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Certain free cyclic group actions on homotopy spheres, bounding parallelizable manifolds

By Yoshinobu KAMISHIMA

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Introduction

The purpose of this paper is to study free cyclic group actions on homotopy spheres constructed by S. Weintraub [3]. He applied an equivariant plumbing technique to construct semifree cyclic group actions on highly-connected $4k$-dimensional manifolds. The boundaries of these manifolds are elements of bP_{4k} and they admit free Z_p-actions for any integer p. We call these "Weintraub's actions".

On the other hand, it is well known that López De Medrano [1] constructed free involutions on homotopy spheres with non-trivial Browder-Liversay invariants. It is apparent that López's construction cannot be applied to get any other free cyclic group actions except involutions. However, we shall prove that certain examples of López's involutions on homotopy spheres of bP_{4k} extend to free Z_{2q}-actions for any q which are realized by "Weintraub's actions" raised above. This is the main motivation of this research.

The results are summarized as follows. One of the properties about Weintraub's actions has been found in [3, Theorem 1.7] and in § 2, we state this in an alternative form for our argument.

THEOREM 1. *Suppose that p is any integer. Choose a unimodular, even, symmetric matrix A and denote $\sigma(A)$ its index. For any $k\geqq 2$ and collection $\{a_1, \cdots, a_k\}$ with $(a_i, p)=1$, there is a free Z_p-action T_A on a homotopy sphere $\Sigma_A\in bP_{4k}$ the Atiyah-Singer invariant of which has the form*

$$\sigma(T_A, \Sigma_A^{4k-1})=\prod_{i=1}^{k}\left(\frac{1+t^{a_i}}{1-t^{a_i}}\right)^2-\sigma(A)\,.$$

Here $\Sigma_A=\sigma(A)/8\Sigma_1$, where Σ_1 is the generator of bP_{4k} and $t=\exp(2\pi i/p)$.

Hereafter, by (T_A, Σ_A) we denote the free Z_p-action on the homotopy sphere constructed for any p and A under the assumptions of Theorem 1. As to the normal cobordism classes of such actions, we shall prove the following result.

THEOREM 2. *Let T_A be a free Z_p-action on Σ_A. Then, there exists a homotopy equivalence*

$$f: \Sigma_A/T_A \longrightarrow L^{4k-1}(p, a_1, \cdots, a_k, a_1, \cdots, a_k)$$

such that the normal invariant $\eta(f)$ is zero in

$$\left[L^{4k-1}(p, a_1, \cdots, a_k, a_1, \cdots, a_k),\ G/O\right],$$

where $L^{4k-1}(p, a_1, \cdots, a_k, a_1, \cdots, a_k)$ is the $(4k-1)$-dimensional standard lens space.

In § 3, we look at the effect on the action of choosing a different matrix with the same index.

THEOREM 4. *Suppose that p is any odd integer or 2, 4 and 6. Let T_{A_i} be a free Z_p-action on a homotopy sphere Σ_{A_i} for $i=1, 2$. Then, Σ_{A_1}/T_{A_1} is h-cobordant to Σ_{A_2}/T_{A_2} if and only if $\sigma(A_1)=\sigma(A_2)$.*

As to the López's involutions, a difficulty lies in determing the differentiable structures of constructed homotopy spheres, and in general they are unknown. But P. Orlik and C. P. Rourke [2] showed, using López's construction, that for each $i\in Z$ there exists a homotopy sphere $\Sigma_i^{4k-1}(=i\Sigma_1)$, bounding a parallelizable manifold M_i, and an involution T_i such that the Browder-Livesay invariant $I(T_i, \Sigma_i)=\sigma(M_i)=8i$. When we concentrate our attension on these López's involutions, in § 4 we have the following.

THEOREM 6. *Suppose that $p=2q$ $(q\geqq 1)$. There exists a free Z_p-action T_A on $\Sigma_A\in bP_{4k}$ which satisfies that if we restrict this action to the Z_2-action on Σ_A, then (T_i, Σ_i) is equivariantly diffeomorphic to (T_A^q, Σ_A).*

S. Weintraub kindly informed me that instead of lens space

$$L^{4k-1}(p, a_1, \cdots, a_k, a_1, \cdots, a_k)$$

one can use lens spaces $L^{4k-1}(p, a_1, \cdots, a_{2k})$ *with* $(a_i, p)=1$ for $i=1, \cdots, 2k$ (see Remark 4. 1).

1. Constructions of Z_p-actions

This section is devoted to the preliminaries of proofs of Theorem 1, 2. We shall present some notations which will be used frequently.

Let $D^{2k}(S^{2k-1})(p, a_1, \cdots, a_k)$ be the unit disk (sphere) in C^k with the Z_p-action $t(z_1, \cdots, z_k)=(t^{a_1}z_1, \cdots, t^{a_k}z_k)$, $t=\exp(2\pi i/p)$.

Let $S^{2k}(p, a_1, \cdots, a_k)$ be the suspension of $S^{2k-1}(p, a_1, \cdots, a_k)$, *i. e.*, the unit sphere in $C^k\times R$ with the Z_p-action $t(z_1, \cdots, z_k, x)=(t^{a_1}z_1, \cdots, t^{a_k}z_k, x)$. By $L^{4k-1}(p, a_1, \cdots, a_k, a_1, \cdots, a_k)$ we denote the $(4k-1)$-dimensional lens space with the type $(a_1, \cdots, a_k, a_1, \cdots, a_k)$. Sometimes, we write simply,

$$\begin{aligned}
L(p) = L^{4k-1}(p) &= L^{4k-1}(p, a_1, a_1) = L^{4k-1}(p, a_1, \cdots, a_k, a_1, \cdots, a_k)\,,\\
&L^{4k-1}(p, -a_1, a_1) = L^{4k-1}(p, -a_1, a_2, \cdots, a_k, a_1, a_2, \cdots, a_k)\,,\\
&D^{2k}(p, a_1) = D^{2k}(p, a_1, \cdots, a_k)\,,\\
&S^{2k}(p, a_1) = S^{2k}(p, a_1, \cdots, a_k)\,, \qquad \text{and so on.}
\end{aligned}$$

The following two lemmas are crucial for our argument. Lemma 1.1 is the special case of [3, Lemma 1.6], but it is sufficient to our need.

LEMMA 1.1. *For any integer $k \geqq 2$ and collection $\{a_1, \cdots, a_k\}$ with $(a_i, p) = 1$, there are D^{2k}-bundles E_+, E_0 and E_- over S^{2k} with semi-free Z_p-actions T satisfying*

(1) *T is a bundle map preserving the 0-section.*

(2) *The action T on the 0-section is $S^{2k}(p, a_1, \cdots, a_k)$ and T has no fixed points outside the 0-section.*

(3) *The normal representations of each fixed point are*

$$\text{(i)}\quad \left\{\begin{matrix} D^{2k}(p, a_1, \cdots, a_k) \times D^{2k}(p, a_1, \cdots, a_k) \\ D^{2k}(p, a_1, \cdots, a_k) \times D^{2k}(p, a_1, \cdots, a_k) \end{matrix}\right\} \quad \textit{for } E_+\,,$$

$$\text{(ii)}\quad \left\{\begin{matrix} D^{2k}(p, a_1, \cdots, a_k) \times D^{2k}(p, a_1, \cdots, a_k) \\ D^{2k}(p, -a_1, a_2, \cdots, a_k) \times D^{2k}(p, a_1, \cdots, a_k) \end{matrix}\right\} \quad \textit{for } E_0\,,$$

$$\text{(iii)}\quad \left\{\begin{matrix} D^{2k}(p, a_1, \cdots, a_k) \times D^{2k}(p, -a_1, a_2, \cdots, a_k) \\ D^{2k}(p, -a_1, a_2, \cdots, a_k) \times D^{2k}(p, a_1, \cdots, a_k) \end{matrix}\right\} \quad \textit{for } E_-\,.$$

Here $D^{2k}(p, -) \times D^{2k}(p, -)$ is a local trivialization around a fixed point.

(4) *The Euler classes of bundles E_+, E_0 and E_- are taken to be 2, 0 and -2 mod any multiple of $2p$ times respectively. Furthermore, these bundles are stably trivial.*

We write simply E for one of the above bundles. E has two isolated fixed points. Let denote N_1, N_2 the equivariant tubular neighborhoods of the fixed points in E. It follows from (3) that N_i/Z_p is diffeomorphic to $L^{4k-1}(p, a_1, a_1)$ or to $L^{4k-1}(p, -a_1, a_1)$. Put $W = E - \mathrm{int}\left\{\bigcup_{i=1}^{2} N_i\right\}/Z_p$.

LEMMA 1.2. *W defines a "normal cobordism" between $\partial E/Z_p$ and $\left\{\bigcup_{i=1}^{2} \partial N_i/Z_p\right\}$, i.e., there is a normal map $H\colon W \longrightarrow L^{4k-1}(p)$ covered by a bundle map $b\colon \nu_W \longrightarrow \nu_{L(p)}$, where ν_W, $\nu_{L(p)}$ are stable normal bundles of W, $L^{4k-1}(p)$ respectively (note that H is not a degree 1 map). Moreover, the map H_- of the boundary components $\partial_- W = \left\{\bigcup_{i=1}^{2} \partial N_i/Z_p\right\}$ onto $L^{4k-1}(p)$*

is either the identity map or the orientation reversing diffeomorphism. Here, the latter diffeomorphism is settled in the proof of the lemma.

PROOF OF THE LEMMA 1.1. Let

$$d: S^{2k}(p, a_1, \cdots, a_k) \longrightarrow S^{2k}(p, a_1, \cdots, a_k) \times S^{2k}(p, a_1, \cdots, a_k)$$

be the diagonal embedding which is invariant under the action. Let

$$H_{2k}(S^{2k} \times S^{2k}) = \langle\alpha\rangle + \langle\beta\rangle$$

with the first factor representing α and the second representing β. For any $l \in Z$, we take $|l|$-embedded spheres S^{2k}'s in the free part of

$$S^{2k}(p, a_1, \cdots, a_k) \times S^{2k}(p, a_1, \cdots, a_k)$$

each of which represents β. Taking their equivariant connected sum with $d(S^{2k}(p, a_1, \cdots, a_k))$, *i.e.*,

$$d\left(S^{2k}(p, a_1, \cdots, a_k)\right) \underset{Z_p}{\#} |l| S^{2k} \subset S^{2k}(p, a_1, \cdots, a_k) \times S^{2k}(p, a_1, \cdots, a_k)\,,$$

we have a stably trivial normal bundle E_+ over S^{2k} which is invariant under the action. E_+ has the Euler class

$$\chi(E_+) = \left(\alpha + (pl+1)\,\beta\right) \cdot \left(\alpha + (pl+1)\,\beta\right) = 2 + 2pl\,.$$

Clearly, E_+ satisfies (1), (2) and (3).

Let g be the equivariant diffeomorphism of $S^{2k-1}(p, a_1, \cdots, a_k)$ onto $S^{2k-1}(p, -a_1, a_2, \cdots, a_k)$ defined by $g(z_1, \cdots, z_k) = (\bar{z}_1, z_2, \cdots, z_k)$. Here $\bar{z}$ is the conjugate of z in C. Denote the $2k$-dimensional sphere with a Z_p-action obtained by attaching $D^{2k}(p, a_1, \cdots, a_k)$ to $D^{2k}(p, -a_1, a_2, \cdots, a_k)$ by means of g by

$$S_1^{2k}(p, a_1, \cdots, a_k) = D^{2k}(p, a_1, \cdots, a_k) \underset{g}{\cup} D^{2k}(p, -a_1, a_2, \cdots, a_k)$$

which is again $S^{2k}(p, a_1, \cdots, a_k)$. We also define equivariant embeddings

$$d' : S_1^{2k}(p, a_1, \cdots, a_k) \longrightarrow S_1^{2k}(p, a_1, \cdots, a_k) \times S_1^{2k}(p, a_1, \cdots, a_k)$$

and

$$\iota : S_1^{2k}(p, a_1, \cdots, a_k) \longrightarrow S_1^{2k}(p, a_1, \cdots, a_k) \times S^{2k}(p, a_1, \cdots, a_k)$$

by setting

$$d'\left((z_1, \cdots, z_k, x)\right) = \left((z_1, \cdots, z_k, x),\ (\bar{z}_1, z_2, \cdots, z_k, x)\right),$$

$$\iota(z) = (z, x_1)\,,$$

where $(z_1, \cdots, z_k, x)$, $z \in S_1^{2k}(p, a_1, \cdots, a_k)$ and x_1 is a fixed point of $S^{2k}(p, a_1, \cdots, a_k)$. Making use of d', ι, we obtain the desired bundles E_-, E_0 accordingly.

PROOF OF LEMMA 1. 2. The fixed points of $S^{2k}(p, a_1, \cdots, a_k)$ are written as $x_1=(\bar{0}, 1)$, $x_2=(\bar{0}, -1)$, $\bar{0}=(0, \cdots, 0)\in C^k$. Denote the equivariant tubular neighborhood of x_i in E by N_i for $i=1, 2$, as before. Make N_i small to be contained in $D^{2k}\times D^{2k}$ of (3) of Lemma 1. 1. From the construction of E, there exists an equivariant embedding

$$i:\ E\subsetneq S^{2k}(p, a_1)\times S^{2k}(p, a_1)\subsetneq D^{2k+1}(p, a_1)\times D^{2k+1}(p, a_1)\,.$$

It is easily seen that the equivariant normal bundle ν_i of E in

$$D^{2k+1}(p, a_1)\times D^{2k+1}(p, a_1)$$

is trivial, *i. e.*, $\nu_i=E\times D^1\times D^1$, and the action on the part $D^1\times D^1$ of ν_i is trivial. Then, we have an embedding

$$\begin{aligned} E-\operatorname{int}\left\{\bigcup_{i=1}^{2} N_i\right\}&\subset D^{2k+1}(p, a_1)\times D^{2k+1}(p, a_1)-(\bar{0}\times D^1)\times(\bar{0}\times D^1)\\ &\cong(D^{2k}(p, a_1)\times D^{2k}(p, a_1)-\bar{0}\times\bar{0})\times D^1\times D^1\,. \end{aligned}$$

It induces an embedding of the quotient spaces

$$i:\ W\subsetneq L^{4k-1}(p)\times I\times D^1\times D^1$$

which has a trivial normal bundle. Hence this defines a "normal cobordism", *i. e.*, there is a normal map H: $W\longrightarrow L^{4k-1}(p)$ which is covered by a bundle map b: $\nu_W\longrightarrow\nu_L$. Comparing with (3) of Lemma 1. 1 and looking at the inclusion maps of the boundary components carefully, the map H of $D^{2k}\times D^{2k}-\operatorname{int} N_i/Z_p$ onto $L^{4k-1}(p)$ is as follows with respect to E_+, E_0 and E_-:

(i) $D^{2k}(p, a_1)\times D^{2k}(p, a_1)-\operatorname{int} N_i/Z_p=L^{4k-1}(p)\times I\longrightarrow L^{4k-1}(p)$

$H=Pr\cdot(1\times 1)$,

$H_-=id$ *on* $\partial N_i/Z_p=L^{4k-1}(p)$ *for* $i=1, 2$.

(ii) $D^{2k}(p, a_1)\times D^{2k}(p, a_1)-\operatorname{int} N_1/Z_p\longrightarrow L^{4k-1}(p)$

$H=Pr\cdot(1\times 1)$, $H_-=id$ *on* $\partial N_1/Z_p=L^{4k-1}(p)$,

$D^{2k}(p, -a_1)\times D^{2k}(p, a_1)-\operatorname{int} N_2/Z_p\longrightarrow L^{4k-1}(p)$

$H=Pr\cdot(c\times 1)$, $H_-=c\times 1$ *on* $\partial N_2/Z_p=L^{4k-1}(p, -a_1, a_1)$,

where c *is the map induced from the map* $\tilde{c}$ *of* $D^{2k}(p, -a_1)$ *onto* $D^{2k}(p, a_1)$ *defined by* $\tilde{c}(z_1, z_2, \cdots, z_k)=(\bar{z}_1, z_2, \cdots, z_k)$.

(iii) $D^{2k}(p, a_1)\times D^{2k}(p, -a_1)-\operatorname{int} N_1/Z_p\longrightarrow L^{4k-1}(p)$

$H=Pr\cdot(1\times c)$, $H_-=1\times c$ *on* $\partial N_1/Z_p=L^{4k-1}(p, a_1, -a_1)$,

$D^{2k}(p, -a_1)\times D^{2k}(p, a_1)-\operatorname{int} N_2/Z_p\longrightarrow L^{4k-1}(p)$

$H=Pr\cdot(c\times 1)$, $H_-=c\times 1$ *on* $\partial N_1/Z_p=L^{4k-1}(p, -a_1, a_1)$.

Next, we consider to plumb bundles equivariantly at a fixed point or at a free point of the actions. In particular, our aim is to consider plumbings on the quotient spaces.

LEMMA 1.3. *Suppose that E^i's are plumbed one after another at a fixed point on each (i.e., the graph is a tree) and denote M' its resulting manifold. Let $N(pts)$ be the tubular neighborhoods of the fixed points in M', so that they are a union of N_i's of Lemma 1.2 for $i=1, 2$. Then the cobordism $V'=M'-\text{int}\, N(pts)/Z_p$ defines a normal cobordism G': $V'\longrightarrow L(p)$ between $\partial M'/Z_p$ and $\{\bigcup_F \partial N_i/Z_p,\ i=1, 2\}$ covered by a bundle map*

$$b' : \nu_{V'} \longrightarrow \nu_{L(p)}\,.$$

Here F is the set of fixed points.

Under the situation of Lemma 1.3, we shall prove

LEMMA 1.4. *If we do further plumbings in the free part of the action in M', and if we denote its manifold by M, then the resulring cobordism $V=M-\text{int}\, N(pts)/Z_p$ defines a normal cobordism G: $V\longrightarrow L(p)$ between $\partial M/Z_p$ and $\{\bigcup_F \partial N_i/Z_p,\ i=1, 2\}$. The map G on the boundary components $\{\bigcup_F \partial N_i/Z_p,\ i=1, 2\}$ is unchanged, i.e., $=G'$.*

PROOF OF LEMMA 1.3. The normal representations (3) of Lemma 1.1 inform us how to plumb two bundles together equivariantly, *i.e.*, around a fixed point, the two spaces $D^{2k}\times D^{2k}$ are equivariantly diffeomorphic by the map $h: D^{2k}\times D^{2k}\longrightarrow D^{2k}\times D^{2k}$, $h(x, y)=(y, x)$. When we consider plumbings on the quotient spaces, plumbing E^1 with E^2 together equivariantly at a fixed point (for example, at $x_2\in N_2\subset E^1$ and $x_1\in N_1\subset E^2$) is equivalent to taking $E^1-\text{int}\,\{N_1\cup N_2\}/T\cup E^2-\text{int}\,\{N_1\cup N_2\}/T$ and identifying $D^{2k}\times D^{2k}-\text{int}\, N_2/Z_p$ with $D^{2k}\times D^{2k}-\text{int}\, N_1/Z_p$ by the induced map h' from h. If we put the manifold M' when E^1 and E^2 are plumbed as above, the resulting cobordism V' is $V'=M'-\text{int}\,\{N_1\cup N_2\cup N_2\}/Z_p$, where the first N_1, N_2 are in E^1 and the last N_2 in E^2, and N_1 in E^2 is identified with N_2 in E^1. In view of (i), (ii) and (iii) in the proof of Lemma 1.2, the following diagram is commutative:

$$\begin{array}{ccc} D^{2k}\times D^{2k}-\text{int}\, N_2/Z_p & \xrightarrow{H} & L(p) \\ \Big\downarrow{\scriptstyle h'} & & \Big\downarrow{\scriptstyle h'} \\ D^{2k}\times D^{2k}-\text{int}\, N_1/Z_p & \xrightarrow{H} & L(p)\,. \end{array} \tag{1}$$

The commutative diagram (1) is compatible with the bundle maps b of the stable normal bundles which cover H. Therefore, V' defines a normal cobordism between $\partial M'/Z_p$ and $\{\partial N_1/Z_p,\ \partial N_2/Z_p,\ \partial N_2/Z_p\}$.

Further, if E^2 is plumbed with E^3 equivariantly at the unused fixed point in E^2, the diagram (1) holds around the unused fixed point, so the resulting cobordism also defines a normal cobordism. Proceeding in this way, we finish the proof of the lemma.

PROOF OF LEMMA 1.4. We do further plumbings in the free part of the action on M'. This can be done by taking two disjoint trivializations $D_i^{2k}\times D_i^{2k}\subset V'$ and then identifying $D_1^{2k}\times D_1^{2k}$ with $D_2^{2k}\times D_2^{2k}$ by the map $h(x, y)=(y, x)$, $h: D_1\times D_1\longrightarrow D_2\times D_2$. Lifting gives p-plumbings in the cover M'. Denote its manifold by M. If $(G', b'): V'\longrightarrow L(p)$ is a normal map of Lemma 1.3, we can arrange, using the homotopy extension theorem, that $G'|D_1\times D_1=(G'|D_2\times D_2)h$ without changing on the boundary components $\{\bigcup_F \partial N_i/Z_p,\ i=1, 2\}$. Let V be the resulting cobordism when we identify $D_1\times D_1$ with $D_2\times D_2$ by h. Then, $V=M-\mathrm{int}\, N(pts)/Z_p$. The above compactibility defines a map $G: V\longrightarrow L(p)$. By choosing a bundle equivalence of $\nu_{V'}|D_1\times D_1$ with $\nu_{V'}|D_2\times D_2$ covering h, we may arrange, using the bundle covering homotopy theorem, that $b'|(\nu_{V'}|D_1\times D_1)$ and $b'|(\nu_{V'}|D_2\times D_2)$ are compatible to give a bundle map $b: \nu_V\longrightarrow\nu_L$. Hence $G: V\longrightarrow L(p)$ is a normal map. Repeating further plumbings in the free part of the action as above, the above argument also holds. Therefore, this proves the lemma.

2. Proofs of Theorem 1 and 2

We shall recall a usefull algebraic result.

DEFINITION. *Suppose that p is any integer. Let A and B be unimodular, even, symmetric matrices of the same rank. We say that A is 'congruent mod p' with B if there exists a matrix H,* $\det H=\pm 1$, *such that* $A\equiv {}^tH\cdot B\cdot H \bmod p$.

Then, by [3, Lemma 1.5], it follows that

($*$) *Any two unimodular, even, symmetric matrices of the same rank are congruent* mod p.

THEOREM 1. *Suppose that p is any integer. Choose a unimodular, even, symmetric matrix A of rank $2m$ $(m\geqq 2)$ in the congruence class* mod $2p$ *and denote $\sigma(A)$ its index. Then, for any $k\geqq 2$ and collection $\{a_1, \cdots, a_k\}$ with $(a_i, p)=1$, there is a free Z_p-action T_A on a homotopy sphere $\Sigma_A\in bP_{4k}$ the Atiyah-Singer invariant of which has the form*

$$\sigma(T_A^j, \Sigma_A^{4k-1})=\prod_{i=1}^{k}\left(\frac{1+(t^j)^{a_i}}{1-(t^j)^{a_i}}\right)^2-\sigma(A)\,.$$

Here, $\Sigma_A=\sigma(A)/8\Sigma_1$, the connected sum of $\sigma(A)/8$-copies of Σ_1's, where Σ_1

is the generator of bP_{4k}, *and* $t=\exp(2\pi i/p)$.

THEOREM 2. *Let* T_A *be a free* Z_p*-action on* Σ_A. *Then, there exists a homotopy equivalence* $f: \Sigma_A/T_A \longrightarrow L^{4k-1}(p)$ *such that the normal invariant* $\eta(f)$ *is zero in* $[L^{4k-1}(p), G/O]$, *i.e.,* Σ_A/T_A *has the same normal cobordism class as* $L^{4k-1}(p)$.

PROOF OF THEOREM 1. Let P_{2m} be the symmetric, unimodular, even, matrix of rank $2m$ ($m \geqq 2$) defined in [3],

$$P_{2m} = \begin{array}{c} m\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} 2 & 1 & & & & & & & \\ 1 & 2 & 1 & & & & & & \\ & 1 & \ddots & & & & & & \\ & & & 2 & 1 & & & & \\ & & & 1 & 0 & 1 & & & \\ & & & & 1 & -2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & -2 & 1 \\ & & & & & & & 1 & -2 \end{bmatrix}.$$

It follows from (*) that there exists a matrix H, $\det H = \pm 1$ such that

(2.1) $$P_{2m} \equiv {}^tH \cdot A \cdot H \mod 2p\,.$$

Rearrange (2.1) as follows.

(2.2) $$X+Y={}^tH\cdot A\cdot H\,, \qquad \text{where}$$

$$X=\begin{bmatrix} x_{11} & 1 & & & \\ 1 & x_{22} & 1 & & \\ & 1 & \ddots & & \\ & & & \ddots & 1 \\ & & & 1 & x_{2m2m} \end{bmatrix}, \qquad Y=\begin{bmatrix} 0 & & & \\ & 0 & & * \\ & & \ddots & \\ * & & & 0 \end{bmatrix}$$

satisfying that $x_{ii}\equiv 2\,(2p)$ for $1\leqq i\leqq m$, $x_{m+1\,m+1}\equiv 0\,(2p)$ and $x_{ii}\equiv -2\,(2p)$ for $m+2\leqq i\leqq 2m$, and each entry of $*$ in Y is a multiple of p-times.

We can take bundles E^i's, $i=1, \cdots, 2m$ from Lemma 1.1 each of which satisfies $\chi(E^i)=x_{ii}$, plumbing E^i with E^{i+1} together equivariantly at a fixed point of the Z_p-action on each. We can do this equivariantly from the normal representations (3) of Lemma 1.1 since E^i is the type E_+ untill $i=m$, E^{m+1} is the type E_0, and E^i is the type E_- for $i=m+2, \cdots, 2m$. Thus we obtain a Z_p-manifold with boundary M' which has the plumbing matrix X.

M' has $(2m+1)$-isolated fixed points. Let $N((2m+1)\,pts)$ be the equivariant tubular neighborhoods of $(2\,m+1)$-fixed points in M' and put

$$V' = M' - \mathrm{int}\, N\big((2m+1)\, pts\big)\big/ Z_p\,.$$

By Lemma 1. 3, there is a normal map $G' : V' \longrightarrow L^{4k-1}(p)$ between $\{\bigcup_F \partial N_i/Z_p,$ $i=1, 2\}$ and $\partial M'/Z_p$. Looking at the boundary components, we see that

$$\begin{aligned}&\left(G'\Big|\{\textstyle\bigcup_F \partial N_i/Z_p\},\ \{\bigcup_F \partial N_i/Z_p\}\right)\\ &\qquad = \Big((m+1)\,\big(L(p),\, id\big) \cup m\big(L(p,\ -a_1,\, a_1),\, c\times 1\big)\Big)\,,\end{aligned}$$

so that G' has degree 1. So far, M' is simply connected, and $\pi_1(V') = Z_p$. Hence $G' : V' \longrightarrow L(p)$ is a normal map in the usual sense. Now, to realize Y, all other plumbings in M' must be done by a multiple of p-times. We can do them equivariantly in the free part of E^i's of the action. We have a manifold with boundary M which admits a Z_p-action with $(2m+1)$-fixed points inside M. We then put

$$V = M - \mathrm{int}\, N\big((2m+1)\, pts\big)\big/ Z_p\,.$$

It follows from Lemma 1.4 that

(1) there is a normal map $G: V \longrightarrow L(p)$ between $\partial M/Z_p$ and

$$\Big\{(m+1)\,\big(L(p),\, id\big) \cup m\big(L(p,\ -a_1,\, a_1),\, c\times 1\big)\Big\}$$

(of course, G has degree 1).

From the standard theory of plumbing, it follows that M is connected, $\pi_1(\partial M) = \pi_1(M)$ is free, and

$$H_i(\partial M) = H_i(M) = 0 \quad \text{for} \quad 1 < i < 2k-1\,, \qquad H_{2k-1}(M) = 0\,.$$

Put $(G|\partial_+ V, \partial_+ V) = (f', \partial M/Z_p)$. Since f' has degree 1, so $\pi_1(f') = 0$. There is no obstruction to doing a normal surgery on a generator in

$$\pi_2(f') = \mathrm{Ker}\,\Big\{f'_* : \ \pi_1(\partial M/Z_p) \longrightarrow \pi_1\big(L(p)\big)\Big\}\,,$$

so there is a trace W and a normal map $F' : W \to L(p)$ between $\partial M/Z_p$ and $\partial_+ W$ such that $f = F|\partial_+ W$ is 2-connected. Then, we set
$V_1 = V \cup W$ and $M_1 = M \cup \widetilde{W} (= \widetilde{V}_1 \cup N(2m+1)\ \mathrm{pts})$ along $\partial M/Z_p$ and ∂M respectively. Put $\partial_+ W = L$.

(2) V_1 is a normal cobordism between

$$\Big((m+1)\,\big(L(p),\, id\big) \cup m\big(L(p,\ -a_1,\, a_1),\, c\times 1\big)\Big) \quad \text{and} \quad L\,.$$

The universal cover $\widetilde{L}$ bounds the parallelizable manifold M_1. Since the

intersection matrix on the bilinear form $H_{2k}(M_1)\times H_{2k}(M_1)\longrightarrow Z$ is the plumbing matrix $(X+Y)$ which is unimodular, and from the above facts, it concludes that $\pi_{i+2}(f)=\pi_{i+1}(\tilde{L})=0$ for all $i\geqq 0$. Hence, f is a homotopy equivalence of L onto $L^{4k-1}(p)$.
Denote the Z_p-action on M_1 by T, and then put $\tilde{L}=\Sigma_A\in bP_{4k}$ and $T|\tilde{L}=T_A$ (we called (T_A, Σ_A) "Weintraub's action" in Introduction). Since generators of $H_{2k}(M_1)$ consist of invariant $(2k)$-spheres and the induced action is trivial on homology, we have

$$\begin{aligned} Sing(T, M_1) &= \text{Index of the intersection matrix on } H_{2k}(M_1) \\ &= \sigma(A)\,, \end{aligned}$$

the local invariants

$$\begin{aligned} L(T, M_1) &= \sum_{i=1}^{2m+1} L(T, x_i)\,, \qquad x_i \text{ the fixed points} \\ &= (m+1)\prod_{i=1}^{k}\left(\frac{1+t^{a_i}}{1-t^{a_i}}\right)^2 - m\prod_{i=1}^{k}\left(\frac{1+t^{a_i}}{1-t^{a_i}}\right)^2 \\ &= \prod_{i=1}^{k}\left(\frac{1+t^{a_i}}{1-t^{a_i}}\right)^2. \end{aligned}$$

It follows that $\Sigma_A=\sigma(A)/8\Sigma_1$ and the Atiyah-Singer invariant

$$\sigma(T_A, \Sigma_A) = \prod_{i=1}^{k}\left(\frac{1+t^{a_i}}{1-t^{a_i}}\right)^2 - \sigma(A)\,.$$

This proves the Theorem 1.

Proof of Theorem 2. By (1), (2) in the proof of Theorem 1, there is a normal cobordism $F: V_1\longrightarrow L(p)$ between

$$\Big((m+1)\big(L(p), id\big)\cup m\big(L(p, -a_1, a_1), c\times 1\big)\Big) \quad\text{and}\quad L=\Sigma_A/T_A\,.$$

Since $c\times 1: L(p, -a_1, a_1)\longrightarrow L(p)$ is the orientation reversing diffeomorphism, there is a normal cobordism W_1 between

$$\Big((m+1)\big(L(p), id\big)\cup m\big(L(p, -a_1, a_1), c\times 1\big)\Big) \quad\text{and}\quad \big(L(p), id\big)\,.$$

Combing these cobordisms V_1, W_1, there exists a normal cobordism $G: V_2\longrightarrow L(p)$ between $(L(p), id)$ and $(\Sigma_A/T_A, f)$ completing the proof of Theorem 2.

Note 1. Clearly, we can take W_1 such that the intersection form on $H_{2k}(\tilde{W}_1)$ does not affect that on $H_{2k}(\tilde{V}_2)$. The intersection form on $H_{2k}(\tilde{V}_2)$ is the same as that on $H_{2k}(M_1)$, *i.e.*, $\sigma(\tilde{V}_2)=\sigma(M_1)=\sigma(A)$, because $H_{2k}(\tilde{V}_2)=H_{2k}(\tilde{V}_1)=H_{2k}(M_1)$, $V_2=V_1\cup W_1$.

NOTE 2. For any $i \in Z$, we can take a unimodular, even, symmetric matrix with index $8i$. So, we can take it as A for the above normal cobordism (G, V_2). Then, we can derive (iii) of Theorem 13 A. 4 [4] for the case of a cyclic group π.

COROLLARY 3. *The transfer* $\tau\colon L_0^h(Z_p) \longrightarrow L_0(1)$ *is onto for any integer* p.

3. Effects on the action

If one fixes the rank of a unimodular, even, symmetric matrix, by (*) in § 2, so many "Weintraub's actions" are constructed. We consider these actions under the calculations of Wall groups $L_0^\varepsilon(Z_p)$ $(\varepsilon = h, s)$.

THEOREM 4. *Suppose that p is any odd integer or 2, 4 and 6. Let A_i be a unimodular, even symmetric matrix for $i=1, 2$. There is a free Z_p-action (T_{A_i}, Σ_{A_i}) as in Theorem 1. Then, Σ_{A_1}/T_{A_1} is h-cobordant to Σ_{A_2}/T_{A_2} if and only if $\sigma(A_1)=\sigma(A_2)$. In particular, Σ_A/T_A is h-cobordant to $L^{4k-1}(p)$ if and only if $\sigma(A)=0$.*

The following is an immediate consequence of the Theorem, since $Wh(Z_2)=0$.

COROLLARY 5. *Let (T_{A_i}, Σ_{A_i}) be a free involution on a homotopy sphere for $i=1, 2$. Then, (T_{A_1}, Σ_{A_1}) is equivariantly diffeomorphic to (T_{A_2}, Σ_{A_2}) if and only if $\sigma(A_1)=\sigma(A_2)$. In particular, (T_A, Σ_A) is equivariantly diffeomorphic to (a, S^{4k-1}) if and only if $\sigma(A)=0$. Here, a is the antipodal map on the standard sphere S^{4k-1}.*

We quote the Theorem of Wall [4]. Let $R(Z_p)$ denote the complex representation ring of Z_p. Using the ideas of Atiyah-Singer, we can define a homomorphism called the "Multi-signature invariant"

$$\rho\colon L_0^\varepsilon(Z_p) \longrightarrow R(Z_p) \qquad \text{by setting}$$

$$\rho(t^i, x) = \operatorname{trace} t_*^i \,\big|\, H_{2k}(\widetilde{W})_+ - \operatorname{trace} t_*^i \,\big|\, H_{2k}(\widetilde{W})_- \,, \qquad i = 1, \cdots, p-1 \,.$$

Here t is a generator of Z_p and $x=\theta(F, W) \in L_0^\varepsilon(Z_p)$, where

$$F\colon W^4 \longrightarrow L^{4k-1}(p) \times I$$

is a normal map. In this case, the following alternative formula is deduced from the definition

$$\rho(t^i, x) = \sigma(t^i, \widetilde{\partial_+ W}) - \sigma(t^i, \widetilde{\partial_- W}) \,, \qquad i = 1, \cdots, p-1 \,.$$

THEOREM (Wall). *Suppose that p is any odd integer or 2, 4 and 6. Then, the multi-signature ρ is injective on the summand $\widetilde{L_0^h(Z_p)}$, where*

$L_0^h(\widetilde{Z_p})$ is the reduced Wall group.

The proof is seen in [4, Theorem 13 A. 4, (ii)] and [5].

PROOF OF THEOREM 4. We have a normal cobordism between $(L(p), id)$ and $(\Sigma_{A_i}/T_{A_i}, f_i)$ as in Theorem 2. Denote the normal cobordism by Y_i for $i=1, 2$, respectively. By note 1 $\sigma(\tilde{Y}_i)=\sigma(A_i)$. Put $X=Y_1\cup -Y_2$. Then, there is a normal cobordism $F: X\longrightarrow L(p)$ between Σ_{A_1}/T_{A_1} and Σ_{A_2}/T_{A_2}. Set the surgery obstruction of F

$$x=\theta(F, X)\in L_0^h(Z_p)\,.$$

For the multi-signature of x, it follows by Theorem 1 that

$$\begin{aligned}\rho(T^j, x) &= \sigma(T_{A_1}^j, \Sigma_{A_1})-\sigma(T_{A_2}^j, \Sigma_{A_2})\\ &= \sigma(A_1)-\sigma(A_2)\,, \qquad i=1, \cdots, p-1\,.\end{aligned}$$

If $\sigma(A_1)=\sigma(A_2)$, by the above Theorem, x lies in the summand $L_0(1)\subset L_0^h(Z_p)$. Hence, x is written $m\chi_R$ for some $m\in Z$, where χ_R is the regular representation of Z_p. Taking a p-fold covering of x, it follows that $\tilde{x}=\theta(\tilde{F}, \tilde{X})=\mathrm{pm}$. Since $\theta(\tilde{F}, \tilde{X})=\sigma(\tilde{X})=\sigma(\tilde{Y}_1)-\sigma(\tilde{Y}_2)=0$, m must be zero, *i. e.*, $\theta(F, X)=0$. Hence, Σ_{A_1}/T_{A_1} is h-cobordant to Σ_{A_2}/T_{A_2}. Conversely, if Σ_{A_1}/T_{A_1} is h-cobordant to Σ_{A_2}/T_{A_2}, then the Atiyah-Singer invariants of these must agree. Hence, from our computations in Theorem 1, $\sigma(A_1)=\sigma(A_2)$. The rest of the Theorem follows from the fact that the Atiyah-Singer invariant of $L(p)=L^{4k-1}(p, a_1, \cdots, a_k, a_1, \cdots, a_k)$ is $\prod_{i=1}^{k}\left(\frac{1+t^{a_i}}{1-t^{a_i}}\right)^2$.

REMARK 3. 1. According to the method of Theorem 1, we have a plumbing manifolds with the plumbing matrix P_{2m} (see Proof of Theorem 1). If we concentrate on the boundary, *i. e.*, on $(T_{P_{2m}}, \Sigma_{P_{2m}})$, $\Sigma_{P_{2m}}$ is S^{4k-1} obtained by attaching $S^{2k-1}\times D^{2k}$ to $D^{2k}\times S^{2k-1}$ by means of

$$\phi(x, y)\longrightarrow\big(x, u(x)\,y\big) \quad \text{on} \quad S^{2k-1}\times S^{2k-1}\,,$$

where $u: S^{2k-1}\longrightarrow SO(2k)$ is the characteristic map of the tangent bundle τ of S^{2k}. ϕ makes sense for $y\in D^{2k}$ and hence extends to an equivariant diffeomorphism of $S^{2k-1}\times D^{2k}$ onto itself.
Thus $(T_{P_{2m}}, \Sigma_{P_{2m}})$ is equivariantly diffeomorphic to the linear Z_p-action on S^{4k-1} which induces just $L^{4k-1}(p, a_1, \cdots, a_k, a_1, \cdots, a_k)$.

4. López's involutions

P. Orlik and C. P. Rourke [2] proved the following theorem using López's construction.

THEOREM. *For each i there exists a homotopy sphere Σ_i^{4k-1}, bounding a parallelizable manifold M_i, and an involution T_i such that*

$$I(T_i, \Sigma_i) = \sigma(M_i) = 8i .$$

First, we show that Σ_i/T_i is normally cobordant to the standard projective space $P^{4k-1}(k \geqq 2)$.

LEMMA 4.1. *There exists a normal cobordism X_i between P^{4k-1} and Σ_i/T_i so that $\sigma(\tilde{X}_i)=8i$.*

This lemma depends only on the proof of the above theorem if one takes care of normal maps. So, we sketch its proof for the necessity of recalling the López's construction. We use the same notations as [2]. It is sufficient to prove the case $i=1$.

Let T_0: $S^{4k-1} \longrightarrow S^{4k-1}$ be the antipodal map and $W=S^{4k-2} \underset{Z_2}{\#} 4(S^{2k-1} \times S^{2k-1})$ be a characteristic submanifold of S^{4k-1}, *i.e.*, $S^{4k-1}=V \cup T_0 V$, $V \cap T_0 V=W$. Since W/T and P^{4k-2} are characteristic submanifolds of P^{4k-1}, there is a characteristic cobordism Y joining them so that $S^{4k-1} \times I=X^{4k} \cup TX^{4k}$, $X \cap TX=\tilde{Y}$ and $\partial X=V \cup \tilde{Y} \cup D^{4k-1}$. If $F: Y \longrightarrow P^{4k-2}$ is a normal map, then $\tilde{F}: \tilde{Y} \longrightarrow S^{4k-2}$ extends to a normal map $G: X \longrightarrow D^{4k-1}$.

Let $\{\alpha_1, \cdots, \alpha_8, \beta_1, \cdots, \beta_8\}$ be a standard basis for $H_{2k-1}(W)$ chosen so that $\alpha_i \in \operatorname{Ker}\{i_*: H_{2k-1}(W) \longrightarrow H_{2k-1}(V)\}$ and $\beta_i \in \operatorname{Ker}\{i_*: H_{2k-1}(W) \longrightarrow H_{2k-1}(T_0 V)\}$. Choose new generators $\alpha_i^* = p_{ij}\alpha_j + q_{ij}\beta_j$, $i=1, \cdots, 8$. The matrices $P=(p_{ij})$ $Q=(q_{ij})$ are given explicitly in [1]. So, we perform surgery on the

$$\alpha_i^* \in \operatorname{Ker}\left\{\tilde{f}_*: H_{2k-1}(W) \longrightarrow H_{2k-1}(S^{4k-2})\right\},$$

obtaining a normal cobordism $h: A \longrightarrow S^{4k-2}$ between $\tilde{f}: W \longrightarrow S^{4k-2}$ and a homotopy equivalence $K \longrightarrow S^{4k-2}$. Then, they showed that $V \underset{W}{\cup} A$ is a $(4k-1)$-disk. Thus K is a standard sphere. Attach a disk D on $V \underset{W}{\cup} A$ so that $V \underset{W}{\cup} A \cup D$ is a sphere bounding a $4k$-disk B (see Figure 1). The normal map $(G|V) \cup h: V \underset{W}{\cup} A \longrightarrow D^{4k-1} \cup S^{4k-2} = D^{4k-1}$ extends to a normal map $H: V \underset{W}{\cup} A \cup D \longrightarrow \partial(D^{4k-1} \times I)$. Again, H extends to a normal map $\bar{H}: B \longrightarrow D^{4k-1} \times I$. Combining with G, there is a normal map

$$\bar{G}: X \underset{V}{\cup} B \longrightarrow D^{4k-1} \cup D^{4k-1} \times I = D^{4k-1} \times I \longrightarrow D^{4k-1}.$$

Put $B' = X \underset{V}{\cup} B$. Let B'^* be another copy of B'. We obtain a parallelizable manifold M' with a free involution T, $M' = B' \underset{(T,\tilde{Y})}{\cup} B'^*$, glued on $(T, \tilde{Y})$. Then, M'/T is a cobordism between P^{4k-1} and a "López's involution Σ_1/T_1".

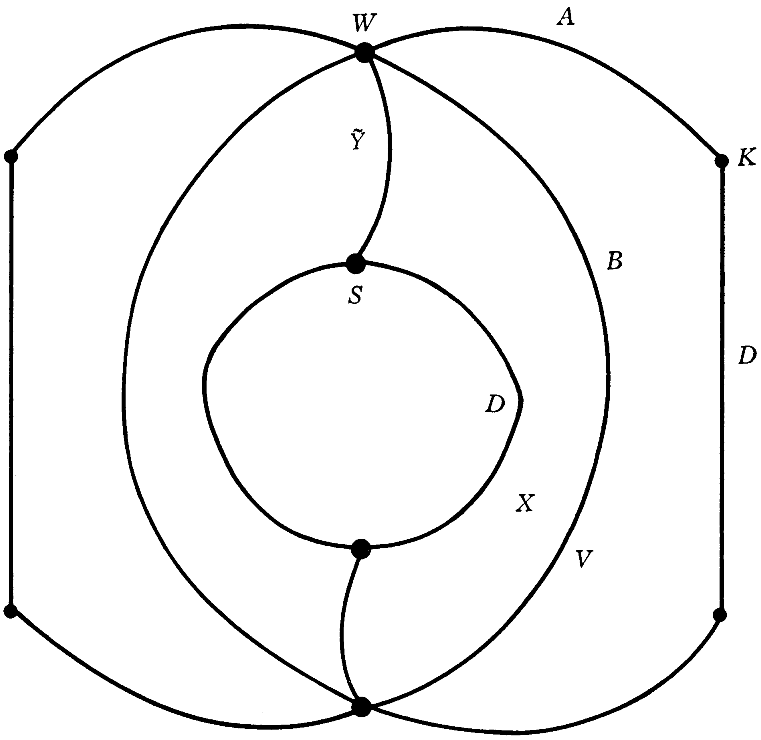

Fig. 1.

Let η_P be the normal bundle of P^{4k-2} in P^{4k-1}, and η_Y the normal bundle of Y in M'/T. Then, $\partial E(\eta_P)=S^{4k-2}$, $\partial E(\eta_Y)=\tilde{Y}$ and $P^{4k-1}=E(\eta_P)\cup D^{4k-1}$, $M'/T=E(\eta_Y)\cup B'$. Since $F: Y\longrightarrow P^{4k-2}$ is a normal map, the same is true for $E(F): E(\eta_Y)\longrightarrow E(\eta_P)$, because η_Y is the pull-back of η_P. Now, $\bar{G}: B'\longrightarrow D^{4k-1}$ is also a normal map. Hence, M'/T defines a normal cobordism between P^{4k-1} and Σ_1/T_1. For the rest of the lemma, the boundary (T_0, S^{4k-1}) of M' bounds a disk D^{4k} with the antipodal map. Put $M=M'\cup D^{4k}$. Then, $M=B\underset{V}{\cup}C\underset{V^*}{\cup}B^*$, where C is the standard $(4k)$-disk with boundary $S^{4k-1}=V\cup T_0V$. Then, it has been shown in [2] that $\sigma(M')=\sigma(M)=8$.

Consequently, we can say that in general case (T_i, Σ_i) bounds an M_i which admits an involution T with only one fixed point, and if we remove the interior of a disk D of the fixed point from M_i, then M_i-int D/T is a normal cobordism between P^{4k-1} and Σ_i/T_i so that $\sigma(M_i-\text{int}\,D)=8i$.

THEOREM 6. *Suppose that $p=2q$ $(q\geqq 1)$. There exists a free Z_p-action T_A on a homotopy sphere $\Sigma_A\in bP_{4k}$ which satisfies that: If we restrict this "Weintraub's action" to the Z_2-action on Σ_A, then the above "López's involution" (T_i, Σ_i) is equivariantly diffeomorphic to (T_A^q, Σ_A) for any q.*

PROOF. By Lemma 4.1, there is a normal cobordism X_i such that $\sigma(\tilde{X}_i)=8i$. Let $F_i: X_i\longrightarrow P^{4k-1}$ be a normal map between P^{4k-1} and Σ_i/T_i. Then, the surgery obstruction of F_i is

$$(1) \qquad \theta(F_i)=\big(\sigma(X_i),\,\sigma(\tilde{X}_i)\big)=(8i,\,8i)\in L_0(Z_2)\,.$$

This follows from the fact that

$$2\sigma(X_i)-\sigma(\tilde{X}_i)=I(T_i,\,\Sigma_i)-I(a,\,S^{4k-1})=8i\,.$$

Take a free Z_p-action T_A on Σ_A from Theorem 1 such that $\sigma(A)=8i$ (for example, a direct sum of i-copies of the well known (8×8)-matrix E_8). By Theorem 2 and Note 1, we have a normal cobordism (G, Y) between $L(p)$ and Σ_A/T_A such that $\sigma(\tilde{Y})=\sigma(A)=8i$. Since the Atiyah-Singer invariant σ and the Browder-Livesay invariant I agree for involutions, so if Y^q is the q-fold covering of Y, so that (G_q, Y_q) is a normal cobordism between P^{4k-1} and Σ_A/T_A^q, then it follows that

$$(2) \qquad \theta(G^q)=\big(\sigma(Y^q),\,\sigma(\tilde{Y})\big)=(8i,\,8i)\in L_0(Z_2)\,.$$

From (1) and (2), there is an h-cobordism between Σ_i/T_i and Σ_A/T_A^q (note that $\Sigma_A=\sigma(A)/8\Sigma_1$). Hence, (T_i, Σ_i) is equivariantly diffeomorphic to (T_A^q, Σ_A).

REMARK 4.1. We have the analogous results for lens spaces $L^{4k-1}(p, a_1, \cdots, a_k)$ instead of lens spaces $L^{4k-1}(p, a_1, \cdots, a_k, a_1, \cdots, a_k)$. Let $\{a_1, \cdots, a_{2k}\}$ be any collection with $(a_i, p)=1$ and b an integer which reduces to $\prod_{i=k+1}^{2k} a_i\Big/\prod_{i=1}^{k} a_i$ mod p. Let $\bar{b}$ reduce b^{-1} mod p. Instead of P_{2m}, we use the following matrix due to S. Weintraub.

$$P_{2m}(b)=\begin{bmatrix}
2b & 1 & & & & & & & & \\
1 & 2\bar{b} & 1 & & & & & & & \\
 & & \ddots & & & & & & & \\
 & & & 2b & 1 & & & & & \\
 & & & 1 & 2\bar{b} & 1 & & & & \\
 & & & & 1 & 0 & 1 & & & \\
 & & & & & 1 & -2\bar{b} & 1 & & \\
 & & & & & & 1 & -2b & & \\
 & & & & & & & & \ddots & \\
 & & & & & & & & -2b & 1 \\
 & & & & & & & & 1 & -2\bar{b}
\end{bmatrix}$$

Then, we can construct the bundles with Z_p-actions $E_\pm$, $\bar{E}_\pm$ and E_0 which have the Euler classes congruent with $\pm 2b$, $\pm 2\bar{b}$, 0 mod $2p$ accordingly. The results follow similarly.

References

[1] LÓPEZ DE MEDRANO: Involutions on manifolds, Springer-Verlag 1971.
[2] P. ORLIK and C. P. ROURKE: Free invalutions on homotopy $(4k+3)$ spheres, Bull. Amer. Math. Soc. 74, 949–953 (1968).
[3] S. WEINTRAUB: Semi-free Z_p-actions on highly-connected manifolds, Math. Z. 145, 163–185 (1975).
[4] C. T. C. WALL: Surgery on compact manifolds, Academic Press, 1971.
[5] C. T. C. WALL: Some L groups of finite groups, Bull. Amer. Math. Soc. 79, 526–529 (1973).

Yoshinobu Kamishima
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo, JAPAN