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On the Dirichlet problem for unbounded boundary functions

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1. We study the boundary behavior of the Dirichlet solution for an unbounded boundary function in this paper. We treat the Dirichlet problem by the Perron-Wiener-Brelot method. Let G be a bounded open set in the complex plane and let $f(b)$ be an extended real-valued function defined on its boundary ∂G. The upper class U_f^G for f is given by

$$U_f^G=\Big\{s\,;\ \textit{superharmonic, bounded below on } G\,,$$

$$\varliminf_{z\to b} s(z)\geqq f(b) \textit{ for all } b\in\partial G\Big\}$$

and set $\bar{H}_f^G(z)=\inf\{s(z)\,;\ s\in U_f^G\}$ and $\underline{H}_f^G(z)=-\bar{H}_{-f}^G(z)$ for $z\in G$. If $\bar{H}_f^G=\underline{H}_f^G$ and both harmonic, then f is called to be resolutive and $H_f^G=\bar{H}_f^G=\underline{H}_f^G$ is called the Dirichlet solution for f. Any bounded continuous function on ∂G is resolutive. A point $b\in\partial G$ is called a regular boundary point if

$$(1)\qquad \lim_{z\to b} H_g^G(z)=g(b)$$

for every bounded continuous function g on ∂G. Otherwise b is an irregular boundary point. The regularity is a local character, that is, if b is a regular boundary point for G and G_0 is an open subset of G for which b is also a boundary point, then b is regular for G_0.

Let $b\in\partial G$ be a regular boundary point. By M. Brelot's example [1], we know that (1) does not always hold for any unbounded function, even if it is continuous and resolutive. In this paper, we give a sufficient condition for (1) to hold. Our result is the following:

THEOREM. *Let f be an extended real-valued continuous and resolutive function on ∂G. Suppose $b_0\in\partial G$ is a regular boundary point. If there is a disk $B(b_0, r_0)=\{z\,;\ |z-b_0|<r_0\}$ such that the Dirichlet integral $D_{G\cap B(b_0,r_0)}(H_f^G)$ of H_f^G on $G\cap B(b_0, r_0)$ is finite, then $\lim_{z\to b_0} H_f^G(z)=f(b_0)$.*

For another sufficient condition, we refer to W. Ogawa [3].

2. **The proof of the theorem.** We refer to the monograph of Constantinescu-Cornea [2] for the definition and the properties of Dirichlet fun-

ctions. We denote by E the set of all irregular boundary points of ∂G.

First assume that $f \geqq 0$ on ∂G. For every positive integer m, define $f_m(b) = \min(f(b), m)$ and $H_m = H^G_{f_m}$. Since f_m is bounded continuous on ∂G,

$$(2) \qquad \lim_{z \to b} H_m(z) = f_m(b) \qquad \textit{for every } b \in \partial G - E.$$

Since $f_m \uparrow f$, $H_m \uparrow H^G_f$ on G. Since $\varliminf_{z \to b_0} H^G_f(z) \geqq \lim_{z \to b_0} H_m(z) = f_m(b_0)$ for every m, we have

$$\varliminf_{z \to b_0} H^G_f(z) \geqq f(b_0).$$

From this it follows that if $f(b_0) = +\infty$, $\lim_{z \to b_0} H^G_f(z) = +\infty$. Therefore we can assume $f(b_0) < +\infty$. Take an integer m_0 such that $f(b_0) < m_0$. Then there is a disk $B_1 = B(b_0, r_1)$ such that $0 < r_1 < r_0$ and $f(b) < m_0$ for every $b \in \partial G \cap B_1$. For every n and m such that $n > m \geqq m_0$, define

$$S_{n,m} = H_n - H_m.$$

Since $f_n(b) = f_m(b) = f(b)$ for every $b \in \partial G \cap B_1$, it follows from (2) that

$$(3) \qquad \lim_{z \to b} S_{n,m}(z) = 0 \qquad \textit{for every } b \in (\partial G - E) \cap B_1.$$

We denote by I the component of ∂G which contains b_0.

Case 1. The case where I is a single point b_0. In this case, there is a Jordan region $D \subset B_1$ such that $b_0 \in D$ and $\partial D \subset G$. Define a function f_0 on $\partial(G \cap D)$ as follows:

$$f_0(b) = \begin{cases} H^G_f(b) & \textit{if } b \in \partial D, \\ 0 & \textit{if } b \in \partial G \cap D. \end{cases}$$

Then f_0 is bounded continuous and so resolutive. Consider $H^{G \cap D}_{f_0} - S_{n,m}$, which is bounded harmonic on $G \cap D$. Since every point of ∂D is regular for $G \cap D$,

$$\lim_{z \to b} \left(H^{G \cap D}_{f_0}(z) - S_{n,m}(z) \right) = H^G_f(b) - \left(H_n(b) - H_m(b) \right) \geqq 0$$

for every $b \in \partial D$, and by (3), $\lim_{z \to b} (H^{G \cap D}_{f_0}(z) - S_{n,m}(z)) = 0$ for every $b \in (\partial G - E) \cap D$. Since E is a polar set and $\partial(G \cap D) - E = \partial D \cup ((\partial G - E) \cap D)$, this shows that

$$S_{n,m}(z) \leqq H^{G \cap D}_{f_0}(z) \qquad on\ G \cap D.$$

Letting $n \to \infty$, we obtain $H^{G \cap D}_{f_0}(z) \geqq H^G_f(z) - H_m(z) \geqq 0$ on $G \cap D$. Since b_0 is regular for $G \cap D$ and G, $\lim_{z \to b_0} H^{G \cap D}_{f_0}(z) = 0$ and $\lim_{z \to b_0} H_m(z) = f_m(b_0) = f(b_0)$. Thus we have $\lim_{G \cap D \ni z \to b_0} H^G_f(z) = \lim_{G \ni z \to b_0} H^G_f(z) = f(b_0)$.

Case 2. The case where I is a continuum. Set $\alpha=D_{G\cap B_1}(H_f^G)<+\infty$. Let f' be a boundary function on $\partial(G\cap B_1)$ which is equal to H_f^G on $\partial(G\cap B_1)\cap G$ and to f on $\partial(G\cap B_1)\cap\partial G$.Then f' is resolutive and $H_{f'}^{G\cap B_1}=H_f^G$ on $G\cap B_1$. Set $H'_m=H^{G\cap B_1}_{\min(f',m)}$ for every positive integer m. Then $H'_m\uparrow H_f^G$ on $G\cap B_1$ as $m\to\infty$ and

$$(4)\qquad \lim_{z\to b} H'_m(z)=\min\bigl(f'(b),m\bigr)=f(b)$$

for every $b\in(\partial G-E)\cap B_1$ and every $m\geqq m_0$. By the harmonic decomposition of Royden (cf. e.g. Satz 7.6 in [2]) we see that

$$D_{G\cap B_1}(H'_m)\leqq D_{G\cap B_1}\bigl(\min(H_{f'}^{G\cap B_1},m)\bigr)\leqq D_{G\cap B_1}(H_{f'}^{G\cap B_1})=\alpha\,.$$

For every n and m such that $n>m\geqq m_0$, define $S'_{n,m}=H'_n-H'_m$. Then

$$(5)\qquad \lim_{z\to b} S'_{n,m}(z)=0 \qquad \textit{for every } b\in(\partial G-E)\cap B_1$$

by (4) and $D_{G\cap B_1}(S'_{n,m})\leqq 2D_{G\cap B_1}(H'_n)+2D_{G\cap B_1}(H'_m)\leqq 4\alpha$.

Let $G_{n,m}=\{z\in G\cap B_1\,;\ S_{n,m}(z)>1\}$. Then $\partial G_{n,m}\cap B_1=(\partial G_{n,m}\cap B_1\cap G)\cup(\partial G_{n,m}\cap B_1\cap\partial G)$, where $\partial G_{n,m}\cap B_1\cap G$ is analytic and $E_{n,m}=\partial G_{n,m}\cap B_1\cap\partial G$ is closed relative to B_1 and has capacity zero by $E_{n,m}\subset E$. Define a function $T_{n,m}$ on B_1 as follows:

$$T_{n,m}(z)=\begin{cases} S'_{n,m}(z)-1 & \textit{if } z\in G_{n,m}\,,\\ 0 & \textit{if } z\in B_1-G_{n,m}\,.\end{cases}$$

Then we see that $T_{n,m}$ is continuous on $B_1-E_{n,m}$, a Tonelli function on B_1 and

$$D_{B_1}(T_{n,m})\leqq D_{G\cap B_1}(S'_{n,m})\leqq 4\alpha\,,$$

that is; $T_{n,m}$ is a Dirichlet function on B_1. Take r_2, r_3 and r_4 such that $r_1>r_2>r_3>r_4>0$ and set $B_i=B(b_0,r_i)$, $i=2, 3, 4$. Since $E_{n,m}$ is a polar set, we may take r_2 such that $\partial B_2\cap(\bigcup_{n,m} E_{n,m})=\phi$. Let I_0 be the component of $I\cap\bar{B}_4$ which contains b_0. Then I_0 is also a continuum and $T_{n,m}=0$ on I_0. Let $w_i(z)$ $(i=2,3)$ be the harmonic measure of ∂B_i with respect to B_i-I_0. Then w_i can be continuously extend to $\bar{B}_i$ such that

$$(6)\qquad w_i=0 \textit{ on } I_0 \quad \textit{and} \quad w_i=1 \textit{ on } \partial B_i\,.$$

In B_2-I_0, there is only one component J_0 such that $\partial J_0\supset\partial B_2$ and the other components $J_i(i\geqq 1)$ satisfy $\partial J_i\subset I_0$. Let $g_{n,m}$ be the restriction of $T_{n,m}$ to $\partial(B_2-I_0)=\partial B_2\cup I_0$. Then $g_{n,m}$ is bounded continuous on $\partial B_2\cup I_0$. Let $U_{n,m}$ be the Dirichlet solution corresponding to B_2-I_0 and the boundary function $g_{n,m}$. Then we have

$$(7)\qquad D_{B_2-I_0}(U_{n,m}) \leqq D_{B_2}(T_{n,m}) \leqq 4\alpha$$

by the Dirichlet principle (p. 155 in [2]). Let $\beta_{n,m}=\min_{z\in\partial B_3} U_{n,m}(z)\geqq 0$. Then $D_{B_3-I_0}(U_{n,m})\geqq D_{B_3-I_0}(\min(U_{n,m}, \beta_{n,m}))$. Since $\min(U_{n,m}, \beta_{n,m})=\beta_{n,m}w_3$ on $\partial B_3 \cup I_0$, $D_{B_3-I_0}(\min(U_{n,m}, \beta_{n,m}))\geqq \beta_{n,m}^2 D_{B_3-I_0}(w_3)$ by the Dirichlet principle. Hence we have

$$(8)\qquad D_{B_3-I_0}(U_{n,m}) \geqq \beta_{n,m}^2 D_{B_3-I_0}(w_3)\,.$$

Consider $U_{n,m}+w_2-S'_{n,m}$ on $G\cap B_2$. By

$$\lim_{z\to b}\left(U_{n,m}(z)+w_2(z)\right)=g_{n,m}(b)+1=S'_{n,m}(b) \qquad \textit{if } b\in\partial B_2\cap G_{n,m}\,,$$
$$\lim_{z\to b} w_2(z)=1\geqq S'_{n,m}(b) \qquad \textit{if } b\in\partial B_2\cap(G-G_{n,m})$$

and (5), we have $\lim_{z\to b}(U_{n,m}(z)+w_2(z)-S'_{n,m}(z))\geqq 0$ for every $b\in\partial(G\cap B_2)-E$. This shows that

$$(9)\qquad S'_{n,m}(z)\leqq U_{n,m}(z)+w_2(z) \qquad \textit{on } G\cap B_2\,.$$

Consider $\{U_{n,m}\}_{n,m}$ on J_0 (Note $U_{n,m}=0$ on $J_i(i\geqq 1)$). Suppose $\sup_{n,m} U_{n,m}(z_0)=+\infty$ for some point $z_0\in J_0$. Then $\sup_{n,m}\min_{z\in\partial B_3} U_{n,m}(z)=+\infty$ by Harnack theorem. Since $D_{J_0}(U_{n,m})\geqq(\min_{z\in\partial B_3} U_{n,m}(z))^2 D_{J_0\cap B_3}(w_3)$ by (8) this implies $\sup_{n,m} D_{J_0}(U_{n,m})=+\infty$. But this contradicts (7). Hence we have $\sup_{n,m} U_{n,m}(z_0)<+\infty$. This shows that $\sup_{n,m}\max_{z\in\partial B_3} U_{n,m}(z)=\gamma<+\infty$ by Harnack theorem. Since $\lim_{z\to b} U_{n,m}(z)=0$ for every $b\in I_0$, we have

$$U_{n,m}(z)\leqq \gamma w_3(z) \qquad \textit{on } B_3-I_0\,.$$

Hence $0\leqq S'_{n,m}(z)\leqq\gamma w_3(z)+w_2(z)$ on $G\cap B_3$ by (9). Letting $n\to\infty$, $0\leqq H_{f'}^{G\cap B_1}(z)-H'_m(z)\leqq\gamma w_3(z)+w_1(z)$. Consequently we conclude from (4) and (6) $\lim_{z\to b_0} H_f^G(z)=\lim_{z\to b_0} H_{f'}^{G\cap B_1}(z)=f(b_0)$.

Finally we remove the assumption that $f\geqq 0$ on ∂G. Since f' is resolutive, $g(b)=\max(f'(b), 0)$ and $h(b)=\max(-f'(b), 0)$ are resolutive and $H_{f'}^{G\cap B_1}=H_g^{G\cap B_1}-H_h^{G\cap B_1}$. Since $D_{G\cap B_1}(H_{f'}^{G\cap B_1})<+\infty$, we see that $D_{G\cap B_1}(H_g^{G\cap B_1})<+\infty$ and $D_{G\cap B_1}(H_h^{G\cap B_1})<+\infty$. From the above reasoning it follows that $\lim_{z\to b_0} H_g^{G\cap B_1}(z)=g(b_0)$ and $\lim_{z\to b_0} H_h^{G\cap B_1}(z)=h(b_0)$. Hence we have $\lim_{z\to b_0} H_f^G(z)=\lim_{z\to b_0} H_{f'}^{G\cap B_1}(z)=g(b_0)-h(b_0)=f(b_0)$. This completes the proof.

References

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