with* $H^{r+1}(T_{\phi_0}\overline{M})$.

PROOF. The first half is reduced to Corollary 3.6. It is trivial that if $\gamma(h)+\beta(V)=0$ and $h\in H^r(S^2\overline{M})$ then $V\in H^{r+1}(T_{\phi_0}\overline{M})$. Conversely, if $V\in H^{r+1}(T_{\phi_0}\overline{M})$ then $\beta(V)\in[H^{r-1}(T_{\phi_0}\overline{M})]^{NJ}$ owing to the formula (3.9). On the other hand, K is contained in J by the following Corollary 3.9. Therefore the formula (3.10) implies that there is $h\in H^r(S^2\overline{M})$ such that $\gamma(h)+\beta(V)=0$. Q. E. D.

To state Corollary 3.9, we show

LEMMA 3.8. *Let I denote the isometry group of (M, g) if $n\neq 2$, or the conformal transformation group if $n=2$. Then I preserves $\mathscr{H}^{r,s}$, i.e., if $(\bar{g}, \phi)\in\mathscr{H}^{r,s}$ and $\gamma\in I$ then $(\bar{g}, \phi\circ\gamma)\in\mathscr{H}^{r,s}$.*

PROOF. Denote by α and τ_ϕ the fundamental from and tension field of ϕ with respect to $\bar{g}$, respectively. Then

$$\begin{aligned}
\alpha_{\phi\circ\gamma}(X, Y) &= \bar{\nabla}_X(\phi_*\gamma_* Y)-\phi_*\gamma_*\nabla_X Y\\
&=\Big\{\bar{\nabla}_X\big(\phi_*(\gamma_* Y)\big)-\phi_*\nabla_X(\gamma_* Y)\Big\}+\Big\{\phi_*\nabla_X(\gamma_* Y)-\phi_*\gamma_*\nabla_X Y\Big\}\\
&=\alpha_\phi(\gamma_* X, \gamma_* Y)+\phi_*\alpha_\gamma(X, Y).
\end{aligned}$$

If $\gamma^*g=\exp f\cdot g$, then

$$\tau_{\psi\circ\gamma}=\exp f\cdot\tau_\psi\circ\gamma+\psi_*\tau_\gamma\,.$$

Assume that $\tau_\psi=0$. By easy computation we see

$$g(\tau_\gamma, X)=\frac{1}{2}(2-n)\,Xf\,.$$

Therefore, if $n=2$ or f is constant then $\tau_{\psi\circ\gamma}=0$. Q. E. D.

COROLLARY 3.9. $\beta(K)=0$.

PROOF. If $X\in K$ then $\exp tX\in I$. Therefore, if we define a variation $\psi(t)$ of ψ_0 by $\psi(t)=\psi_0\circ\exp tX$, then $\tau_{\psi(t)}=0$, and so $\tau'=0$. Q. E. D.

Lemma 3.8 shows that we have to consider the coset space $\mathscr{H}^{r,s}/I$. But the action of I on $\mathscr{H}^{r,s}$ is not differentiable, hence $\mathscr{H}^{r,s}/I$ does not become a Hilbert manifold. Here, we consider a submanifold of $\mathscr{H}^{r,s}$ instead of $\mathscr{H}^{r,s}/I$. Let $(\bar{g}_0, \psi_0)\in\mathscr{H}$ and W be a sufficiently small neighbourhood of $(\bar{g}_0, \psi_0)$ in $\mathscr{H}^{r,s}$. The set $\tilde{\mathscr{V}}^{r,s}$ is defined by

$$(3.11)\qquad \tilde{\mathscr{V}}^{r,s}=\left\{(\bar{g}, \psi)\in\mathscr{H}^{r,s}\cap W;\ \psi=\exp_{\bar{g}_0}\xi\circ\psi_0,\ \xi\in H^s(T_{\psi_0}\bar{M}),\ \langle\xi, K\rangle=0\right\}.$$

PROPOSITION 3.10. *Assume that $s>n/2+3$ and $r>s+\bar{n}/2-1$. If W is sufficiently small then $\tilde{\mathscr{V}}^{r,s}$ becomes a $C^{r-s-[\bar{n}/2]}$-Hilbert submanifold of $\mathscr{H}^{r,s}$. We denote this manifold by $\mathscr{V}^{r,s}$.*

PROOF. We define the map: $(\bar{g}, \exp_{\bar{g}_0}\xi\circ\psi_0)\in\mathscr{H}^{r,s}\to\xi^K$, where ξ^K is the K-component of ξ. This map is $C^{r-s-[\bar{n}/2]}$ and the derivative at $(\bar{g}_0, \psi_0)$ is surjective owing to Lemma 3.7. Apply the implicit function theorem on this map. Q. E. D.

We denote by $\mathscr{D}^s$ the H^s-diffeomorphism group of M and $\mathscr{D}$ the C^∞-diffeomorphism group of M. (See Omori [**5**, 1.8 Lemma].) Let $\mathfrak{h}$ be a finite dimensional Lie algebra of C^∞-vector fields on M. Then, by Palais [**8**, Chapter IV Theorem III], there is a Lie transformation group H whose Lie algebra is $\mathfrak{h}$.

LEMMA 3.11. *Assume $s>n/2$. If $\mathfrak{h}$ and H are as above then the inclusion: $H\to\mathscr{D}^s$ is an imbedding.*

PROOF. There is a neighbourhood W of if in H such that $\exp^{-1}|W\colon W\to\mathfrak{h}$ is a diffeomorphism onto an open set of $\mathfrak{h}$. Owing to Omori [**5**, 1.15 Theorem], for each positive integer k there is a sufficiently large integer t such that $\exp\colon H^t(TM)\to\mathscr{D}^s$ is C^k. Therefore the inclusion: $W\to\mathscr{D}^s$, which coinsides with $\exp\circ(\exp^{-1}|W)$, is C^∞. We easily see that the derivative of the inclusion at $id\in H$ is injective. Since the right multiplication of $\eta\in\mathscr{D}$ for $\mathscr{D}^s$ is C^∞ (see Ebin [**1**, Proposition 3.4; **2**, 4.18 Théorème fondamental]), the above information implies that the inclusion: $H \to \mathscr{D}^s$ is an immersion.

To show that the inclusion is a homeomorphism onto its image, we use Palais [8, Chapter IV Theorem VI], that is the topology of H coinsides with the compact-open topology. Since $s > n/2$, the topology of $\mathscr{D}^s$ is stronger than the compact-open topology. Q. E. D.

REMARK 3.12. For the case that H is the isometry group, see Ebin [1, Corollary 5.4; 2, 7.6 Théoréme].

COROLLARY 3.13. *Denote by I the isometry group of (M, g) if $n \neq 2$, or conformal transformation group if $n=2$. Then I becomes a submanifold of $\mathscr{D}^s$ for all $s > n/2$.*

Now, we give two lemmas which make clear the meaning that we may consider $\mathscr{V}^{r,s}$ instead of $\mathscr{K}^{r,s}/I$.

LEMMA 3.14. *Assume that $t > n/2+3$, $s > t+n/2$ and $r > s+\bar{n}/2-1$. Then the composition $c: \mathscr{V}^{r,s} \times I \to \mathscr{K}^{r,t}$ is $C^{s-t-[n/2]-1}$.*

PROOF. $\mathscr{V}^{r,s}$, $\mathscr{K}^{r,t}$ and I are submanifolds of $\mathscr{M}^r \times \mathscr{P}^s$, $\mathscr{M}^r \times \mathscr{P}^t$ and $\mathscr{D}^t$ respectively. Therefore the proof reduces to the differentiability of the composition: $\mathscr{P}^s \times \mathscr{D}^t \to \mathscr{P}^t$. But it is easy to check owing to Lemma 2.5. Q. E. D.

REMARK 3.15. The composition $c: \mathscr{V}^{r,s} \times I \to \mathscr{K}^{r,s}$ is continuous owing to Ebin [1, Lemma 3.1; 2, 4.3 Proposition].

LEMMA 3.16. *Assume that $t > n/2+3$, $s > t+n/2$ and $r > s+\bar{n}/2-1$. Then there is a local $C^{s-t-[n/2]-1}$-map $d: \mathscr{K}^{r,s} \to \mathscr{V}^{r,t} \times I$ such that $c \circ d$ is the identity map.*

PROOF. Let $\{K_p\}$ be basis of K. Difine a function on $\mathscr{P}^s$ by

$$G_{pq}(\phi) = \int_M \bar{g}_0(\phi_* K_p, \phi_* K_q)\, v_{g_0}.$$

We see that G_{pq} are C^∞ owing to Lemma 2.4. The map which corresponds $\gamma \in I$ and $\xi \in H^s(T_{\phi_0 \circ \gamma} \bar{M})$ to $\xi^K \in H^s(T_{\phi_0 \circ \gamma} \bar{M})$ is given by

$$\xi^K = \phi_{0*} \gamma_* \sum_{p,q} G^{pq}(\phi_0 \circ \gamma) \cdot \langle \xi, \phi_{0*} \gamma_* K_p \rangle \cdot K_q,$$

where (G_{pq}) is the inverse matrix of (G_{pq}). Since G_{pq} are C^∞ owing to Lemma 2.4, this map is a C^∞-submersion of the vector bundle $T\mathscr{P}^s|_{\phi_0 \circ I}$ over $\phi_0 \circ I$ to the subbundle $\bigcup_{\gamma \in I} \phi_{0*} \gamma_* K$ of the tangent vector bundle of $\phi_0 \circ I$. Hence the kernel bundle ν is a C^∞-bundle over $\phi_0 \circ I$. The fiber of ν at ϕ_0 is $[H^s(T_{\phi_0} \bar{M})]^{NK}$ and the derivative of $\mathrm{Exp}_{\bar{g}_0}|\nu$ at ϕ_0 is the identity and surjective onto $T_{\phi_0} \mathscr{P}^s$. Therefore $\mathrm{Exp}_{\bar{g}_0}|\nu$ is a local diffeomorphism.

Define a map $\pi: \mathscr{P}^s \to I$ by

$$\phi=\left(\mathrm{Exp}_{\bar{g}_0}\circ(\mathrm{Exp}_{\bar{g}_0}|\nu)^{-1}(\phi)\right)\circ\phi_0\circ\pi(\phi)\,.$$

Since the map: $\gamma\to\phi_0\circ\gamma$ is injective immersion, π is C^∞. Then the decomposition: $\phi\to(\phi\circ(\pi(\phi))^{-1}, \pi(\phi))\in\mathscr{P}^t\times I$ is $C^{s-t-[n/2]-1}$ near ϕ_0, owing to Lemma 3.14. We define the map d by

$$d(\bar{g}, \phi)=\left(\left(\bar{g}, \phi\circ\left(\pi(\phi)\right)^{-1}\right), \pi(\phi)\right)\,. \qquad \text{Q. E. D.}$$

4. Variations of harmonic mappings caused by deformations of riemannian metrics

In this section we assume that $s>n/2+3$, $r>s+\bar{n}/2-1$ and so Theorem 3.4 and Proposition 3.10 hold. First we consider the case that $(\bar{g}_0, \phi_0)\in\mathscr{H}$ has the property that $J=K$.

LEMMA 4.1. *Assume that $(\bar{g}_0, \phi_0)\in\mathscr{H}$ has the property that $J=K$. Then the differential of the projection $\pi: \mathscr{V}^{r,s}\to\mathscr{M}^r$ at $(\bar{g}_0, \phi_0)$ is bijective.*

PROOF. Owing to Lemma 3.7 and the formula (3.10), the projection: $T_{(\bar{g}_0,\phi_0)}\mathscr{H}^{r,s}\to T_{\bar{g}_0}\mathscr{M}^r$ is surjective. Therefore, for each $h\in H^r(S^2\bar{M})$ there is $V\in T_{\phi_0}\mathscr{P}^s$ such that $(h, V)\in T_{(\bar{g}_0,\phi_0)}\mathscr{H}^{r,s}$, and so $(h, V^{NK})\in T_{(\bar{g}_0,\phi_0)}\mathscr{V}^{r,s}$, where V^{NK} means the $[H^s(T_{\phi_0}\bar{M})]^{NK}$-component of V. On the other hand, if $(0, V)\in T_{(\bar{g}_0,\phi_0)}\mathscr{V}^{r,s}$ then $V\in J$. But here V is orthogonal to J, and so $V=0$.

Q. E. D.

THEOREM 4.2. *Assume that $s>n/2+3$ and $r>s+\bar{n}/2-1$. If $(\bar{g}_0, \phi_0)\in\mathscr{H}$ satisfies $J=K$, then the projection $\pi: \mathscr{V}^{r,s}\to\mathscr{M}^r$ is a local $C^{r-s-[n/2]}$-diffeomorphism around $(\bar{g}_0, \phi_0)$.*

PROOF. Apply the inverse function theorem to Lemma 4.1.

Q. E. D.

COROLLARY 4.3. *Under the above assumption, let $\bar{g}(t)$ be a deformation of $\bar{g}_0$, i.e., a C^∞-curve in $\mathscr{M}^r$ such that $\bar{g}(0)=\bar{g}_0$. Then for sufficiently small t, there exists unique $\phi(t)\in\mathscr{P}^s$ such that $(\bar{g}(t), \phi(t))\in\mathscr{V}^{r,s}$. Moreover $\phi(t)$ is a $C^{r-s-[\bar{n}/2]}$-curve in $\mathscr{P}^s$, and if we set $\bar{g}'(0)=h$ and $\phi'(0)=V$ then*

$$(4.1) \qquad \gamma(h)+\beta(V)=0\,, \qquad V\in\left[H^s(T_{\phi_0}\bar{M})\right]^{NK}$$

holds.

Next we consider the case that $(\bar{g}_0, \phi_0)\in\mathscr{H}$ has the property that $J\supsetneqq K$. Let S be a finite dimensional C^∞-submanifold of $\mathscr{M}^r$ containing $\bar{g}_0$. Assume that $j\circ\gamma|T_{\bar{g}_0}S$: $T_{\bar{g}_0}S\to J^{NK}$ is an isomorphism, where j is the projection map to the J-part.

LEMMA 4.4. *Let π be the projection*: $\mathscr{N}^{r,s}\to\mathscr{M}^r$. *Then there is a neighbourhood W of $(\bar g_0, \psi_0)$ in $\mathscr{N}^{r,s}$ such that $\pi^{-1}(S)\cap W$ is a finite dimensional $C^{r-s-[\bar n/2]}$-submanifold of $\mathscr{X}^{r,s}$. The tangent space of $\pi^{-1}(S)\cap W$ at $(\bar g_0, \psi_0)$ is $(0, J^{NK})$.*

PROOF. Set $(T_{(\bar g_0,\psi_0)}\pi)(T_{(\bar g_0,\psi_0)}\mathscr{N}^{r,s})=E$. Owing to Lemma 3.7 and the formula (3.10), $H^r(S^2\bar M)$ is the direct sum of E and $T_{\bar g_0}S$. Therefore there is a local diffeomorphism ϕ: $\mathscr{M}^r\to H^r(S^2\bar M)$ around $\bar g_0$ such that $\phi(S)$ is contained in $T_{\bar g_0}S$ and $T_{\bar g_0}\phi$ is the identity. Then, if we denote by p: $H^r(S^2\bar M)\to E$ the projection with respect to the decomposition $H^r(S^2\bar M)=T_{\bar g_0}S\oplus E$, $\pi^{-1}(S)$ coinsides with $(p\circ\phi\circ\pi)^{-1}(0)$. Since the differential of $p\circ\phi\circ\pi$ at $(\bar g_0, \psi_0)$ is surjective, there is a neighbourhood W of $(\bar g_0, \psi_0)$ such that $\pi^{-1}(S)\cap W$ is a submanifold of $\mathscr{X}^{r,s}$. Let $(h, V)\in T_{(\bar g_0,\psi_0)}\mathscr{N}^{r,s}$. If $T_{(\bar g_0,\psi_0)}(p\circ\phi\circ\pi)(h, V)=0$, then $h\in T_{\bar g_0}(S)$. But here Theorem 3.4 and the formula (3.9) implies that $[\gamma(h)]^J=0$. Therefore $h=0$, which implies that the tangent space of $\pi^{-1}(S)\cap W$ is $(0, J^{NK})$. Q. E. D.

Set $\pi^{-1}(S)\cap W=S'$ and $(\pi|S')-\bar g_0=\bar\pi$. $\bar\pi(\bar g_0, \psi_0)=0$ and $T_{(\bar g_0,\psi_0)}\bar\pi=0$.

LEMMA 4.5. *Let $(\bar g(t), \psi(t))$ be a curve in S' such that $(\bar g(0), \psi(0))=(\bar g_0, \psi_0)$. If we set $\bar g''(0)=h$ and $\psi'(0)=V$ then*

$$\sum_a\Big\{4\bar R_0(V, X_a)\bar\nabla_{X_a}V+(\bar\nabla_{X_a}\bar R_0)(V, X_a)V+(\bar\nabla_V\bar R_0)(V, X_a)X_a\Big\}+\gamma(h)=0\,, \tag{4.2}$$

where $\bar R_0$ is the curvature tensor of $\bar g_0$ and $\{X_a\}$ is an orthonormal frame of M.

PROOF. If we set $\bar g'(t)=\bar h(t)$ and $\psi'(t)=V(t)$ then, by Theorem 3.4, we have

$$\gamma_{(\bar g(t),\psi(t))}\big(\bar h(t)\big)+\beta_{(\bar g(t),\psi(t))}\big(V(t)\big)=0\,.$$

We give the differential of this equation at $(\bar g_0, \psi_0)$. The differential for the direction to $\mathscr{M}^r$ is given by $\gamma(\bar h'(0))$ and the differential for the direction to $\mathscr{P}^s$ is given as $\bar\nabla_V[\beta_{(\bar g_0,\psi(t))}(V(t))]$. We compute this form. Omit t in the following computation. Recal the definition (3.4).

$$(\bar\nabla\bar\nabla V)(X, X)=\bar\nabla_X\bar\nabla_X V-\bar\nabla_{\bar\nabla_X X}V\,,$$

and

$$\begin{aligned}\bar\nabla_V(\bar\nabla_X\bar\nabla_X V)&=\bar R(V, X)\bar\nabla_X V+\bar\nabla_X\bar\nabla_V\bar\nabla_X V\\&=\bar R(V, X)\bar\nabla_X V+\bar\nabla_X\big(\bar R(V, X)V+\bar\nabla_X\bar\nabla_V V\big)\\&=\bar R(V, X)\bar\nabla_X V+(\bar\nabla_X\bar R)(V, X)V+\bar R(\bar\nabla_X V, X)V+\bar R(V, \bar\nabla_X X)V\\&\quad+\bar R(V, X)\bar\nabla_X V+\bar\nabla_X\bar\nabla_X\bar\nabla_V V\,.\end{aligned}$$

And

$$\bar{\nabla}_V(\bar{\nabla}_{\bar{\nabla}_X X} V)=\bar{R}(V,\bar{\nabla}_X X)\,V+\bar{\nabla}_{\bar{\nabla}_X X}\bar{\nabla}_V V,$$

$$\begin{aligned}\bar{\nabla}_V\big(\bar{R}(V,X)X\big)&=(\bar{\nabla}_V\bar{R})(V,X)\,X+\bar{R}(\bar{\nabla}_V V,X)\,X+\bar{R}(V,\bar{\nabla}_V X)\,X\\&\qquad+\bar{R}(V,X)\,\bar{\nabla}_V X\\&=(\bar{\nabla}_V\bar{R})(V,X)X+\bar{R}(V,\bar{\nabla}_X V)\,X+\bar{R}(V,X)\,\bar{\nabla}_X V+\bar{R}(\bar{\nabla}_V V,X)\,X.\end{aligned}$$

Therefore

$$\begin{aligned}&\bar{\nabla}_V\big[(\bar{\nabla}\bar{\nabla}V)(X,X)+\bar{R}(V,X)\,X\big]\\&\quad=4\bar{R}(V,X)\,\bar{\nabla}_X V+(\bar{\nabla}_X\bar{R})(V,X)\,V+(\bar{\nabla}_V\bar{R})(V,X)\,X+\bar{\nabla}_X\bar{\nabla}_X(\bar{\nabla}_V V)\\&\qquad-\bar{\nabla}_{\bar{\nabla}_X X}(\bar{\nabla}_V V)+\bar{R}(\bar{\nabla}_V V,X)\,X\\&\quad=4\bar{R}(V,X)\,\bar{\nabla}_X V+(\bar{\nabla}_X\bar{R})(V,X)\,V+(\bar{\nabla}_V\bar{R})(V,X)\,X+\beta(\bar{\nabla}_V V).\end{aligned}$$

Set $X=X_a$ and take summation over a. Q. E. D.

LEMMA 4.6. *Let ϕ: $\boldsymbol{R}^p\to\boldsymbol{R}^p$ be a C^r-map ($r\geqq 2$) such that $\phi(0)=0$ and $d\phi(0)=0$. Set $2\tilde{\phi}(v)=(\mathrm{Hess}\,\phi)(v,v)$, where* Hess ϕ *is the Hessian of ϕ at the origin, i.e.,* $(\mathrm{Hess}\,\phi)(v,v')=v\,[v'(\phi)]$. *Assume that* Im $\tilde{\phi}$ *contains an open set of $\boldsymbol{R}^p$. Let w be an element of $\boldsymbol{R}^p$ such that $\tilde{\phi}^{-1}(w)=\{\pm v_a\}_{1\leqq a\leqq q}$ for some q and that the linear map: $v\to(\mathrm{Hess}\,\phi)(v_a,v)$ is non-degenerate for each a. If $w(t)$ is a C^r-curve in $\boldsymbol{R}^p$ such that $w(0)=0$ and $w'(0)=w$, then there are a neighbouhood W of $0\in\boldsymbol{R}^p$ and C^{r-1}-curves $v_a(t)$ in $\boldsymbol{R}^p$ such that $v_a(0)=0$, $v_a'(0)=v_a$ and*

$$\phi^{-1}\big(w(t^2)\big)\cap W=\big\{v_a(\pm t)\big\}_{1\leqq a\leqq q} \tag{4.3}$$

holds for sufficiently small $t>0$.

PROOF. We may assume that $w(t^2)=t^2w$, by changing coordinate system of $\boldsymbol{R}^p$ if necessary. Let $\{x^i\}$ and $\{y^i\}$ be the coordinates of the domain and the image, respectively. By the consition of ϕ we can assume that ϕ has the form as $\phi^k(x)=\phi^k{}_{ij}x^ix^j$, where $\phi^k{}_{ij}$ are C^{r-2}-function. Moreover we see easily that the equation $\phi^k{}_{ij}(x)\,z^iz^j=w^k$ for (z^i) has $2p$ solutions depending C^{r-2}-ly on x, which coinsides with $\{\pm v_a\}$ at $x=0$. Let $\lambda_a(x)$ be a solution such that $\lambda_a(0)=v_a$. We may assume that $\lambda_a^i(x)\neq 0$ for all i if x is sufficiently small. In fact this is satisfied by changing coordinate system of $\{y^j\}$ if necessary. Set $t_a^i(x)=x^i/\lambda_a^i(x)$. Then t_a^i are C^{r-2}-functions and the transformation matrix $(\partial t_a^j/\partial x^i)$ at 0 is non-degenerate. Therefore there are curves $\zeta_a(t)$ in $\boldsymbol{R}^p$ such that $t_a^i(\zeta_a(t))=t$. $\zeta_a(t)$ satisfies

$$\zeta_a^i(t)=t\cdot\lambda_a^i\big(\zeta_a(t)\big)\qquad\text{for any } a$$

and

$$\phi^k\Big(\zeta_a(t)\Big)=\phi^k{}_{ij}\Big(\zeta_a(t)\Big)\cdot\zeta_a^i(t)\cdot\zeta_a^j(t)$$
$$=\phi^k{}_{ij}\Big(\zeta_a(t)\Big)\cdot t\cdot\lambda_a^i\Big(\zeta_a(t)\Big)\cdot t\cdot\lambda_a^j\Big(\zeta_a(t)\Big)$$
$$=t^2\cdot w^k. \qquad \text{Q. E. D.}$$

THEOREM 4.7. *Assume that* $s>n/2+3$ *and* $r>s+\bar{n}/2+1$. *Let* $(\bar{g}_0, \phi_0)$ *be an element of* $\mathscr{H}$ *such that* $J\supsetneqq K$. *Denote by* $\Phi(V, W)$ *a* J^{NK}*-valued symmetric* 2*-form on* J^{NK} *associated with* (4.2), *that is,*

$$\begin{aligned}(4.4)\qquad \Phi(V, W) = -\sum_a\Big\{&4\bar{R}_0(V, X_a)\bar{\nabla}_{X_a}W+(\bar{\nabla}_{X_a}\bar{R}_0)(V, X_a)W\\ &+(\bar{\nabla}_V\bar{R}_0)(W, X_a)X_a+4\bar{R}_0(W, X_a)\bar{\nabla}_{X_a}V\\ &+(\bar{\nabla}_{X_a}\bar{R}_0)(W, X_a)V+(\bar{\nabla}_W\bar{R}_0)(V, X_a)X_a\Big\}^{J^{NK}},\end{aligned}$$

and set $\tilde{\phi}(V)=(1/2)\Phi(V, V)$. *Let* $h\in C^\infty(S^2\overline{M})$. *Assume that* $\operatorname{Im}\tilde{\phi}$ *contains an open set of* J. *If* $\tilde{\phi}^{-1}([\gamma(h)]^J)=\{\pm V_a\}_{1\leqq a\leqq q}$ *has the property that the linear map*: $V\to\Phi(V, V_a)$ *is non-degenerate for each* a, *then, for a* C^∞*-curve* $\bar{g}(t)$ *in* $\mathscr{M}^r$ *such that* $\bar{g}(0)=\bar{g}_0$ *and* $\bar{g}'(0)=h$ *there are a neighbourhood* W *of* $(\bar{g}_0, \phi_0)$ *in* $\mathscr{V}^{r,s}$ *and* $C^{r-s-[\bar{n}/2]-1}$*-curves* $\phi_a(t)$ *in* W *such that* $\phi_a(0)=\phi_0$, $\phi_a(0)=V_a$ *and*

$$(4.5)\qquad \pi^{-1}\Big(\bar{g}(t^2)\Big)\cap W=\Big\{\Big(\bar{g}(t^2),\ \phi_a(\pm t)\Big)\Big\}_{1\leqq a\leqq q}$$

for sufficiently small $t>0$, *where* π *is the projection*: $\mathscr{V}^{r,s}\to\mathscr{M}^r$.

PROOF. Since the condition implies that $[\gamma(h)]^J\neq 0$, we can construct the set S introduced above Lemma 4.5, so as to include $\{\bar{g}(t)\}$. Then, by Lemma 4.4 and Lemma 4.5, the proof reduces to Lemma 4.6.

Q. E. D.

REMARK 4.8. When the variational completeness is satisfied, this theorem cannot be applied. In fact, Hess π vanishes in this case.

ADDED IN PROOF. After this paper was written, the auther received a preprint of J. Eells and L. Lemaire: Deformations of metrics and associated harmonic maps (to appear in the Patodi memorial volume). Some of the results have overlap with this paper.

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