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Remarks on the spaces of type $H+AP$

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§ 1. Introduction.

For a LCA group G, $AP(G)$ and $M(G)$ denote the space of all almost periodic functions and the space of all bounded regular measures on G respectively.

Let R be the reals. In [4], S. Power proved that the sum of the Hardy space and the space of all almost periodic functions on $R(H^{\infty}(R)+AP(R))$ is a closed subspace of $L^{\infty}(R)$ but not an algebra.

For a LCA group G, in [6], W. Rudin proved that $H+C_{u}(G)$ is a closed subspace of $L^{\infty}(G)$ for a translation-invariant weak*-closed subspace H of $L^{\infty}(G)$ and the space of all bounded uniformly continuous functions $C_{u}(G)$.

In this paper, we shall prove that $H+AP(G)$ is a closed subspace of $L^{\infty}(G)$ for every translation-invariant weak*-closed subspace H of $L^{\infty}(G)$. Moreover, we shall investigate whether a space of type $H+AP(G)$ becomes an algebra.

DEFINITION 1. *For any subset Φ of $L^{\infty}(G)$, the spectrum of Φ is defined as the set $\sigma(\Phi)$ of all $\gamma\in\hat{G}$ that belong to the smallest translation-invariant weak*-closed subspace of $L^{\infty}(G)$ containing Φ.*

Easily, we have the following:

$$\sigma(\Phi)=\cap\left\{\hat{f}^{-1}(0)\,;\,f\in L^{1}(G),\ f_{*}\Phi=0\right\}.$$

§ 2. Main Theorem

Let $\bar{G}$ denote the Bohr compactification of G. Then we can identify $AP(G)$ with $C(\bar{G})$. Let $d\bar{x}$ denote the Haar measure on $\bar{G}$. For f, $g\in AP(G)$, we define $f*g$, $||f||_{1}$ and $\hat{f}$ with respect to $d\bar{x}$. The symbol $B(L^{\infty}(G))$ denotes the Banach algebra of bounded linear operators on $L^{\infty}(G)$.

LEMMA. *There exists a linear map*

$$f\longmapsto\lambda_{f}\,;\ AP(G)\longmapsto B\left(L^{\infty}(G)\right)$$

satisfying the following conditions for f, $g\in AP(G)$ and $\phi\in L^{\infty}(G)$:

(a) $\lambda_f(\psi)\in AP(G)$, $\left\|\lambda_f(\psi)\right\|_\infty \leqq \|f\|_1\|\psi\|_\infty$ *and* $\sigma\left(\lambda_f(\psi)\right)\subset\sigma(f)\cap\sigma(\psi)$,

(b) $\lambda_f(g)=f*g$.

PROOF. Let N be the family of all neighborhoods of $0\in\hat{G}$ directed by set inclusion. For each $V\in N$, choose $h_V\in L^1(G)$ such that

(1) $h_V\geqq 0$, $\|h|\|_1=1$ and $\operatorname{supp}(\hat{h}_V)\subset V$,

and define

(2) $\lambda_V(\phi,\psi)=(\phi h_V)*\psi$ for $\phi,\psi\in L^\infty(G)$.

Then, λ_V is a bilinear operator on $L^\infty(G)$ and

(3) $\left\|\lambda_V(\phi,\psi)\right\|_\infty \leqq \|\phi h_V\|_1\|\psi\|_\infty \leqq \|\phi\|_\infty\|\psi\||_\infty$.

Since the unit ball of $L^\infty(G)$ is weak* compact, we can find a subnet $\{\lambda_{V_i}\}$ of $\{\lambda_V\}$ such that for each ϕ, $\psi\in L^\infty(G)$, $\{\lambda_{V_i}(\phi,\psi)\}$ converges weak* to an element of $L^\infty(G)$, which will be denoted by $\lambda(\phi,\psi)$. Evidently, λ is a bilinear operator on $L^\infty(G)$.

Now, we claim that

(4) $\sigma\left[\lambda(\phi,\psi)\right]\subset\sigma(\phi)\cap\sigma(\psi)$ for ϕ, $\psi\in L^\infty(G)$.

In fact, notice that $\operatorname{supp}(\phi h)$ is contained in the closure of $\sigma(\phi)+\operatorname{supp}(\hat{h})$ for $\phi\in L^\infty(G)$ and $h\in L^1(G)$. Therefore, (4) is an easy consequence of the definition of λ combined with the fact that $\sigma(g*\psi)\subset\operatorname{supp}(\hat{g})\cap\sigma(\psi)$ for $g\in L^1(G)$ and $\psi\in L^\infty(G)$. Next, we claim that

(5) $\left\|\lambda(f,\psi)\right\|_\infty \leqq \|f\|_1\|\psi\|_\infty$ for $f\in AP(G)$ and $\psi\in L^\infty(G)$.

To see this, we regard each $h_V dx$ as a measure on $\bar{G}$. Then, the net $\{h_V dx\}$ converges to $d\bar{x}$ in the weak* topology of $M(\bar{G})$ by (1). Therefore $f\in AP(G)$ implies

$$\lim_v \|fh_v\| = \lim_v \int_G |f| h_v\, dx = \int_{\bar{G}} |f|\, d\bar{x} = \|f\|_1 .$$

Thus (5) follows from the first inequality in (3).

In order to complete the proof, it will suffice to check that $\lambda(f,\psi)\in AP(G)$ if $f\in AP(G)$ and $\psi\in L^\infty(G)$, and that $\lambda(f,g)=f*g$ if f, $g\in AP(G)$.

By the continuity and Bilinearity of λ, we may assume that $f=\gamma$ and

$g=\chi$ for some γ, $\chi\in\hat{G}$. But then $\sigma[\lambda(\gamma, \phi)]\subset\sigma(\gamma)=\{\gamma\}$ by (4). It follows from 7. 8. 3 (e) of [5] that $\lambda(\gamma, \phi)$ is a constant multiple of γ; hence $\lambda(\gamma, \phi)\in AP(G)$.

Finally notice that $\gamma*\chi$ is eather χ (if $\gamma=\chi$) or 0 (if $\gamma\neq\chi$), and that $\hat{h}_v(\gamma-\chi)=0$ if $\gamma\neq\chi$, provided that $V\in N$ is small enough. We therefore conclude that

$$(\gamma h_v)*\chi=\hat{h}_v(\chi-\gamma)\,\chi=\gamma*\chi$$

for all sufficiently small $V\in N$, which completes the proof. Q. E. D.

THEOREM 1. *Let G be a LCA group. For every translation-invariant weak*-closed subspace H of* $L^\infty(G)$, $H+AP(G)$ *is norm-closed in* $L^\infty(G)$.

PROOF OF THEOREM 1. Let H be a translation-inrariant weak*-closed suspace of $L^\infty(G)$, and let λ_f, $f\in AP(G)$, be as in Lemma 1. Thus each λ_f maps $L^\infty(G)$ into $AP(G)$. Moreover, λ_f maps H into H. In fact, if $\phi\in H$, then $\lambda_f(\phi)$ is an element of $AP(G)$ with spectrum contained in $\sigma(\phi)\subset\sigma(H)$, so that $\lambda_f(\phi)$ is in H (cf. 7. 8. 3 (e) [5]). Finally, choose a net $\{f_i\}$ in $AP(G)$ such that $\|f_i\|_1\leqq1$ for all i and $\lim_i\|f_i*g-g\|_\infty=0$ for all $g\in AP(G)$. Then we have $\|\lambda_{f_i}\|\leqq\|f_i\|_1\leqq1$ for all i and $\lim_i\|\lambda_{f_i}(g)-g\|_\infty=0$ for all $g\in AP(G)$. Therefore, by Theorem 1. 2 of [6], $H+AP(G)$ is norm-closed in $L^\infty(G)$.

Q. E. D.

Next, we investigate whether a space of type $H+AP(G)$ becomes an algebra.

THEOREM 2. *Let G be a noncompact LCA group. Suppose*

(i) *H is a translation-invariant, weak*-closed, proper subspace of* $L^\infty(G)$ *such that* $\sigma(H)$ *has nonempty interior; and*

(ii) *S is a norm-closed proper subalgebra of* $C_u(G)$ *such that* $AP(G)\subset S$ *and* $L^1(G)*S\subset S$.

Then the norm closure of H+S in $L^\infty(G)$ *does not form an algebra.*

PROOF. By (i), there exists a neigborhood V of $0\in\hat{G}$ and γ_1, $\gamma_2\in\hat{G}$ such that $\gamma_1+V\subset\sigma(H)$ and $(\gamma_2+V)\cap\sigma(H)=\phi$. Choose and fix any $f\in C_u(G)\cap S^c$. There is no lose of generality in assuming that $\sigma(f)$ is compact. Indeed, every $g\in C_u(G)$ can be approximated in norm by functions of the form $v*g$, where $v\in L^1(G)$ and supp $(\hat{v})$ is compact.

Choose $\{\chi_1, \chi_2, \cdots, \chi_n\}\subset\hat{G}$ so that $\sigma(f)\subset\{\chi_1, \chi_2, \cdots, \chi_n\}+V$. We can find $k_1, k_2, \cdots, k_n\in L^1(G)$ such that supp $(\hat{k}_j)\subset\chi_j+V$ for all j and $\sum_{j=1}^n\hat{k}_j=1$ in a neighborhood of $\sigma(f)$. Then, $\sum_{j=1}^n k_j*f=f\notin S$, so $k_j*f\notin S$ for some j. Replacing f by $\bar{\chi}_j(k_j*f)$, we may therefore assume that $\sigma(f)$ is a compact subset of V. Now notice that $\sigma(\gamma_1 f)=\gamma_1+\sigma(f)\subset\gamma_1+V\subset\sigma(H)$, so $\gamma_1 f$ belongs to H by (i).

Since $\gamma_2+\sigma(f)$ is a compact subset of $\hat{G}$ disjoint from $\sigma(H)$ there exists $k\in L^1(G)$ such that $\hat{k}=1$ in a neighborhood of $\gamma_2+\sigma(f)$ and $\operatorname{supp}(\hat{k})\cap\sigma(H)=\phi$. Then, $h\in H$ and $s\in S$ implies $\gamma_2 f+k*s=k*(\gamma_2 f+h+s)$. Hence, by (ii), we have

$$\begin{aligned}\inf\left\{\|\gamma_2 f+h+s\|_\infty;\ h\in H,\ s\in S\right\}\\ \geqq \inf\left\{\|\|\gamma_2 f+k*s\|_\infty/\|k\|_1;\ s\in S\right\}\\ >0\,.\end{aligned}$$

In other words, the element $\gamma_2 f=(\gamma_2\bar{\gamma}_1)(\gamma_1 f)$ is in SH but not in the closure of $S+H$. This completes the proof. Q. E. D.

After the first draft of this paper was written, Dr. S. Saeki pointed out that the results in that draft had nothing to do with the order structure of the group under the consideration. The present version was reorganized following his suggestions. The author appreciates his various comments and advices.

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