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Conformally flat Riemannian manifolds of constant scalar curvature

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Introduction

N. Ejiri [3] showed the existence of compact Riemannian manifolds of constant scalar curvature which admit non-homothetic conformal transformations. This is related to solutions of a non-linear differential equation (*) and he did not give any concrete solutions.

Here we give explicit solutions of (*) for the case of $n=3$ (Lemma 4). This problem is also related to examples of compact or complete conformally flat Riemannian manifolds of constant scalar curvature S. In § 2 we show concrete examples of such Riemannian manifolds (Theorems 6 and 7). These show that $S=$ constant (as one condition of weaker type of local homogeneity) on a conformally flat Riemannian manifold does not imply local homogeneity.

A Kählerian analogue of conformal flatness is the vanishing of the Bochner curvature tensor. In § 3 we study some conditions weaker than local homogeneity. Contrary to the conformally flat case, Theorems 8 and 11 show that $S=$ constant or constancy of length of the Ricci curvature tensor on a Kählerian manifold with vanishing Bochner curvature tensor implies local homogeneity.

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§ 1. Warped products.

Let (F, h) be an n-dimensional Riemannian manifold and f be a positive function on an open interval I of a real line $\boldsymbol{R}$. Consider the product $I\times F$ with the projections $\pi: I\times F\rightarrow I$, and $\eta: I\times F\rightarrow F$. The space $I\times F$ with the Riemannian metric

$$\langle X, Y\rangle_{(t,x)}=(\pi X, \pi Y)_{t}+f^{2}(\pi x)\, h_{x}(\eta X, \eta Y)$$

is called the warped product and is denoted by $I\times_{f}F$, where X, Y are tangent vectors at $(t, x)\in I\times F$, and π, η denote also their differentials (cf. R. L.

Bishop and B. O'Neill [2]). Let d/dt be a canonical unit vector field on $\boldsymbol{R}$ and on $I\times_f F$. Let R and R^* denote the Riemannian curvature tensors of (F, h) and $I\times_f F$, respectively, and let $(d/dt, e_1, \cdots, e_n)$ be an orthonormal basis of the tangent space $(I\times_f F)_{(t,x)}$ at (t, x). The function f on I is naturally lifted to a function on $I\times F$ and we denote it by the same letter f. The following Lemma 1 is verified by using relations between R and R^* given in [2].

LEMMA 1. *We put* $f'=df/dt$ *and* $f''=d^2f/dt^2$.

$$\langle R^*(e_a, e_b)e_c, e_d\rangle = (1/f)^2 h\Big(R(\eta f e_a, \eta f e_b)\eta f e_c, \eta f e_d\Big) - (f'/f)^2(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc}),$$

$$\langle R^*(e_a, e_b)e_c, d/dt\rangle = 0,$$

$$\langle R^*(d/dt, e_a)d/dt, e_b\rangle = -(f''/f)\delta_{ab}.$$

LEMMA 2. (Y. Ogawa [8], N. Ejiri [3]). *Let* S *and* S^* *be the scalar curvatures of* (F, h) *and* $I\times_f F$, *respectively. Then*

$$(*)\qquad S^* = -2n(f''/f) - n(n-1)(f'/f)^2 + S(1/f)^2.$$

The theorem of N. Ejiri [3] is stated as follows: Let (F, h) be a compact n-dimensional Riemannian manifold. Assume that the scalar curvature S is constant and positive. Then for any positive real number S^*, there exists a non-constant periodic and positive solution f of (*) for $I=\boldsymbol{R}$ with period t_0, and $(M, \langle\,,\,\rangle) = (\boldsymbol{R}/(t_0 Z))\times_f F$ is a compact Riemannian manifold whose scalar curvature is S^*. Furthermore, the vector field $X=f\,d/dt$ is an infinitesimal non-homothetic conformal transformation on $(M, \langle\,,\,\rangle)$. Since M is compact, X generates a 1-parameter group of conformal transformations of $(M, \langle\,,\,\rangle)$. So it admits a non-homothetic conformal transformation.

By Lemma 4 in § 2 we get an explicit example:

$$(M, \langle\,,\,\rangle) = \Big(\boldsymbol{R}/(2\pi Z)\Big)\times_f F,$$

$$f^2 = \alpha \sin t + S/3,$$

where (F, h) is a compact 3-dimensional Riemannian manifold of positive constant scalar curvature S (for example, a Euclidean unit 3-sphere $S^3(1)$, where $S=6$) and α is a constant such that $0<\alpha<S/3$.

§ 2. Conformally flat Riemannian manifolds.

REMARK 3. (J. Kato) If S and S^* are constant, (*) is reducible to the first order differential equation

$$(**)\qquad (f')^2 = Af^{1-n} - \big(S^*/n(n+1)\big)f^2 + \big(1/n(n-1)\big)S,$$

where A is a constant.

To prove this we put $z=f^{n-1}(f')^2$. Since $d(f')^2/df=(d(f')^2/dt)(dt/df)=2f''$, (*) implies

$$dz/df = -(S^*/n)f^n + (S/n)f^{n-2}.$$

Integrating the last equation we get

$$z = -\big(S^*/n(n+1)\big)f^{n+1} + \big(S/n(n-1)\big)f^{n-1} + A,$$

and hence we get (**).

Looking at (**) we see that (*) is explicitly solved if $n=3$.

LEMMA 3. *For $n=3$, if S and S^* are constant, positive solutions of (*) are of the following forms:*

$$f = \big[\alpha\sin\theta(t+\beta) + S/S^*\big]^{1/2} \qquad \text{for } S^*>0,$$

$$f = \big[(S/6)t^2 + \alpha t + \beta\big]^{1/2} \qquad \text{for } S^*=0,$$

$$f = [\alpha e^{\theta t} + \beta e^{-\theta t} + S/S^*]^{1/2} \qquad \text{for } S^*<0,$$

where $\theta=(|S^*|/3)^{1/2}$ and α, β are constant.

Furthermore, f is periodic and non-constant on $I=\mathbf{R}$, if and only if $S>0$, $S^*>0$, and $0<|\alpha|<S/S^*$.

PROOF. We put $w=f^2$. Then (*) is

$$3w'' + S^*w = S.$$

This is a linear differential equation and we get solutions. Q. E. D.

Let (M, g) be a conformally flat m-dimensional Riemannian manifold. Then the following results are known:

[i] If (M, g) is reducible, then (M, g) is locally one of the following spaces:

$$E^m, \qquad E^1\times S^{m-1}(c), \qquad E^1\times H^{m-1}(-c),$$

$$S^p(c)\times H^{m-p}(-c); \qquad 2\leq p\leq m-2,$$

where E^m, $S^m(c)$ and $H^n(-c)$ denote simply connected space forms of constant curvature 0, $c>0$, and $-c$, respectively (M. Kurita [5]).

[ii] If M is compact, the fundamental group of M is finite and the scalar curvature S is constant, then S is positive and (M, g) is of constant curvature (S. Tanno [13]).

[iii] If M is compact, S is constant, and the Ricci curvature tensor is

positive semi-definite, then (M, g) is covered by one of the following spaces;

$$E^m, \qquad E^1 \times S^{m-1}(c), \qquad S^m(c).$$

Here, compactness of M is replaced by constancy of length of the Ricci curvature tensor (P. J. Ryan [9]).

LEMMA 5. *Let $I \times_f F$ be a warped product of an open interval I of $\boldsymbol{R}$ and a 3-dimensional Riemannian manifold (F, h). If it is conformally flat and has constant scalar curvature S^*, then (F, h) is of constant curvature $S/6$ and f is one of the functions in Lemma 4.*

PROOF. Since I is 1-dimensional, $I \times_f F$ is conformal to $I \times_1 F$. Thus, $I \times_1 F$ is conformally flat and (F, h) is of constant curvature by [i]. By Lemma 2, f satisfies (*), and so f is one of the functions in Lemma 4.

Q. E. D.

THEOREM 6. *For positive real numbers S, S^*, and $\alpha < S/S^*$, we have a compact conformally flat Riemannian manifold*

$$S^1 \times_f \left(S^3(S/6)/\Gamma\right)$$

of constant scalar curvature S^*, where S^1 is a circle of length $2(3/S^*)^{1/2}\pi$, $f^2 = \alpha \sin (S^*/3)^{1/2} t + S/S^*$, and $S^3(S/6)/\Gamma$ is a space form of positive curvature $S/6$.

Conversely, among warped products $S^1 \times_f F$ (with non-constant f, constant S, $\dim F = 3$), any compact conformally flat Riemannian manifold of constant scalar curvature S^* is of the above form or its finite covering manifold.

PROOF. This follows from Lemmas 4 and 5.

REMARK 7. The Ricci curvature of the space in Theorem 6 satisfies the following:

$$R_1^*(e_a, e_a) > 0 \qquad (1 \leq a \leq 3),$$

$$R_1^*(e_a, e_b) = 0 \qquad (a \neq b),$$

$$R_1^*(e_a, d/dt) = 0,$$

and $R_1^*(d/dt, d/dt)$ takes positive and negative values depending on t. The sectional curvature $K^*(e_a, e_b)$ is positive and $K^*(e_a, d/dt)$ takes positive and negative values depending on t. These are verified by Lemma 1 and the explicity form of f.

THEOREM 7. *Let (F, h) be a complete 3-dimensional Riemannian manifold of constant scalar curvature S. Then $\boldsymbol{R} \times_f F$ is a complete conformally flat Riemannian manifold of constant scalar curvature S^*, if and only if*

(F, h) is of constant curvature $S/6$ and, putting $\theta=(|S^*|/3)^{1/2}$,

(i) *for the case of* $S^*>0$;

$$f^2=\alpha\sin\theta t+S/S^*, \qquad S>0,\ 0\leq\alpha<S/S^*,$$

(ii) *for the case of* $S^*=0$;

$$f^2=(S/6)\,t^2+\alpha t+\beta,$$

α, β satisfying one of (ii-1), (ii-2):

$$\text{(ii-1)} \qquad 3\alpha^2<2\beta S, \qquad S>0,$$

$$\text{(ii-2)} \qquad \alpha=0, \qquad \beta>0, \qquad S=0,$$

(iii) *for the case of* $S^*<0$;

$$f^2=ae^{\theta t}+be^{-\theta t}+S/S^*,$$

a, b satisfying one of (iii-1), (iii-2), (iii-3), (iii-4):

$$\text{(iii-1)} \quad a>0, \qquad b=0, \qquad S\leq 0,$$

$$\text{(iii-2)} \quad a=0, \qquad b>0, \qquad S\leq 0,$$

$$\text{(iii-3)} \quad a>0, \qquad b>0, \qquad 2\sqrt{ab}>-S/S^*.$$

$$\text{(iii-4)} \quad a=0,\ b=0,\ S<0.$$

PROOF. Completeness of $\boldsymbol{R}\times_f F$ follows from completeness of $\boldsymbol{R}$ and (F, h). The remainder of proof follows from Lemmas 4 and 5.

Q. E. D.

By Theorems 6 and 7 we see that the condition of weaker type of local homogeneity; S=constant, on a complete Riemannian manifold does not imply local homogeneity.

So a question which is still open is: Is a complete conformally flat Riemannian manifold with constant scalar curvature and constant length of the Ricci curvature tensor locally homogeneous?

P. J. Ryan's result [iii] gives a partial answer for the case where the Ricci curvature tensor is positive semi-definite.

U. Simon [10] gives also a partial answer.

If M is compact and the fundamental group of M is finite, constancy of S implies local homogeneity as [ii] shows.

Locally homogeneous conformally flat Riemannian manifolds are locally classified (cf. H. Takagi [11], D. V. Alekseevskii and B. N. Kimel'fel'd [1]).

§ 3. Kählerian manifolds with vanishing Bochner curvature tensor.

Let (M, J, g) be a Kählerian manifold of real dimension $m=2n$ with almost complex structure tensor J and Kählerian metric tensor g. By R, R_1 and S we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature of (M, g), respectively. The Bochner curvature tensor B has properties similar to those of the Weyl conformal curvature tensor of a Riemannian manifold.

By (R, R), (R_1, R_1) we denote the local inner products of R, R_1, respectively. By (CP^n, H), $(CE^n, 0)$ and $(CD^n, -H)$ we denote simply connected complex space forms with constant holomorphic sectional curvature $H>0$, 0, and $-H$.

THEOREM 8. *Let (M, J, g) be a complete and simply connected Kählerian manifold with vanishing Bochner curvature tensor. If one of S, (R_1, R_1), and (R, R) is constant, then (M, J, g) is one of the following spaces*:

$$(CP^n, H), \qquad (CE^n, 0), \qquad (CD^n, -H),$$
$$(CP^p, H)\times(CD^{n-p}, -H);\ 1\leq p\leq n-1.$$

If $m=2n=2$ Theorem 8 is trivial. So we assume that (M, J, g) is a Kählerian manifold with $B=0$ and $m\geq 4$. It is known that the condition $B=0$ implies the following (cf. M. Matsumoto [6], p. 26)

$$(1)\qquad 2(m+2)(\nabla_Z R_1)(X, Y)=\nabla_X S\cdot g(Y, Z)+\nabla_Y S\cdot g(X, Z) -\nabla_{JX}S\cdot g(Y, JZ)-\nabla_{JY}S\cdot g(X, JZ)+2\nabla_Z S\cdot g(X, Y),$$

where X, Y, and Z are vector fields on M. Calculating $(\nabla_Z R_1, R_1)$ we get

$$(2)\qquad (m+2)\nabla_Z(R_1, R_1)=4R_1(Z, \operatorname{grad} S)+2S\nabla_Z S,$$

where we have used $R_1(JX, JY)=R_1(X, Y)$.
On the other hand, $B=0$ implies (cf. S. Tanno [14], p. 260)

$$(3)\qquad (R, R)-16(m+4)^{-1}(R_1, R_1)+8(m+2)^{-1}(m+4)^{-1}S^2=0.$$

LEMMA 9. *Let q be a real number such that $q\neq 2(m+6)/(m+2)$. If*

$$(4)\qquad 2(m+2)(R_1, R_1)-qS^2=constant,$$

then S is constant.

PROOF. Operating ∇_Z to (4) and applying (2) we get

$$(5)\qquad 4R_1(Z, \operatorname{grad} S)=(q-2)S\nabla_Z S.$$

Operating ∇_Y to (5) and applying (1) we obtain

$$2(m+2)^{-1}\Big[(\nabla S, \nabla S)\, g(Y, Z)+3\nabla_Y S\nabla_Z S-\nabla_{JY} S\nabla_{JZ} S\Big]$$
$$+4R_1(Z, \nabla_Y \operatorname{grad} S)=(q-2)\Big[Sg(\nabla_Y \operatorname{grad} S, Z)+\nabla_Y S\nabla_Z S\Big].$$

Putting $Y=Z=\operatorname{grad} S$ and applying (5) we obtain

$$8(\nabla S, \nabla S)^2=(q-2)(m+2)(\nabla S, \nabla S)^2.$$

Thus, $q\neq 2(m+6)/(m+2)$ implies $\nabla S=0$. Q. E. D.

LEMMA 10. *The following are equivalent*:

(i) $S=$*constant,*

(ii) $(R_1, R_1)=$*constant,*

(iii) $(R, R)=$*constant.*

PROOF. Assume (i). Then (ii) follows from (2). Assume (ii). Then (i) follows from Lemma 9 and $q=0$. (i) and (ii) imply (iii) by (3). Finally assume (iii). Then (3) implies (4) for $q=1$, and (i) follows from Lemma 9. Q. E. D.

THEOREM 11. *Let* (M, J, g) *be a Kählerian manifold with* $B=0$. *Assume one of the conditions* (i), (ii) *and* (iii) *of Lemma* 10. *Then* (M, J, g) *is either*

(a) *of constant holomorphic sectional curvature, or*

(b) *locally a product* $(CP^p, H)\times(CD^{n-p}, -H)$.

PROOF. If S is constant, this is a Theorem of M. Matsumoto and S. Tanno [7]. Thus, the remainder of proof follows from Lemma 10. Q. E. D.

Proof of Theorem 8 is completed by Theorem 11.

REMARK 12. Theorem of K. Yano and S. Ishihara [15] follows from our Theorem 8 under the weaker assumptions. Their method of proof is based on Ryan's one and so they assume compactness of M and positive semi-definiteness of the Ricci curvature tensor.

Theorem 1 and Theorem 2 of Y. Kubo [4] follows from our Theorem 8, because the assumption $(R_1, R_1)<S^2/(m-1)$ is stronger than the assumption that the Ricci curvature tensor is definite (cf. S. Tanno [12], p. 42).

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