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Partially admissible shifts on linear topological spaces

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§ 1. Introduction

Quasi-invariant cylinder measures on real Banach spaces were studied in [8] by W. Linde. On the other hand, partially admissible shifts of measures on real linear spaces were studied in [4] and [5] by R. M. Dudley, and in case of Hilbert spaces more complete results were given in [16] by A. V. Skorohod.

In this paper we introduce partially admissible shifts of cylinder measures on real linear topological spaces. The definition generalizes the notion of partially admissible shifts of measures. Section 3 contains some results on partially admissible shifts of cylinder measures. The main result of this section is the following theorem.

THEOREM. *Let $F\subset E$ be linear topological spaces, and μ be a cylinder measure on E. Suppose that the inclusion map $F\rightarrow E$ be continuous, and $1\leqq p<\infty$. Also suppose that one of the following two conditions be satisfied:*

(1) *$F\subset\tilde{M}_{\mu}$ and the linear topological space F is barrelled,*

(2) *$F\cap\tilde{M}_{\mu}$ is second category in F,*

where we denote by $\tilde{M}_{\mu}$ the set of partially admissible shifts of the cylinder measure μ. Then there exists a neighbourhood V of zero in F such that the inequality

$$\sup_{x\in V}|\langle x^{*}, x\rangle|\leqq\left(\int_{E}|\langle x^{*}, x\rangle|^{p}d\mu(x)\right)^{1/p}\qquad \text{for all } x^{*}\in E^{*}$$

holds.

This generalizes the results of W. Linde [8] and D. Xia [25]. Furthermore, using this theorem, it is shown that a Banach space E is isomorphic to a Hilbert space iff it admits a cylinder measure μ of type 2 such that $\tilde{M}_{\mu}$ is second category in E. The remainder parts of this section generalize the results of R. M. Dudley [4], [5] and W. Linde [8].

In Section 4 we study the partially admissible shifts of measures, and then our results generalize the ones of A. V. Skorohod [16] and D. Xia [25].

Throughout the paper, we assume that linear spaces are with real coefficients.

§ 2. Basic definitions and well known results

1°. p-absolutely summing operators and $\mathscr{L}_p$-spaces

Let E and F be Banach spaces, and denote their dual spaces by E^* and F^*, respectively. Let $1 \leqq p < \infty$.

A sequence $\{x_i\}$ with values in E is called weakly p-summable if for each $x^* \in E^*$, the sequence $\{x^*(x_i)\} \in l_p$.

A sequence $\{x_i\}$ with values in E is called absolutely p-summable if the sequence $\{\|x_i\|\} \in l_p$.

DEFINITION 2.1.1. A linear operator T from E into F is called p-absolutely summing if for each $\{x_i\} \subset E$ which is weakly p-summable, $\{T(x_i)\} \subset F$ is absolutely p-summable.

We shall say "absolutely summing" instead of "1-absolutely summing".

The following theorems are due to J. S. Cohen. For the definitions of p^*-strongly summing operators and $\mathscr{L}_p$-spaces; see [2] and [9].

THEOREM 2.1.1. (c.f. [2])

Let $1/p + 1/p^* = 1$. A linear operator T from E into F is p^*-strongly summing iff the adjoint operator T^* from F^* into E^* is p-absolutely summing.

THEOREM 2.1.2. (c.f. [2])

Let E be a Banach space which is isomorphic to the dual of an $\mathscr{L}_p$-space. For any Banach space F and an operator T from E into F, if T is p-absolutely summing then the adjoint operator T^* from F^* into E^* is p-absolutely summing.

REMARK 2.1.1. It is easily seen that if E is isomorphic to the quotient space of the dual of an $\mathscr{L}_p$-space then Theorem 2.1.2. is true. For the related results; see [20].

It is well known (c.f. [12]) that if an operator T from E into F is p-absolutely summing then it is q-absolutely summing (for $1 \leqq p \leqq q < \infty$). Hence, if an operator T from E into F is p^*-strongly summing then it is q^*-strongly summing (for $1 < q^* \leqq p^* \leqq \infty$).

PROPOSITION 2.1.1. (c.f. [12])

Let H be a Hilbert space and E be a Banach space. For a linear operator T from H into E the followings are equivalent.

(1) *T is p-absolutely summing (for $1 \leqq p \leqq 2$).*

(2) *There exists a Hilbert space G such that*

$$H \xrightarrow[U]{} G \xrightarrow[V]{} E$$

$T=V\circ U$ where the linear operators U is of Hilbert-Schmidt type and V is continuous, respectively.

An operator T from E into F is said to be Hilbertian if there exist a Hilbert space H and continuous linear operators $A: E\to H$ and $B: H\to F$ such that $T=B\circ A$.

COROLLARY 2.1.1. *If a linear operator T from E into F is p-absolutely summing (for $1\leqq p\leqq 2$), then it is Hilbertian.*

PORPOSITION 2.1.2. (c.f. [3], [10])

Any continuous linear operator from $\mathcal{L}_\infty$-space into $\mathcal{L}_1$-space is 2-absolutely summing. Hence, it is Hilbertian.

COROLLARY 2.1.2. *Let E be a Banach space which is isomorphic to a quotient space of $\mathcal{L}_\infty$-space. Then any continuous linear operator from E into $\mathcal{L}_1$-space is Hilbertian.*

Next, we shall give a necessary and sufficient condition such that a Banach space E is isometric to a subspace of $L_p(\mu)$, for some measure μ. The key notion here is that of a negative definite function.

DEFINITION 2.1.2. A function f from a linear space X into the non-negative reals is said to be negative definite if $f(0)=0$, $f(x)=f(-x)$ for all $x\in X$ and

$$\sum_{i,j=1}^{n} f(x_i-x_j)\,\alpha_i\alpha_j\leqq 0$$

for every choice of $\{x_i\}_{i=1}^{n}\subset X$, and every choice of scalars $\{\alpha_i\}_{i=1}^{n}$ with

$$\sum_{i=1}^{n}\alpha_i=0\,.$$

THEOREM 2.1.3. (c.f. [1])

Let $1\leqq p\leqq 2$. A Banach space E is isometric to a subspace of $L_p(\mu)$, for some measure μ, iff the map $x\to||x||^p$ is negative definite.

COROLLARY 2.1.3. *Let $1\leqq p\leqq 2$. Let E be a linear space and $||\cdot||$ be a seminorm on E. Denote the associated Banach space of the seminormed space $(E, ||\cdot||)$ by $\hat{E}$. If the map $x\to||x||^p$ is negative definite, then $\hat{E}$ is isometric to a subspace of $L_p(\mu)$, for some measure μ.*

It is well known that if f is negative definite then f^α is also negative definite for every $0<\alpha\leqq 1$. Thus we have

COROLLARY 2.1.4. *Let $1\leqq q\leqq p\leqq 2$. Then $L_p(\mu)$ is isometric to a subspace of $L_q(\nu)$ for some measure ν.*

REMARK 2.1.2. Since every $\mathcal{L}_p$-space (for $1\leqq p<\infty$) is isomorphic to a subspace of $L_p(\mu)$ for some measure μ (c.f. [10]), hence by the above corollary, every $\mathscr{L}_p$-sapce (for $1\leqq p\leqq 2$) is isomorphic to a subspace of $L_1(\mu)$ for some measure μ.

2°. Partially admissible shifts of cylinder measures and measures

Let E be a real linear topological space, and denote the dual space of E by E^*.

First, we introduce partially admissible shifts of cylinder measures. For the definition of cylinder sets and cylinder measures, and the related results: see [7], [11], [23] and [25].

If a cylinder measure μ is given on E, then μ_x (for $x\in E$) denotes the cylinder measure on E defined by

$$\mu_x(Z)=\mu(Z-x) \qquad \text{for any cylinder set } Z \text{ of } E.$$

DEFINITION 2.2.1. An element $x\in E$ is called an admissible shift for the cylinder measure μ if for any $\varepsilon>0$ there is a $\delta>0$ such that

$$\mu_x(Z)<\varepsilon$$

for any cylinder set Z of E for which

$$\mu(Z)<\delta .$$

The set of admissible shifts of the cylinder measure μ will be denoted by M_μ.

DEFINITION 2.2.2. An element $x\in E$ is called a partially admissible shift for the cylinder measure μ if there is an $\varepsilon>0$ and $\delta>0$ such that

$$\mu_x(Z^c)>\varepsilon$$

for any cylinder set Z of E for which

$$\mu(Z)<\delta ,$$

where Z^c denotes the complement of Z in E.

The set of partially admissible shifts of the cylinder measure μ will be denoted by $\widetilde{M}_\mu$.

It is easily seen that $M_\mu\subset\widetilde{M}_\mu$, but in general M_μ does not coincide with $\widetilde{M}_\mu$.

REMARK 2.2.1. In general, the cylinder measure μ is not σ-additive. But if it happens that μ is σ-additive, then using well known technique, we can extend μ to a probability measure on the σ-algebra generated by cylinder sets. Then, it is easily seen that an element $x\in E$ is an admissible shift for the measure μ in a sense of Definition 2.2.1. iff μ_x is absolutely continuous with respect to μ, and also seen that an element $x\in E$ is a partially admissible shift for the measure μ in a sense of Definition 2. 2. 2. iff μ_x contains a component absolutely continuous with respect to μ. Thus, in case of measures Definition 2. 2. 1. and Definition 2. 2. 2. coincide with the definitions of Skorohod, respectively (c. f. [16]).

Next, we introduce the continuity of cylinder measures.

For a cylinder measure μ on E, the Fourier transform $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(x^*)=\int_E e^{i\langle x^*,x\rangle}d\mu(x) \qquad \text{for } x^*\in E^* .$$

Let τ be a linear topology on E^*, and denote a linear topological space E^* equipped with the topology τ by E^*_τ.

DEFINITION 2. 2. 3. The cylinder measure μ is said to be continuous with respect to τ if the function $\hat{\mu}(x^*)$ is continuous on E^*_τ.

PROPOSITION 2. 2. 1. (c. f. [25])

The cylinder measure μ is continuous with respect to τ iff for any $\varepsilon>0$ there exists a neighbourhood V of zero in E^*_τ such that

$$\mu\Big(\Big\{x\Big|\,\big|\langle x^*, x\rangle\big|>1\Big\}\Big)<\varepsilon \qquad \textit{for all } x^*\in V .$$

Now, we shall show examples of a linear topology τ such that the cylinder measure μ is continuous with respect to τ.

On E^*, we let τ_μ be the topology of convergence in μ-cylinder measure, metrized by the semimetric

$$d(x^*, y^*)=\int_E\frac{|\langle x^*, x\rangle-\langle y^*, x\rangle|}{1+|\langle x^*, x\rangle-\langle y^*, x\rangle|}\,d\mu(x) \qquad \text{for } x^*, y^*\in E^* .$$

Then, it is easily seen that the cylinder measure μ is continuous with respect to τ_μ, namely $\hat{\mu}(x^*)$ is a continuous function on $E^*_{\tau_\mu}$. In the ensuing discussions, we denote the linear topological space E^* equipped with the topology τ_μ by E^*_μ instead of $E^*_{\tau_\mu}$.

Another example is the following.

Let $1\leqq p<\infty$. Define

$$||x^*||_p=\Big(\int_E\big|\langle x^*, x\rangle\big|^p d\mu(x)\Big)^{1/p} \qquad \text{for } x^*\in E^* .$$

Here, if $||x^*||_p<\infty$ for all $x^*\in E^*$ then $||x^*||_p$ is a seminorm on E^*. Denote the seminormed space E^* equipped with the seminorm $||\cdot||_p$ by E^*_p, and denote the associated Banach space of the seminormed space E^*_p by $\hat{E}^*_p$. Then, obviously the identity map $E^*_p\rightarrow E^*_\mu$ is continuous, so we have that μ is continuous with respect to the seminorm $||\cdot||_p$.

Now, let $1 \leqq p \leqq 2$. Then, the above example gives that of a negative definite function. Indeed, the map $x^* \to \|x^*\|_p^p$ is negative definite. Thus, by Corollary 2.1.3., we have that $\hat{E}_p^*$ is isometric to a subspace of $L_p(\nu)$ for some measure ν.

Finally, we introduce the cylinder measure of type p $(1 \leqq p < \infty)$.

Let E be a real Banach space.

DEFINITION 2.2.4. A cylinder measure μ on E is of type p if there is a constant C such that

$$\left(\int_E \left|\langle x^*, x\rangle\right|^p d\mu(x)\right)^{1/p} \leqq C\|x^*\| \qquad \text{for all } x^* \in E^* .$$

REMARK 2.2.2.

(1) Let $1 \leqq q \leqq p < \infty$. If μ is type p then μ is type q.

(2) Let $1 \leqq p < \infty$. If μ is type p then, by the previous arguments, μ is continuous with respect to the seminorm $\|\cdot\|_p$, and so it is continuous with respect to the norm topology of E^*.

(3) Let $1 \leqq p \leqq 2$. If μ is type p then $\hat{E}_p^*$ is isometric to a subspace of $L_p(\nu)$ for some measure ν.

§ 3. The cylinder measure case

In this section we shall discuss the partially admissible shifts of cylinder measures. Let E be a real linear topological space, and denote the dual space of E by E^*. For an element $x \in E$, define

$$e_x(x^*) = \langle x^*, x\rangle \qquad \text{for } x^* \in E^* .$$

PROPOSITION 3.1. *Let μ be a cylinder measure on E, and let τ be a linear topology on E^* such that μ is continuous with respect to τ. Then for each $x \in \widetilde{M}_\mu$, e_x is a continuous linear functional on E_τ^*.*

PROOF. Let $x \in \widetilde{M}_\mu$. Then, from the definition of $\widetilde{M}_\mu$, there exists an $\varepsilon > 0$ and $\delta > 0$ such that $\mu_x(Z^c) > \varepsilon$ for every cylinder set Z of E for which $\mu(Z) < \delta$. Here, we may assume that $0 < \delta < \varepsilon < 1$.

On the other hand, since the cylinder measure μ is continuous with respect to τ, hence there exists a neighbourhood V of zero in E_τ^* such that

$$\mu\left(\left\{y \,\middle|\, \left|\langle x^*, y\rangle\right| > 1\right\}\right) < \delta \qquad \text{for all } x^* \in V .$$

For each $x^* \in V$, put

$$Z_{x^*} = \left\{y \,\middle|\, \left|\langle x^*, y\rangle\right| > 1\right\} .$$

Then, $\mu(Z_{x^*}) < \delta$ implies that $\mu_x(Z_{x^*}^c) > \varepsilon$. Since $\mu(E) = 1$, hence it is easily seen that there exists an element $z\in E$ such that the inequalities

$$\left|\langle x^*, z\rangle\right|\leqq 1 \quad\text{and}\quad \left|\langle x^*, z+x\rangle\right|\leqq 1$$

holds.

From this it follows

$$\left|\langle x^*, x\rangle\right|\leqq 2 \qquad \text{for all } x^*\in V\,.$$

This shows that e_x is continuous on E^*_τ, and we complete the proof.

COROLLARY 3.1. *For each* $x\in\widetilde{M}_\mu$, e_x *is a continuous linear functional on* E^*_μ.

By the same method as the proof of Proposition 3.1., we have

PROPOSITION 3.2. *Let* μ *be a cylinder measure on E. If a sequence of cylinder sets* Z_n *of E satisfies that*

$$\lim_{n\to\infty}\mu(Z_n)=1\,,$$

then, the following inclusion

$$\widetilde{M}_\mu\subset\bigcup_{n=1}^{\infty}(Z_n-Z_n)$$

holds.

Let H be a real Hilbert space, and let μ_H be a canonical Gaussian cylinder measure on H. Then, μ_H is a quasi-invariant cylinder measure c.f. [8]), and hence $M_\mu=H$, in particular, $\widetilde{M}_\mu=H$. Thus, we have

COROLLARY 3.2. *If a sequence of cylinder sets* Z_n *of H satisfies that*

$$\lim_{n\to\infty}\mu_H(Z_H)=1\,,$$

then the identity

$$H=\bigcup_{n=1}^{\infty}(Z_n-Z_n)$$

holds.

Now, let μ be a cylinder measure on E, and let $1\leqq p<\infty$. Recall that $||x^*||_p$ (for $x^*\in E^*$) be defined by

$$||x^*||_p=\left(\int_E\left|\langle x^*, x\rangle\right|^p d\mu(x)\right)^{1/p} \qquad \text{for } x^*\in E^*\,.$$

Let U_p and U_p^0 be defined by

$$U_p=\left\{x^*\in E^*\,\middle|\,||x^*||_p\leqq 1\right\} \quad\text{and}\quad U_p^0=\left\{x\in E\,\middle|\,\left|\langle x^*, x\rangle\right|\leqq 1 \text{ for all } x^*\in U_p\right\}.$$

Then, we have the following.

LEMMA 3.1. $\widetilde{M}_\mu \subset \bigcup_{n=1}^{\infty} nU_p^0$.

PROOF. Assume the contrary. Then, there exists an element $x \in \widetilde{M}_\mu$ and sequence $x_n^* \in U_p$ such that the inequality

$$\left|\langle x_n^*, x\rangle\right| > n \qquad (n=1, 2, \cdots)$$

holds.

Since $\frac{1}{n} x_n^*$ tends to zero with respect to $\|\cdot\|_p$, hence tends to zero in μ-cylinder measure. Thus, by Corollary 3.1., we have

$$\lim_{n\to\infty}\left|\left\langle \frac{1}{n} x_n^*, x\right\rangle\right| = 0\,.$$

That is a contradiction, and we complete the proof.

By this lemma, we obtain the following main theorem. That generalizes a result of Linde [8]. We prove it for partially admissible shifts of cylinder measures instead of admissible shifts of cylinder ones.

THEOREM 3.1. *Let $F \subset E$ be linear topological spaces, and μ be a cylinder measure on E. Suppose that the inclusion map $F \to E$ be continuous, and $1 \leqq p < \infty$. Then we have the followings.*

(1) *If $F \subset \widetilde{M}_\mu$ and the linear topological space F is barrelled, then there exists a neighbourhood V of zero in F such that the inequality*

$$\sup_{x\in V}\left|\langle x^*, x\rangle\right| \leqq \|x^*\|_p \qquad \textit{for all } x^* \in E^*$$

holds.

(2) *If $F \cap \widetilde{M}_\mu$ is second category in F, then there exists a negihbourhood V of zero in F such that the inequality*

$$\sup_{x\in V}\left|\langle x^*, x\rangle\right| \leqq \|x^*\|_p \qquad \textit{for all } x^* \in E^*$$

holds.

PROOF. (1): It is easily seen that the set U_p^0 is convex, balanced and closed in E. Since the inclusion map $F \to E$ is continuous, hence by Lemma 3.1., $F \cap U_p^0$ is a barrel in F.

Thus, by the assumption of F, there exists a neighbourhood V of zero in F such that $V \subset U_p^0$.

This shows that the inequality holds for V, and we complete the proof.

(2): The assertion can be proved in a similar way as in the proof of (1), so we omit it.

REMARK 3.1. Let $\mathfrak{F}$ be a σ-algebra of subsets of E which is invariant under E (i.e. for any $x\in E$ and any $Z\in\mathfrak{F}$, $Z-x\in\mathfrak{F}$ holds), and contains all cylinder sets in E, and let μ be a non-trivial (i.e. $\mu(E)>0$) measure on $(E, \mathfrak{F})$. Then Theorem 3.1. is also true.

Thus, our theorem generalizes a result of Xia [25], and then we proved it for partially admissible shifts instead of admissible shifts.

PROPOSITION 3.3. *Let* $1\leqq p\leqq 2$. *Let* $F\subset E$ *be Banach spaces, and let the inclusion map* $T: F\to E$ *be continuous. If there exists a cylinder measure* μ *of type* p *on* E *such that* $F\cap\widetilde{M}_\mu$ *is second category in* F, *then the adjoint map* $T^*: E^*\to F^*$ *can be decomposed as follows;*

$$E^*\xrightarrow[J]{} G\xrightarrow[K]{} F^*$$

$T^*=K\circ J$ *where* G *is a Banach space which is isomorphic to a subspace of* $L_p(\nu)$ *for some measure* ν, *and* J *and* K *are continuous linear maps, respectively.*

PROOF. Let μ be a cylinder measure of type p on E such that $F\cap\widetilde{M}_\mu$ is second category in F. Then, by Theorem 3.1., there exists positive constants C_1 and C_2 such that the inequalities

$$||T^*x^*||_{F^*}\leqq C_1||x^*||_p\leqq C_2||x^*||_{E^*} \qquad \text{for } x^*\in E^*$$

holds.

Thus, it is easily seen that the adjoint map $T^*: E^*\to F^*$ can be decomposed as follows;

$$E^*\xrightarrow[J]{} \hat{E}_p^*\xrightarrow[K]{} F^*$$

$T^*=K\circ J$ where the natural maps J and K are continuous.

Since a Banach space $\hat{E}_p^*$ is isometric to a subspace of $L_p(\nu)$, for some measure ν (c.f. Remark 2.2.2.), hence we complete the proof.

REMARK 3.2. In the above proposition, if $2<p<\infty$, then μ is of type 2. Hence the adjoint map $T^*: E^*\to F^*$ is Hilbertian.

COROLLARY 3.3. *Let* E *be a reflexive Banach space, and* F *be a closed subspace of* E. *If there exists a cylinder measure* μ *of type* p $(1\leqq p\leqq 2)$ *on* E *such that* $F\cap\widetilde{M}_\mu$ *is second category in* F, *then* F *is isomorphic to a quotient space of the dual of* $L_p(\nu)$ *for some measure* ν.

COROLLARY 3.4. *Let* E *be a Banach space, and* F *be a closed subspace of* E. *If there exists a cylinder measure* μ *of type* 2 *on* E *such that* $F\cap\widetilde{M}_\mu$ *is second category in* F, *then* F *is isomorphic to a Hilbert space.*

EXAMPLE 3. 1. Let $1 \leqq p < 2$, or $2 < p < \infty$. Let F be an infinite dimensional closed subspace of l_p, and let μ be a cylinder measure of type 2 on l_p. Then, $F \cap \widetilde{M}_\mu$ is first category in F.

PROOF. Assume the contrary. Then by Corollary 3. 4., F is isomorphic to a Hilbert space. However, l_p does not contain any infinite dimensional closed subspace which is isomorphic to a Hilbert space (c. f. [10]). That is a contradiction.

EXAMPLE 3. 2. Let $2 < p \leqq \infty$. Let μ be a cylinder measure of type 2 on l_∞. Then $l_p \cap \widetilde{M}_\mu$ is first category in l_p.

PROOF. Assume the contrary. Then, by Renark 3. 2., the inclusion map: $l_p \rightarrow l_\infty$ is Hilbertian, and so by the theorem of Pitt [14], it is compact. That is a contradiction.

The following theorem is essentially the same as a result of Linde [8].

THEOREM 3. 2. *A Banach space E is isomorphic to a Hilbert space iff there exists a cylinder measure μ of type 2 on E such that $\widetilde{M}_\mu$ is second category in E.*

PROOF. By Corollary 3. 4. and the result of Linde [8], it is obvious.

PROPOSITION 3. 4. *Let $F \subset E$ be Banach spaces, and the inclusion map from F into E be continuous. Suppose that E is isomorphic to a subspace of $\mathscr{L}_1$-space. Also suppose that there exists a cylinder measure μ of type 1 on E such $F \subset \widetilde{M}_\mu$ is second category in F. Then, the inclusion map T: $F \rightarrow E$ is Hilbertian.*

PROOF. By Proposition 3. 3., the adjoint map $T^*: E^* \rightarrow F^*$ can be decomposed as follows;

$$E^* \xrightarrow[J]{} G \xrightarrow[K]{} F^*$$

$T^* = K \circ J$ where G is a Banach space which is isomorphic to a subspace of $L_1(\nu)$ for some measure ν, and J and K are continuous linear maps, respectively.

Since E^* is isomorphic to a quotient space of $\mathscr{L}_\infty$-space, hence by Corollary 2. 1. 2., the map J is Hilbertian. Thus the map T: $F \rightarrow E$ is Hilbertian, and that completes the proof.

As an easy consequence of Proposition 3. 4., we have

COROLLARY 3. 5. *Let E be a Banach space which is isomorphic to a subspace of $\mathscr{L}_1$-space, and let F be a closed subspace of E. Suppose that there exists a cylinder measure μ of type 1 on E such that $F \cap \widetilde{M}_\mu$ is second category in F. Then F is isomorphic to a Hilbert space.*

EXAMPLE 3. 3. Let $1 \leqq p < 2$.

(1) Let μ be a cylinder measure of type 1 on l_p. Then, $l_1 \cap \widetilde{M}_\mu$ is first category in l_1.

(2) Let F be an infinite dimensional closed subspace of l_p, and let μ be a cylinder measure of type 1 on l_p. Then, $F \cap \widetilde{M}_\mu$ is first category in F.

PROOF. (1): Assume the contrary. Since l_p is isomorphic to a subspace of $\mathscr{L}_1$-space (c. f. [10]), hence it follows from Proposition 3. 4. that the inclusion map: $l_1 \to l_p$ is Hilbertian, and so it is compact (c. f. [14]). That is a contradiction.

(2) Assume the contrary. Then, it follows from Corollary 3. 5. that F is isomorphic to a Hilbert space. However l_p does not contain any in finitite dimensional closed subspace which is isomorphic to a Hilbert space (c. f. [10]). That is a contradiction.

§ 4. The measure case

In this section we shall discuss the partially admissible shifts of measures.

THEOREM 4. 1. *Let $F \subset E$ be linear topological spaces, and let the inclusion map: $F \to E$ be continuous. Suppose that E is a separable linear metric space. Also suppose that there exists a finite Borel measure μ (non-trivial) on E such that $F \cap \widetilde{M}_\mu$ is second category in F. Then, there exists a neighbourhood V of zero in F such that V is precompact in E.*

PROOF. Since E is a separable linear metric space, hence it follows (c. f. [25]) that there exists a sequence of precompact sets B_n in E such that

$$\lim_{n \to \infty} \mu(B_n) = \mu(E) .$$

Hence, by Proposition 3. 2., we have

$$\widetilde{M}_\mu \subset \bigcup_{n=1}^{\infty} (B_n - B_n) .$$

Let K_n be a closure of the set $(B_n - B_n)$ in E. Then, K_n is closed and precompact in E.

Since the inclusion map: $F \to E$ is continuous, and $F \cap \widetilde{M}_\mu$ is second category in F, it follows that $F \cap K_n$ is closed in F and the set S defined by

$$S = \bigcup_{n=1}^{\infty} (F \cap K_n)$$

is second category in F.

From this it follows that there exists n such that $F \cap K_n$ contains some open set in F. Hence it follows that there exists a neighbourhood V of zero in F such that V is precompact in E. That completes the proof.

COROLLARY 4.1. *Let E be a separable linear metric space. If there exists a finite Borel measure μ (non-trivial) on E such that $\widetilde{M}_\mu$ is second category in E, then E is finite dimensional.*

PROOF. Since locally precompact linear topological space is finite dimensional (c.f. [22]), hence by Theorem 4.1. we complete the proof.

REMARK 4.1. Let E be a complete separable linear metric space, and μ be a finite Borel measure (non-trivial) on E. If E is infinite dimensional, then by the above corollary $\widetilde{M}_\mu$ can not coincide with E. Namely, there exists an element x in E such that μ and μ_x are mutually singular.

From now on, we assume that a linear space E be infinite dimensional and a Borel measure μ on E be non-trivial.

REMARK 4.2. If a Banach space E is separable, then it is obvious that Theorem 4.1. and Corollary 4.1. are true. However, if E is not separable, then in general Theorem 4.1. is not true (for example $E=l_\infty$).

On the other hand, Skorohod [16] has shown that if E is a Hilbert space, then Theorem 4.1. and Corollary 4.1. are true.

In the ensuing discussions, we shall show that for any reflexive Banach space E Theorem 4.1. is true, and for any Banach space E Corollary 4.1. is true.

THEOREM 4.2. *Let E be a Banach space, F be a barrelled space and T be a continuous linear map from F into E. Let μ be a finite Borel measure on E, and suppose that $T(F)\subset\widetilde{M}_\mu$. Then, the map T can be decomposed as follows;*

$$F \xrightarrow[J]{} G \xrightarrow[K]{} E$$

$T=K\circ J$ where G is a Banach space, J is a continuous linear map and K is a ∞-strongly summing map, respectively.

Moreover if a Banach space E is reflexive, then the map T is compact.

Before proving the above theorem, we give the following notation.

The normed space E_B: Let E be a locally convex Hausdorff space, and B a bounded convex balanced subset of E. Let E_B be the vector subspace of E spanned by B; note that B is absorbing subset of E_B. For $x\in E_B$, define

$$||x||_B=\inf_{x\in\lambda B}|\lambda|\ .$$

Then, $||x||_B$ is a norm on E_B, and we obtain a normed space E_B equipped with the norm $||\cdot||_B$.

It is well known that the inclusion map: $E_B \to E$ is continuous, moreover, if the set B is complete, then E_B is a Banach space (c. f. [22]).

Now, we return to the proof of Theorem 4. 2..

PROOF. We may assume that μ satisfies the condition

$$\int_E \|x\| \, d\mu(x) < \infty$$

for otherwise, we can replace μ by the equivalent measure

$$\mu_1(A) = \int_A \exp\left(-\|x\|\right) d\mu(x) \qquad \text{for Borel set } A \text{ in } E$$

which certainly satisfies this condition, and $\widetilde{M}_\mu = \widetilde{M}_{\mu_1}$.

Recall that the seminorm $\|\cdot\|_1$ on E^* be defined as follows;

$$\|x^*\|_1 = \int_E \left|\langle x^*, x\rangle\right| d\mu(x) \qquad \text{for } x^* \in E^* .$$

Let U_1 and U_1^0 be defined as before (c. f. § 3. Lemma 3. 1.). Then, it is easily seen that U_1^0 is a bounded convex balanced complete subset of E. Hence, $E_{U_1^0}$ is a Banach space and the inclusion map: $E_{U_1^0} \to E$ is continuous.

It follows from Theorem 3. 1. that $\widetilde{M}_\mu \subset E_{U_1^0}$, and so by the assumption we get $T(F) \subset E_{U_1^0}$.

Now we prove that the map $J: F \to E_{U_1^0}$ is continuous, where J is defined by $Jx = Tx$ for $x \in F$.

Denote the inclusion map: $E_{U_1^0} \to E$ by K. Then, $T = K \circ J$. Since the maps T and K are continuous, and K is one-to-one, hence it follows from Proposition 17. 2. in [22] that the graph of J is closed in $F \times E_{U_1^0}$. Thus, by the closed graph theorem (c. f. [24]), J is continuous from F into $E_{U_1^0}$.

Next, we prove that the map K: $E_{U_1^0} \to E$ is ∞-strongly summing.

In order to prove this, by Theorem 2. 1. 1., we may show that the adjoint map $K^*: E^* \to (E_{U_1^0})^*$ is absolutely summing.

By the definition of U_1^0, for $x^* \in E^*$ we have

$$\begin{aligned} \|K^* x^*\|_{(E_{U_1^0})^*} &= \sup_{x \in U_1^0} \left|\langle K^* x^*, x\rangle\right| \\ &= \sup_{x \in U_1^0} \left|\langle x^*, x\rangle\right| \leqq \|x^*\|_1 . \end{aligned}$$

Let $\{x_n^*\}$ be a weakly summable sequence in E^*, then it is easily seen that there exists a positive constant C such that

$$\sum_{n=1}^{\infty} \left|\langle x_n^*, x\rangle\right| \leqq C \|x\| \qquad \text{for all } x \in E .$$

Hence, we have

$$\begin{aligned}\sum_{n=1}^{\infty}||K^* x_n^*||_{(E_{U_1^0})^*} &\leqq \sum_{n=1}^{\infty}||x_n^*||_1 \\ &= \int_E \sum_{n=1}^{\infty}\left|\langle x_n^*, x\rangle\right| d\mu(x) \\ &\leqq C\int_E ||x||\, d\mu(x) < \infty\,.\end{aligned}$$

This shows that the map $K^*: E^* \to (E_{U_1^0})^*$ is absolutely summing. Thus we have the first assertion.

Finally, we prove the second assertion. Suppose that a Banach space E is reflexive. Then, by the first argument, we may show that the map $K: E_{U_1^0} \to E$ is compact, and it is equivalent to that the adjoint map K^*: $E^* \to (E_{U_1^0})^*$ is compact.

Let $\{x_n^*\}$ be a bounded sequence in E^*. Since E is reflexive, hence by Eberlein's theorem there exists a subsequence $\{x_{n_j}^*\}$ of $\{x_n^*\}$ and $x^* \in E^*$ such that $w^* - \lim_{j\to\infty} x_{n_j}^* = x^*$.

Thus, by the first argument and Lebesgue's dominated convergence theorem, we have

$$\overline{\lim_{j\to\infty}}\, ||K^* x_{n_j}^* - K^* x^*||_{(E_{U_1^0})^*} \leqq \overline{\lim_{j\to\infty}} \int_E \left|\langle x_{n_j}^* - x^*, x\rangle\right| d\mu(x) = 0\,.$$

This shows that the map $K^*: E^* \to (E_{U_1^0})^*$ is compact, and we complete the proof.

REMARK 4.3. In the above theorem, it is easily seen that we can replace the condition "F is barrelled and $T(F) \subset \widetilde{M}_\mu$" by the condition "$T^{-1}(\widetilde{M}_\mu)$ is second category in F".

Since a ∞-strongly summing operator from a Banach space into a Hilbert space is decomposed through a Hilbert-Schmidt operator, we have the following corollary. That generalizes the results of Xia [25] and the author [19].

COROLLARY 4.2. *Let H be a Hilbert space, and μ be a finite Borel measure on H. Then the following* (1) *and* (2) *holds.*

(1) *There exists a Hilbert space G such that*

$$\widetilde{M}_\mu \subset G \subset H$$

where the inclusion map: $G \to H$ is of Hilbert-Schmidt type.

(2) *Let F be a linear subspace of H such that $F \subset \widetilde{M}_\mu$. Suppose that F itself is barrelled, and the inclusion map $T: F \to H$ is continuous. Then, there exists a Hilbert space G such that*

$$F \underset{J}{\subset} G \underset{K}{\subset} H$$

$T=K\circ J$ where the inclusion maps J is continuous and K is of Hilbert-Schmidt type, respectively.

Corollary 4.3. *Let E be a Banach space which is isomorphic to a subspace of $\mathscr{L}_1$-space. Let Φ be a linear subspace of E, and suppose that Φ itself is a complete σ-Hilbert space with respect to the sequence of inner products $(x,y)_n$, $n=1,2,\cdots$.*

Also, suppose that the inclusion map T: $\Phi\to E$ is continuous. For each n, let Φ_n denote the completion of Φ with respect to the inner product $(x,y)_n$. Then the following implication (1)$\Rightarrow$(2) *holds.*

(1) *There exists a finite Borel measure μ on E such that $\Phi\cap\widetilde{M}_\mu$ is second category in Φ.*

(2) *There exists n such that the inclusion map $T:\Phi\to E$ can be extended to a Hilbert-Schmidt map from Φ_n into E.*

Proof. Assume that condition (1). Then, by the remark of Theorem 4.2., there exists n such that the map T can be extended to a ∞-strongly summing map from Φ_n into E. Since a Banach space E is isomorphic to a subspace of $\mathscr{L}_1$-space, it follows from Theorem 2.1.1., Theorem 2.1.2. and Proposition 2.1.1. that the map T: $\Phi_n\to E$ can be decomposed through a Hilbert-Schmidt map. That completes the proof.

Finally, we obtain the following corollaries. These generalize the results of Skorohod [16].

Corollary 4.4. *Let E be a Banach space, and let μ be a finite Borel measure on E. Then, the following* (1) *and* (2) *holds.*

(1) *$\widetilde{M}_\mu$ is first category in E.*

(2) *If the Banach space E is reflexive, then for any infinite dimensional closed subspace F of E, $F\cap\widetilde{M}_\mu$ is first category in F.*

Proof. (2): Assume the contrary. Then, by the remark of Theorem 4.2., the inclusion map: $F\to E$ is compact. This shows that the Banach space F is locally compact, and it follows that F is finite dimensional. That is a contradiction.

(1): Assume the contrary. Since we may assume that the measure μ is of type 2 (c.f. the proof of Theorem 4.2.), it follows from Theorem 3.2. that E is isomorphic to a Hilbert space. Hence, E is reflexive, and so by (2) that is a contradiction.

Corollary 4.5. *Let E be a Fréchet space, and μ be a finite Borel measure on E. Then, there exists an element x in E such that μ and μ_x are mutually singular.*

PROOF. Assume the contrary. Since by the theorem of Sato [15] the measure μ has a Banach support G, here the inclusion map: $G \rightarrow E$ is continuous, and it follows from Proposition 3. 2. and the closed graph theorem (c. f. [24]) that E is isomorphic to a Banach space G. Thus, from Corollary 4. 4. that is a contradiction.

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