Hokkaido Mathematical Journal Vol. 8 (1979) p. 191-198

Some properties of locally conformal Kähler manifolds

By Toyoko KASHIWADA

(Received April 17, 1978; Revised September 4, 1978)

Introduction.

A Hermitian manifold whose metric is locally conformal to a Kähler metric is called locally conformal Kähler manifold (l. c. K-manifold). I. Vaisman ([3]) has given its characterization as follows:

A Hermitian manifold $M(\varphi, g)$ is a l. c. K-manifold if and only if there exists on M a global closed 1-form α such that

$$d\varphi=2\alpha\wedge\varphi . \qquad (*)$$

α is called the *Lee form* and introduced actually as follows: Consider at first a 1-form $\alpha=d\rho$ in a neighbourhood U at any point, where $e^{-2\rho}g$ is a Kähler metric in U. Since in intersections of such neighbourhoods, Kähler metrics must be homothetic, so α is defined namely globally. In [3], the Hopf manifold is given as a typical example of a l. c. K-manifold which admits no Kähler metric.

In this paper, at first we shall show several formulas with tensor analysis because such method seems not to have been yet used in this manifold, and then treat with the Betti number under some conditions.

The author wishes to express her hearty gratitude to Prof. S. Tachibana for his kind criticisms.

§ 1. Preliminaries.

Let $M(\varphi_j{}^k, g_{ij}, \alpha)$ be a l. c. K-manifold. By its definition, at any point there exists a neighbourhood in which a conformal metric $g^*=e^{-2\rho}g$ is Kähler one, *i. e.*,

$$\nabla_k^*(e^{-2\rho}\varphi_{ji})=0\,, \qquad d\rho=\alpha$$

where ∇^* is the covariant derivative with respect to g^*. Since

$$\nabla_k^*(e^{-2\rho}\varphi_{ji})=e^{-2\rho}(\nabla_k\varphi_{ji}+\alpha_j\varphi_{ki}-\alpha^r\varphi_{ri}g_{kj}+\alpha_i\varphi_{jk}-\alpha^r\varphi_{jr}g_{ki})\,,$$

we get

$$\nabla_k\varphi_{ji}=-\alpha_j\varphi_{ki}+\alpha^r\varphi_{ri}g_{kj}-\alpha_i\varphi_{jk}+\alpha^r\varphi_{jr}g_{ki}\,.$$

Conversely a Hermitian structure (φ, g) and closed α in this relation satisfy (*) evidently. Then we can denote

PROPOSITION 1.1. *A Hermitian manifold $M(\varphi, g)$ is a l.c. K-manifold if and only if*

$$\nabla_k \varphi_{ji} = -\beta_j g_{ki} + \beta_i g_{kj} - \alpha_j \varphi_{ki} + \alpha_i \varphi_{kj}$$
$$\nabla_i \alpha_j = \nabla_j \alpha_i \quad (or\ \varphi_{jr} \nabla_i \beta^r = \varphi_{ir} \nabla_j \beta^r),$$

where α is a global closed 1-form and $\beta_i = \alpha^r \varphi_{ri}$.

Hence in a l.c. K-manifold, we get easily following formulas:

$$\nabla_j \beta_i = -\beta_j \alpha_i + \beta_i \alpha_j - |\alpha|^2 \varphi_{ji} + \nabla_j \alpha^r \varphi_{ri},$$
$$\nabla_r \beta^r = 0, \qquad \nabla_i \beta^r \varphi_{jr} = \nabla_j \beta^r \varphi_{ir},$$
$$(n-2)\,\alpha_i = \varphi_{ri} \nabla_h \varphi^{rh} = \varphi^{hr} \nabla_h \varphi_{ri}, \qquad (n-2)\,\beta_i = \nabla^r \varphi_{ri},$$
$$\alpha^r \nabla_r \varphi_{jk} = \beta^r \nabla_r \varphi_{jk} = 0,$$
$$\alpha^r \nabla_i \varphi_{rj} = -\beta_i \alpha_j + \alpha_i \beta_j - |\alpha|^2 \varphi_{ij}, \quad \beta^r \nabla_i \varphi_{rj} = \beta_i \beta_j + \alpha_i \alpha_j - |\alpha|^2 g_{ij}.$$

Furthermore taking account of the Ricci's identity, we get

$$\begin{aligned} F_{khij} &= -R_{khi}{}^r \varphi_{rj} + R_{khj}{}^r \varphi_{ri} \\ &= P_{ki}\varphi_{hj} - P_{hi}\varphi_{kj} + g_{ki} P_{hr}\varphi^r{}_j - g_{hi} P_{kr} \varphi^r{}_j \\ &\quad - P_{kj}\varphi_{hi} + P_{hj}\varphi_{ki} - g_{kj} P_{hr} \varphi^r{}_i + g_{hj} P_{kr} \varphi^r{}_i, \end{aligned}$$

where we put $P_{ij} = -\nabla_i \alpha_j - \alpha_i \alpha_j + \frac{1}{2} \alpha_r \alpha^r g_{ij}$. We shall use also the tensor $G_{khij} = F_{khis} \varphi_j{}^s = R_{khrs} \varphi_i{}^r \varphi_j{}^s - R_{khij}$. If we make use of the tensor $H_{ij} = \frac{1}{2} R_{ijrs} \varphi^{rs}$ ([5]), from above equation we can write

$$\begin{aligned} F_{kr}{}^r{}_j &= -R_k{}^r \varphi_{rj} - H_{kj} \\ &= -(n-3)\, P_{kr} \varphi^r{}_j - P_{jr} \varphi^r{}_k - P_r{}^r \varphi_{kj}, \\ G_{kr}{}^r{}_j &= -(n-3)\, P_{kj} + P_{rs} \varphi_k{}^r \varphi_j{}^s - P_r{}^r g_{kj}. \end{aligned} \tag{1.1}$$

§ 2. Conditions to be a Kähler manifold.

In a Kähler manifold, tensor F, G vanish naturally. As to the converse matter, we have

THEOREM 2.1. *In a compact l.c. K-manifold $M^n(\varphi, g, \alpha)$ $(n \neq 2)$, if*

$$\tilde{H}_r{}^r - R_r{}^r (= G_{sr}{}^{rs}) \geqq 0$$

holds good where $\tilde{H}_{ij} = -H_{ir} \varphi_j{}^r$, then it is a Kähler manifold. The inequality $\geqq$ in this case is naturally reduced to $=$.

PROOF. From (1. 1), it follows

$$\tilde{H}_{ij}-R_{ij}=-(n-3)P_{ij}+P_{rs}\varphi_i^r\varphi_j^s-P_r^r g_{ij}=G_{ir}{}^r{}_j \tag{2.1}$$
$$\tilde{H}_r^r-R_r^r=(n-2)\left(2\nabla_r\alpha^r-(n-2)\alpha_r\alpha^r\right).$$

Therefore taking the integral, we have $\alpha=0$, which means the manifold is Kählerian. q. e. d.

Now, on account of the skew-symmetric property of H_{ij} in (1. 1), we get the following formula

$$R_{ir}\varphi_j^r+R_{jr}\varphi_i^r-(n-2)(P_{ir}\varphi_j^r+P_{jr}\varphi_j^r)=0\,. \tag{2.2}$$

Thus it follows

PROPOSITION 2.2. *If an n-dimensional l. c. K-manifold* $M^n(\varphi, g, \alpha)$ $(n\neq 2)$ *is an Einstein space, then* P_{ij} *is hybrid, i. e.,*

$$(\nabla_i\alpha_r+\alpha_i\alpha_r)\varphi_j^r+(\nabla_j\alpha_r+\alpha_j\alpha_r)\varphi_i^r=0\,. \tag{2.3}$$

Especially if we suppose $\nabla\alpha=0$, then (2. 3) implies $\alpha=0$. Hence we obtain that

COROLLARY 2. 3. *If a l. c. K-manifold* $M^n(\varphi, g, \alpha)$ $(n\neq 2)$ *with* $\nabla\alpha=0$ *is an Einstein space, it must be a Kähler manifold.* (*See the remark of Lemma* 3. 2.)

We continue to consider under the condition $\nabla\alpha=0$, $\alpha\neq 0$. Since

$$\tilde{H}_{ij}-R_{ij}=(n-3)\alpha_i\alpha_j-\beta_i\beta_j-(n-3)|\alpha|^2 g_{ij}\,,$$

transvecting it with a vector X,

$$(\tilde{H}_{ij}-R_{ij})X^iX^j=(n-3)\left((\alpha_rX^r)^2-|\alpha|^2|X|^2\right)-(\beta_rX^r)^2$$

we know the tensor $\tilde{H}_{ij}-R_{ij}$ is negative semidefinite.

PROPOSITION 2. 4. *In a l. c. K-manifold* $M^n(\varphi, g, \alpha)$ $(n\geqq 4)$ *with* $\nabla\alpha=0$, $\alpha\neq 0$, *it is valid for any vector field* X *that*

$$\tilde{H}_{ij}X^iX^j-R_{ij}X^iX^j\leqq 0\,.$$

The equality holds only if for $X=f\alpha$, ($f\in \boldsymbol{C}^\infty(M)$).

§ 3. Riemannian curvature tensor.

In this section we treat with the following relation:

$$R_{abcd}\varphi_i^a\varphi_j^b\varphi_k^c\varphi_l^d=R_{ijkl}\,. \tag{3.1}$$

It seems worthy to consider the relation (3. 1) because the Bochner curvature tensor in a Kähler manifold has been generalized into an almost Hermitian manifold with this equality ([4]).

Now let

$$g^* = e^{-2\rho} g\,, \qquad P_{ij} = -\nabla_i \rho_j - \rho_i \rho_j + \frac{1}{2}\rho_r \rho^r g_{ij}$$

where ρ_i is the differential of ρ. Since as well known, it is valid

$$e^{2\rho} R^*{}_{ijkl} - R_{ijkl} = P_{ik} g_{jl} - P_{jk} g_{il} + g_{ik} P_{jl} - g_{jk} P_{il}$$

where R^* is the Riemannian curvature tensor of g^*, the relation (3.1) is preserved with a conformal transformation whose P is hybrid, *i.e.*, $P_{rs}\varphi_i{}^r \varphi_j{}^s - P_{ij} = 0$. Now, we can get

THEOREM 3.1. *In a l.c. K-manifold* $M^n(\varphi, g, \alpha)$ $(n \neq 2)$, (3.1) *holds good if and only if* $\nabla_i \alpha_j + \alpha_i \alpha_j$ *(or Ricci tensor) is hybrid.*

PROOF. If $\nabla_i \alpha_j + \alpha_i \alpha_j$ (then P) is hybrid, (3.1) is valid because it holds in a Kähler manifold and g is conformal to a Kähler metric g^* with $P_{ij} = -\nabla_i \alpha_j + \alpha_i \alpha_j + \frac{1}{2}|\alpha|^2 g_{ij}$.

Conversely we assume (3.1). With respect to a φ-base $\{e_1, \cdots, e_m, e_{m+1}, \cdots, e_{2m}\}$, $e_{m+a} = \varphi e_a$, $a = 1, 2, \cdots, m\left(=\frac{n}{2}\right)$, we can write $R_{ji^*} = \sum_{k=1}^{2m} R_{kji^*k}$ where i^*-component means the component for φe_i, $i = 1, \cdots, 2m$, so that

$$R_{ji^*} = \sum_k R_{kji^*k} = -\sum_k R_{k^*j^*ik^*} = -R_{j^*i}$$

i.e. the Ricci tensor is hybrid. Hence, if we notice that by virture of (2.2), the hybrid property of the Ricci tensor is equivalent to that of P, the proof is completed. q.e.d.

We known easily

LEMMA 3.2. *In* $M(\varphi, g, \alpha)$, *the followings are equivalent to one another*:

(i) P *is hybrid, i.e.,* $P_i{}^r \varphi_{rj} = -P_j{}^r \varphi_{ri}$.

(ii) $\varphi_i{}^r \nabla_r \alpha_j + \varphi_j{}^r \nabla_r \alpha_i = \alpha_i \beta_j + \beta_i \alpha_j$.

(iii) $\nabla_i \beta_j + \nabla_j \beta_i = -(\alpha_i \beta_j + \beta_i \alpha_j)$.

REMARK. Evidently it is known from (ii) that $M(\varphi, g, \alpha)$ with $\nabla\alpha = 0$ is a Kähler manifold if its P is hybrid.

From now on in this section, we consider under the condition P to be hybrid.

For brevity's sake we define a operator $\triangle\!\!\!\!\bigcirc$ for a symmetric 2-tensor τ, a skew-symmetric 2-tensor ω as follows ([1]):

$$(\tau \,\triangle\!\!\!\!\bigcirc\, g)_{ijkl} = \tau_{il} g_{jk} - \tau_{ik} g_{jl} + g_{il} \tau_{jk} - g_{ik} \tau_{jl}\,,$$

$$(\omega \,\triangle\!\!\!\!\bigcirc\, \varphi)_{ijkl} = \omega_{il} \varphi_{jk} - \omega_{ik} \varphi_{jl} + \varphi_{il} \omega_{jk} - \varphi_{ik} \omega_{jl} - 2(\omega_{ij} \varphi_{kl} + \varphi_{ij} \omega_{kl})\,.$$

We remark that these tensors satisfy the 1-st Bianchi's identity.

Since $\tilde{P}_{ij}=P_i{}^r\varphi_{rj}$ is skew-symmetric from the assumption, G is denoted as

$$G_{ijkl}=-(P\textcircled{\scriptsize$\wedge$}g+\tilde{P}\textcircled{\scriptsize$\wedge$}\varphi)_{ijkl}+2\tilde{P}_{ij}\varphi_{kl}+2\varphi_{ij}\tilde{P}_{kl}\,.$$

Obviously it is known that

LEMMA 3.3. *In $M(\varphi, g, \alpha)$ with hydrid P, the followings are valid:*

(i) F_{ijkl}, $G_{ijkl}(=F_{ijkr}\varphi_l{}^r)$ *are skew-symmetric for i, j and for k, l.*

(ii) $G_{ijkl}=G_{klij}$, $F_{ijkl}=-F_{klij}$.

(iii) $G_{ijkr}\varphi_l{}^r=G_{ijrl}\varphi_k{}^r=-G_{irkl}\varphi_j{}^r$.

Let $H(X)$ be the holomorphic sectional curvature of the section $\{X, \varphi X\}$ at $p\in M$, *i.e.*,

$$H(X)=\frac{R_{ijkl}X^i\varphi_r{}^jX^r\varphi_s{}^kX^sX^l}{|X|^4}\qquad\big(X\in T_p(M)\big)\,. \tag{3.2}$$

We assume now the holomorphic sectional curvature at $p\in M$ has the constant value H. Then, from (3.2), by straightforward computations, we know this assumption is equivalent to the following relation at $p\in M$ for any base:

$$\begin{aligned}&2(R_{ij^*k^*l}+R_{ik^*j^*l}+R_{il^*k^*j})+G_{klij}+G_{jlik}+G_{kjil}\\&\quad=2H(g_{ij}g_{kl}+g_{ik}g_{jl}+g_{il}g_{jk})\,.\end{aligned} \tag{3.3}$$

Interchanging $k\leftrightarrow k^*$, $l\leftrightarrow l^*$ in (3.3) and subtracting it from (3.3), we have

$$\begin{aligned}&2(R_{ik^*j^*l}+R_{ikjl})\\&\quad=H(g_{ik}g_{jl}+g_{il}g_{jk}-\varphi_{ik}\varphi_{jl}-\varphi_{il}\varphi_{jk})-G_{ijkl}-2G_{ikjl}\,.\end{aligned} \tag{3.4}$$

Interchanging $k\leftrightarrow j$ in (3.4) and subtracting it from (3.4), we get finally

$$\begin{aligned}4R_{iljk}&=H(g_{ik}g_{jl}-g_{ij}g_{kl}+\varphi_{ik}\varphi_{lj}-\varphi_{ij}\varphi_{lk}-2\varphi_{il}\varphi_{jk})\\&\quad+G_{iklj}+G_{ijkl}-2G_{iljk}\,.\end{aligned} \tag{3.5}$$

Hence the following Theorem is obtained:

THEOREM 3.4. *In $M^n(\varphi, g, \alpha)$ whose $\nabla_i\alpha_j+\alpha_i\alpha_j$ is hybrid, if the holomorphic sectional curvature at $p\in M$ is constant H, then* (3.5) *holds good, i.e.,*

$$4R=\frac{1}{2}H(g\textcircled{\scriptsize$\wedge$}g+\varphi\textcircled{\scriptsize$\wedge$}\varphi)+3P\textcircled{\scriptsize$\wedge$}g-\tilde{P}\textcircled{\scriptsize$\wedge$}\varphi$$

at $p\in M$ where $P_{ij}=-\nabla_i\alpha_j-\alpha_i\alpha_j+\dfrac{|\alpha|^2}{2}g_{ij}$, $\tilde{P}_{ij}=P_{ir}\varphi^r{}_j$.

REMARK. Under the above assumption for $n\neq 4$, M is an Einstein manifold if and only if

(3. 6) $$P=\frac{1}{n}P_r{}^r g$$

because of $4R_{ih}=\frac{1}{n}(4R_r{}^r-3(n-4)P_r{}^r)g_{ih}+3(n-4)P_{ih}$.

In the following we shall see that the relation (3. 6) in above Remark has another meaning in the Bochner curvature tensor.

The Bochner curvature tensor in a Kähler manifold $M^n(\varphi, g)$ is defined as ([2])

(3. 7) $$B_g=R-\frac{1}{n+4}(R_1 \bigotriangleup g+\tilde{R}_1 \bigotriangleup \varphi)+\frac{l}{2(n+2)(n+4)}(g \bigotriangleup g+\varphi \bigotriangleup \varphi)$$

where R_1, l are the Ricci tensor, the scalar curvature respectively, and $\tilde{R}_1$ is the tensor $\varphi\circ R_1$ *i. e.*, $(\tilde{R}_1)_{ij}=R_{ir}\varphi^r{}_j$. Furthermore as the generalization from the concept of the decomposition of curvature tensors, the Bochner cutvature tensor in an almost Hermitian manifold $M^n(\varphi, g)$ with the relation (3. 1) is defined as ([4])

$$B_g=R-(A \bigotriangleup g+\tilde{A} \bigotriangleup \varphi)+\frac{\mathring{l}-l}{8n(n-2)}(3g \bigotriangleup g-\varphi \bigotriangleup \varphi)$$

where

$$A=\frac{1}{4(n+4)}\left(R_1+3\mathring{R}_1-\frac{l+3\mathring{l}}{2(n+2)}g\right),\quad \tilde{A}=\varphi\circ A$$
$$(\mathring{R}_1)_{ij}=R_{ij}+G_{rij}{}^r,\qquad \mathring{l}=(\mathring{R}_1)_{rs}g^{rs}.$$

Now, in our manifold, as

$$G_{rij}{}^r=-(n-4)P_{ij}-P_r{}^r g_{ij},$$

we get by direct computations

(3. 8) $$B_g=R-\frac{1}{n+4}(R_1 \bigotriangleup g+\tilde{R}_1 \bigotriangleup \varphi)+\frac{3(n-4)}{4(n+4)}(P \bigotriangleup g+\tilde{P} \bigotriangleup \varphi)+\frac{2nl-3(n^2+2n+8)P_r{}^r}{4n(n+2)(n+4)}g \bigotriangleup g+\frac{2nl+(n^2+18n+8)P_r{}^r}{4n(n+2)(n+4)}\varphi \bigotriangleup \varphi.$$

LEMMA 3. 5. *In a l. c. K-manifold $M^n(\varphi, g, \alpha)$ with hybrid $\nabla_i\alpha_j+\alpha_i\alpha_j$ the Bochner curvature tensor is written as*

$$R-\frac{1}{n+4}(R_1\circledwedge g+\tilde{R}_1\circledwedge\varphi)+\frac{3(n-4)}{4(n+4)}(P\circledwedge g+\tilde{P}\circledwedge\varphi)$$
$$+\frac{2nl-3(n^2+2n+8)P_r{}^r}{4n(n+2)(n+4)}g\circledwedge g+\frac{2nl+(n^2+18n+8)P_r{}^r}{4n(n+2)(n+4)}\varphi\circledwedge\varphi\,.$$

Hence we obtain the following:

THEOREM 3.6. *In a l.c. K-manifold $M(\varphi, g, \alpha)$, if the tensor $\nabla_i\alpha_j+\alpha_i\alpha_j$ is proportional to g, then its Bochner curvature tensor $B_{ijk}{}^l$ is equal to that of Kähler metrics conformal to g in locally.*

PROOF. By virtue of (3.7) for the Kähler metric g^* and (3.8) we have

$$(B_g-B_{g^*})_{ijk}{}^l=\frac{1}{4}\left(3P\circledwedge g-\frac{3P_r{}^r}{n}g\circledwedge g-\tilde{P}\circledwedge\varphi+\frac{P_r{}^r}{n}\varphi\circledwedge\varphi\right)_{ijk}{}^l\,.$$

Hence it is clear $(B_g)_{ijk}{}^l=(B_{g^*})_{ijk}{}^l$ under the condition (3.6).

§ 4. Conformally flat case.

If a l.c. K-manifold $M(\varphi, g, \alpha)$ is conformally flat, the locally corresponding Kähler manifold with $g^*=e^{-2\rho}g$ is conformally flat, so that flat necessarily. Then in this case, the Ricci tensor of $M^n(\varphi, g, \alpha)$ is represented as

$$R_{ij}=(n-2)(-\nabla_i\alpha_j-\alpha_i\alpha_j+\alpha_r\alpha^r g_{ij})-\nabla_r\alpha^r g_{ij}\,.$$

If we assume here $\nabla\alpha=0$, $\alpha\neq0$, then transvecting it with a vector X, we have

$$R_{ij}X^iX^j=(n-2)\left(-(\alpha_rX^r)^2+|\alpha|^2|X|^2\right).$$

Therefore we get

PROPOSITION 4.1. *In a conformally flat l.c. K-manifold with $\nabla\alpha=0$, $\alpha\neq0$, the Ricci tensor is positive semidefinite. The equality $R_{ij}X^iX^j=0$ holds only for the vector $X^r=f\alpha^r(f\in \boldsymbol{C}^\infty(M))$.*

From the above Proposition and by virtue of the well known property that in a compact Riemannian manifold with positive semidefinite Ricci tensor, a harmonic 1-form X must satisfy

$$R_{ij}X^iX^j=0\,,\qquad \nabla_iX_j=0\,,$$

harmonic 1-forms in our present space are only $c\alpha^i$, c=constant. Consequently we obtain

THEOREM 4.2. *In a compact conformally flat l.c. K-manifold with $\nabla\alpha=0$, $\alpha\neq0$, the first Betti number=1.*

Bibliography

[1] J. P. BOURGUIGNON: On harmonic forms of curvature type, to appear.

[2] S. TACHIBANA: On the Bochner curvature tensor, Natur. Sci. Rep. Ochanomizu Univ., 18 (1967), 15-19.

[3] I. VAISMAN: On locally conformal almost Kähler manifolds, Israel J. Math., 24 (1976), 338-351.

[4] L. VANHECKE: The Bochner curvature tensor on almost hermitian manifolds, Rend. Sem. Mat. Univ. e Politec. Torino, 34 (1975-76), 21-38.

[5] K. YANO: Differential geometry on complex and almost complex space, Pergamon Press, 1965.

Department of Mathematics,
Ochanomizu University,
Tokyo, Japan