Hokkaido Mathematical Journal Vol. 9 (1980) p. 63-78

Polynomial rings over Krull orders in simple Artinian rings

Dedicated to professor Goro Azumaya
for his 60th birthday

By Hidetoshi MARUBAYASHI

(Received May 18, 1979)

Introduction

Let Q be a simple artinian ring. An order R in Q is called Krull if there are a family $\{R_i\}_{i\in I}$ and $S(R)$ of overrings of R satisfying the following:

(K 1) $R=\cap_{i\in I}R_i\cap S(R)$, where R_i and $S(R)$ are essential overrings of R (cf. Section 2 for the definition), and $S(R)$ is the Asano overrig of R;

(K 2) each R_i is a noetherian, local, Asano order in Q, and $S(R)$ is a noetherian, simple ring;

(K 3) if c is any regular element of R, then $cR_i\neq R_i$ for only finitely many i in I and $R_kc\neq R_k$ for only finitely many k in I.

If $S(R)=Q$, then R is said to be bounded. Author mainly investigated the ideal theory in bounded Krull orders in Q (cf. [10], [11], [12] and [13]). The class of Krull orders contains commutative Krull domains, maximal orders over Krull domains, noetherian Asano orders and bounded noetherian maximal orders. It is well known that if D is a commutative Krull domain, then the polynomial and formal power series rings $D[\boldsymbol{x}]$ and $D[[\boldsymbol{x}]]$ are both Krull, where the set $\boldsymbol{x}$ of indeterminates is finite or not.

The purpose of this paper is to show how the results above can be carried over to non commutative Krull orders by using prime v-ideals and localization functors. After giving some fundamental properties on polynomial rings (Section 1), we shall show, in Section 2, that if R is a Krull order in Q and if $\boldsymbol{x}$ is a finite set, then so is $R[\boldsymbol{x}]$. In case $\boldsymbol{x}$ is an infinite set, we can not show whether $R[\boldsymbol{x}]$ is Krull or not. But we shall show that $R[\boldsymbol{x}]$ satisfies some properties interesting in multiplicative ideal theory as follows:

(i) $R[\boldsymbol{x}]=\cap_P R[\boldsymbol{x}]_P\cap S(R[\boldsymbol{x}])$, where P ranges over all prime v-ideals of $R[\boldsymbol{x}]$, the local ring $R[\boldsymbol{x}]_P$ is a noetherian and Asano order in the quotient ring of $R[\boldsymbol{x}]$ and the Asano overring $S(R[\boldsymbol{x}])$ is a simple ring.

(ii) The integral v-ideals of $R[\boldsymbol{x}]$ satisfies the maximum condition.

In Section 4, we shall discuss on Krull orders over commutative Krull domains. If Λ is a Krull D-order, where D is a commutative Krull domain, then it is shown that $\Lambda[\boldsymbol{x}]$ and $\Lambda[[\boldsymbol{x}]]$ are both Krull $D[\boldsymbol{x}]$ and $D[[\boldsymbol{x}]]$-orders, respectively, where $\boldsymbol{x}$ is finite or not.

1. Preliminaries

Throughout this paper, each ring will be assumed to have an identity, Q will denote a simple artinian ring and R will denote an order in Q[1]. We refer to N. Jacobson [9] concerning the terminology on orders.

Let $\boldsymbol{x}=\{x_\alpha\}_{\alpha\in\boldsymbol{A}}$ be an arbitrary set of indeterminates over R subject to the condition that $rx_\alpha=x_\alpha r$ for any $r\in R$ and for any $x_\alpha\in\boldsymbol{x}$, where $\boldsymbol{A}$ is an index set. The polynomial ring $R[\boldsymbol{x}]$ is defined to be the union of the rings $R[\boldsymbol{x}']=R[x_{\alpha_1},\cdots,x_{\alpha_n}]$, where $\boldsymbol{x}'=\{x_{\alpha_i}\}_{i=1}^n$ ranges over all finite subsets of $\boldsymbol{x}$. If x is an indeterminate over R, then $Q[x]$ is a principal ideal ring by Example 6.3 of [16] and so it has a simple artinian quotient ring $Q(Q[x])$[2]. Since $Q[x]$ is an essential extension of $R[x]$ as $R[x]$-modules and $R[x]$ is a prime ring, $Q(R[x])=Q(Q[x])$. So $R[x]$ is an order in $Q(R[x])$. Therefore $R[\boldsymbol{x}']$ is also an order in $Q(R[\boldsymbol{x}'])$ for any finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$. Finally if $\boldsymbol{x}'$ and $\boldsymbol{x}''$ are subsets of $\boldsymbol{x}$ and if $\boldsymbol{x}'\subseteq\boldsymbol{x}''$, then we note that $Q(R[\boldsymbol{x}'])\subseteq Q(R[\boldsymbol{x}''])$.

LEMMA 1.1. *Let R be an order in Q. Then $R[\boldsymbol{x}]$ has a simple artinian quotient ring and $Q(R[\boldsymbol{x}])=\cup Q(R[\boldsymbol{x}'])$, where $\boldsymbol{x}'$* runs over all finite subsets of $\boldsymbol{x}$, *and* $\dim R=\dim R[\boldsymbol{x}]$ ($\dim R$ *is always the Goldie dimension of* R).

PROOF. The lemma will be proved in four steps.

(i) Let A and B be any non-zero ideals of $R[\boldsymbol{x}]$. There exists a finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$ such that $A\cap R[\boldsymbol{x}']\neq 0$ and $B\cap R[\boldsymbol{x}']\neq 0$, because $A=\cup(A\cap R[\boldsymbol{x}''])$, where $\boldsymbol{x}''$ ranges over all finite subsets of $\boldsymbol{x}$. Since $R[\boldsymbol{x}']$ is a prime ring, we get $0\neq(A\cap R[\boldsymbol{x}'])(B\cap R[\boldsymbol{x}'])\subseteq AB$. Hence $R[\boldsymbol{x}]$ is a prime ring. It is evident that the ring $S=\cup Q(R[\boldsymbol{x}'])$ is an essential extension of $R[\boldsymbol{x}]$ as $R[\boldsymbol{x}]$-modules.

(ii) If $\dim R=n$, then we shall prove that $\dim R[\boldsymbol{x}']=n$ for any finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$. It suffices to prove that $\dim R[x]=n$. Since Q is the total matrix ring $(K)_n$ over a division ring K, we have $Q[x]=(K)_n[x]\simeq(K[x])_n$ and $K[x]$ is an Ore domain. Hence $n=\dim Q[x]=\dim R[x]$, because $Q(Q[x])=Q(R[x])$.

---

1) Conditions assumed on rings will always be assumed to hold on both sided; for example, an order always means a right and left order.

2) The quotient ring of a ring T will be denoted by $Q(T)$.

(iii) If U is a uniform right ideal of R, then $U[\boldsymbol{x}']$ is also a uniform right ideal of $R[\boldsymbol{x}']$ by (ii) for any finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$ and so $UQ(R[\boldsymbol{x}'])$ is a minimal right ideal of $Q(R[\boldsymbol{x}'])$. It follows that US is a minimal right ideal of S.

(iv) If $U_1\oplus\cdots\oplus U_n$ is an essential right ideal of R, where U_i are uniform right ideals of R, then, since $(U_1\oplus\cdots\oplus U_n)Q=Q$, we have $S=(U_1\oplus\cdots\oplus U_n)S=U_1S\oplus\cdots\oplus U_nS$ and the U_iS are minimal right ideals of S by (iii). Hence S is a simple artinian ring and is the classical right quotient ring of $R[\boldsymbol{x}]$. Similarly, it is the classical left quotient ring of $R[\boldsymbol{x}]$. It is evident that $\dim R=\dim R[\boldsymbol{x}]$.

LEMMA 1. 2. *Let A be a non-zero ideal of $R[x]$ and let $r(x)$, $c(x)=c_nx^n+\cdots+c_0$ be elements of $R[x]$ such that c_n is a regular element of R. If $r(x)A\subseteq c(x)A$ and $\deg r(x)<\deg c(x)$, then $r(x)=0$ ($\deg r(x)$ is the degree of the polynomial $r(x)$).*

PROOF. Let k be the minimum number of the set $\{\deg f(x)\mid A\ni f(x)\neq 0\}$ and $A_0=\{f(x)\in A\mid \deg f(x)=k\}\cup\{0\}$. Then it is an (R, R)-bimodule and so $A_0R[x]$ is an ideal of $R[x]$. If $r(x)\neq 0$, then $0\neq r(x)A_0R[x]\subseteq c(x)A$. For any non-zero element $r(x)f(x)$ ($f(x)\in A_0$), we have $\deg r(x)f(x)\leq \deg r(x)+k$. But the degree of non-zero element of $c(x)A$ is larger that $\deg r(x)+k$, because c_n is a regular element of R and $\deg r(x)<\deg c(x)$. This contradiction implies that $r(x)=0$.

PROPOSITION 1. 3. *If R is a maximal order in Q, then $R[\boldsymbol{x}]$ is a maximal order in $Q(R[\boldsymbol{x}])$.*

PROOF. Firstly we shall prove the assertion in case $\boldsymbol{x}=\{x\}$. Let A be any non-zero ideal of $R[x]$ and let B be the ideal of all leading coefficients of polynomials in A. Let $q=c(x)^{-1}r(x)$ be any non-zero element of $O_l(A)=\{q\in Q(R[x])\mid qA\subseteq A\}$, the left order of A, and let $c(x)=c_nx^n+\cdots+c_0$, $r(x)=r_mx^m+\cdots+r_0$ be non-zero elements in $R[x]$. By the same way as in Lemma 2 of [17], we may assume that c_n is a regular element of R. Since $r(x)A\subseteq c(x)A$, we have $r_mB\subseteq c_nB$ and $c_n^{-1}r_m\in O_l(B)=R$ by Lemma 1. 2 of [2]. Thus $r_m=c_ns_{m-n}$ for some $s_{m-n}\in R$. By Lemma 1. 2, $n\leq m$ and so $r(x)=c(x)t_1(x)+r_1(x)$, where $t_1(x)=s_{m-n}x^{m-n}$, $r_1(x)\in R[x]$ and $\deg r_1(x)<m$, *i. e.*, $c(x)^{-1}r(x)=t_1(x)+c(x)^{-1}r_1(x)$. Hence $(t_1(x)+c(x)^{-1}r_1(x))A\subseteq A$ and $c(x)^{-1}r_1(x)A\subseteq A$. If $n\leq \deg r_1(x)$, then the process is repeated and we get $r_1(x)=c(x)t_2(x)+r_2(x)$ ($t_2(x), r_2(x)\in R[x]$), $\deg r_1(x)>\deg r_2(x)$ and $c(x)^{-1}r_2(x)A\subseteq A$. Continuing the process we obtain $r_i(x)=c(x)t_{i+1}(x)+r_{i+1}(x)$ ($t_{i+1}(x)$, $r_{i+1}(x)\in R[x]$), $\deg r_{i+1}(x)<\deg c(x)$ and $c(x)^{-1}r_{i+1}(x)A\subseteq A$. Then, by Lemma 1. 2, $r_{i+1}(x)=0$ and therefore $c(x)^{-1}r(x)=t_1(x)+\cdots+t_{i+1}(x)\in R[x]$. This implies that $O_l(A)=R[x]$ and, by symmetry, $R[x]=O_r(A)$, the right order of A. Hence $R[x]$ is a maximal order in $Q(R[x])$ by the remark to Lemma 1.2 of [2]. By induction, $R[\boldsymbol{x}']$ is a maximal order in $Q(R[\boldsymbol{x}'])$ for any finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$. Nextly we shall prove the assertion in case $\boldsymbol{x}$ is arbitrary. Let A be any non-zero ideal of $R[\boldsymbol{x}]$ and let q be any element of $O_l(A)$. Then there exists a finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$ such that $q \in Q(R[\boldsymbol{x}'])$ and $0 \neq A \cap R[\boldsymbol{x}']$. It follows that $q(A \cap R[\boldsymbol{x}']) \subseteq A \cap Q(R[\boldsymbol{x}']) = A \cap R[\boldsymbol{x}']$ and so $q \in O_l((A \cap R[\boldsymbol{x}'])) = R[\boldsymbol{x}']$. Hence $O_l(A) = R[\boldsymbol{x}]$ and, by symmetry, $O_r(A) = R[\boldsymbol{x}]$. This implies that $R[\boldsymbol{x}]$ is a maximal order in $Q(R[\boldsymbol{x}])$.

Let I be a right R-ideal. Following [1], we define $I^* = (I^{-1})^{-1}$. If $I = I^*$, then it is said to be a *right v-ideal*. In the same way one defines *left v-ideals* and *v-ideals*.

LEMMA 1.4. *If R is a maximal order in Q and if I is a (one-sided) R-ideal, then $I^{-1}[\boldsymbol{x}] = (I[\boldsymbol{x}])^{-1}$. In particular, if I is a (one-sided) v-ideal, then so is $I[\boldsymbol{x}]$.*

PROOF. We shall prove the lemma when I is a right R-ideal. Since $I^{-1}[\boldsymbol{x}]\, I[\boldsymbol{x}] \subseteq R[\boldsymbol{x}]$, *we get* $I^{-1}[\boldsymbol{x}] \subseteq (I[\boldsymbol{x}])^{-1}$. To prove the inverse inclusion, let q be any element in $(I[\boldsymbol{x}])^{-1}$, *i.e.*, $qI[\boldsymbol{x}] \subseteq R[\boldsymbol{x}]$. Since $qc \in R[\boldsymbol{x}]$ for any regular element c in I, $q \in Rc^{-1}[\boldsymbol{x}] \subseteq Q[\boldsymbol{x}]$. Therefore all coefficients of q (as polynomials over Q) are contained in I^{-1} and so $q \in I^{-1}[\boldsymbol{x}]$. Hence $I^{-1}[\boldsymbol{x}] = (I[\boldsymbol{x}])^{-1}$, as desired.

2. $\boldsymbol{R}[x]$

Let R be an order in Q and let F be a right additive topology on R. We denote by R_F the ring of quotients of with respect to F (cf. [18]). An overring R' of R is said to be *right essential* if it satisfies the following two conditions:

(i) There is a perfect right additive topology F on R such that $R' = R_F$ (cf. p 74 of [18]).

(ii) If $I \in F$, then $R'I = R'$.

If R_F is a right essential overring of R, then F consists of all right ideals I of R such that $IR_F = R_F$. So each element of F is an essential right ideal of R. So if R is a maximal order in Q, then $R_F = \cup^{-1}(I \in F)$.

An overring R' of R is said to be *essential* if it is right and left essential. If P is a prime ideal of R, then we denote by $C(P)$ those elements of R which are regular mod. (P). If R satisfies the Ore condition with respect to $C(P)$, then we denote by R_P the ring of quotients of R with respect to $C(P)$. We call an order R an *Asano order* if its R-ideals form a group under multiplication. An order R is said to be *local* if its Jacobson radical J is the unique maximal ideal and R/J is an artinian ring. Let R be a noetherian, local and Asano order. Then, by Proposition 1.3 of [8], R is a bounded, hereditary, principal right and left ideal ring. Following [8], we define $S(R)=\cup B^{-1}$, where B ranges over all non-zero ideals of R and call it an *Asano overring* of R.

Let R be a maximal order in Q and let P be an ideal of R. Then the following are equivalent (cf. p. 11 and Theorem 4.2 of [1]):

(i) P is a prime v-ideal of R.

(ii) P is a maximal element in the lattice of integral v-ideals of R.

(iii) P is a meet-irreducible in the lattice of integral v-ideals of R.

If P satisfies one of the conditions above, then it is a minimal prime ideal of R by Theorem 1.6 of [2]. The set $D(R)$ of all v-ideals becomes an abelian group under the multiplication "$\circ$" defined by $A^*\circ B^*=(AB)^*=(A^*B)^*=((AB^*))=(A^*B^*)^*$ for any R-ideals A and B (cf. Lemma 2 of [12]). If the integral v-ideals satisfies the maximum condition, then $D(R)$ is a direct product of infinite cyclic groups with prime v-ideals as their generators (cf. Theorem 4.2 of [1]). These results are frequently used in this paper without references.

An order R in Q is called *Krull* if there are a family $\{R_i\}_{i\in \boldsymbol{I}}$ and $S(R)$ of overrings of R satisfying the following:

(K 1) $R=\cap_{i\in \boldsymbol{I}} R_i \cap S(R)$, where R_i and the Asano overring $S(R)$ are essential overrings of R,

(K 2) each R_i is a noetherian, local, Asano order, and $S(R)$ is a noetherian, simple ring, and

(K 3) for every regular element c in R we have $cR_i\neq R_i$ for only finitely many i in $\boldsymbol{I}$ and $R_kc\neq R_k$ for only finitely many k in $\boldsymbol{I}$.

If R is a Krull order in Q, then it is a Krull ring in the sense of [10]. In non-commutative rings, it seems to me that the definition above is more natural than one of Krull rings in [10].

In this section, P_i' will denote the unique maximal ideal of R_i and $P_i=P_i'\cap R$ $(i\in \boldsymbol{I})$. By Proposition 1.1 of [10], P_i is a prime ideal of R and $R_i=R_{P_i}$.

PROPOSITION 2.1. *Let R be a Krull order in Q. Then*

(1) *R is a maximal order in Q.*

(2) *The integral right and left v-ideals satisfy the maximum condition.*

(3) *If A is a non-zero ideal of R, then $AS(R)=S(R)A=S(R)$.*

(4) *Let P be an ideal of R. Then it is a prime v-ideal of R if and only if $P=P_i$ for some i in $\boldsymbol{I}$.*

PROOF. Since a simple ring is a maximal order, (1) follows from the same way as in Proposition 1.3 of [11].

(2) Let I be any right v-ideal. Then $I=\cap_i IR_i \cap IS(R)$ by Corollary 4.2 of [10]. So (2) is evident from the definition of Krull orders.

(3) Let $S(R)=R_F=R_{F_l}$, where F and F_l are perfect right and left additive topologies on R, respectively. Since $S(R)AS(R)=S(R)$, we write $1=\Sigma_{i=1}^n t_i a_i s_i$, where t_i, $s_i \in S(R)$ and $a_i \in A$. There are elements B and C in F and F_l respectively, such that Ct_i, $s_i B \subseteq R$. So $CB \subseteq A$, which implies that $S(R) \supseteq S(R)A \supseteq S(R)CB=S(R)$. Hence $S(R)=S(R)A$ and, by symmetry $S(R)=AS(R)$.

(4) Let P be a prime v-ideal. Then $P=\cap_i PR_i \cap S(R)$. There are finitely many $1, \cdots, k \in \boldsymbol{I}$ only such that $PR_i \neq R_i$ $(1 \leq i \leq k)$. Since R_i is bounded, there are natural numbers n_i such that $P_i'^{n_i} \subseteq PR_i$. It follows that $P_1^{n_1} \cap \cdots \cap P_k^{n_k} \subseteq P$. Hence $P_i \subseteq P$ for some i and thus $P_i=P$. The fact that each P_i is a prime v-ideal follows from the same way as in Lemma 1.5 of [11].

LEMMA 2.2. *Let R be a maximal order in Q and let $S(R)$ be the Asano overring of R. If $AS(R)=S(R)=S(R)A$ for every non-zero ideal A of R, then $S(R)$ is an essential overring of R and is a simple ring.*

PROOF. Let $F=\{I \mid I$ is a right ideal of R and contains a non-zero ideal of $R\}$. We shall prove that F is a right additive topology on R. To prove this let I be any element of F and let A be a non-zero ideal of R such that $I \supseteq A$. Then, for any $r \in R$, we have $r^{-1}I=\{x \in R \mid rx \in I\} \supseteq r^{-1}A \supseteq A$ and so $r^{-1}I \in F$. If $I \in F$ and J is a right ideal of R such that $a^{-1}J \in F$ for all $a \in I$, then we obtain $S(R) \supseteq JS(R) \supseteq \Sigma_{a \in I} a(a^{-1}J)S(R)=\Sigma_{a \in I} aS(R)=IS(R)$ $=S(R)$. Hence $S(R)=JS(R)$. Put $1=\Sigma_{i=1}^n a_i t_i$, where $a_i \in J$ and $t_i \in S(R)$. There is a non-zero ideal B of R such that $t_i B \subseteq R$. It follows that $B \subseteq J$ and $J \in F$. Thus F is a right additive topology on R by Lemma 3.1 of [18]. By the assumption, it is clear that $S(R)=R_F$ and that it is a right essential overring of R. By symmetry, $S(R)$ is a left essential overring of R and therefore it is an essential overring of R. It is clear that $S(R)$ is a simple ring.

LEMMA 2.3. *Let R be an order in Q and let R be a simple ring. Then*

(1) *The correspondence*

$$(*) \qquad P \longrightarrow P'=PQ[x]$$

is one-to-one between the family of all maximal ideals of $R[x]$ and the family of all maximal ideals of $Q[x]$. The inverse of (*) is given by *the correspondence* $P' \to P' \cap R[x]$.

(2) $R[x]_P = Q[x]_{P'}$, *and is a noetherian, local, Asano order for every maximal ideal* P *of* $R[x]$.

(3) $S(R[x])$ *is an essential overring of* $R[x]$, *is a simple ring and* $S(R[x]) \subseteq S(Q[x])$. *In particular, if* R *is noetherian, then so is* $S(R[x])$.

PROOF. The same proof as in Example 6.1 of [16] gives that $R[x]$ is an ipri and ipli-ring. So $R[x]$ is an Asano order in $Q(R[x])$.

(1) Let P' be a maximal ideal of $Q[x]$ and $P = P' \cap R[x]$. It is evident that P is a maximal ideal of $R[x]$. Since $Q[x]$ is an essential overring of $R[x]$ by Lemma 5.3 of [10], we have $P' = PQ[x] = Q[x]P$. Conversely let P be a maximal ideal of $R[x]$ and let $P' = Q[x]PQ[x]$. Assume that $P' = Q[x]$ and write $1 = \Sigma_{i=1}^{n} q_i p_i g_i$, where q_i, $g_i \in Q[x]$ and $p_i \in P$. There are regular elements c, d in R such that cq_i, $g_i d \in R[x]$. It follows that $R = RcdR \subseteq P$, which is a contradiction. Hence P' is a proper ideal of $Q[x]$ so that $P' \cap R[x]$ is also a proper ideal of $R[x]$. This implies that $P = P' \cap R[x]$ and thus $P' = PQ[x] = Q[x]P$, since $Q[x]$ is an essential overring of $R[x]$. It is clear that P' is a maximal ideal of $Q[x]$.

(2) By Example 6.3 of [16], $Q[x]$ is a Dedekind prime ring. So $Q[x]_{P'}$ is a noetherian, local, Asano order in $Q(R[x])$ by Theroem 2.6 of [8]. Since $P = P' \cap R[x]$, we get $Q[x]_{P'} = R[x]_P$ by Proposition 1.1 and Lemmas 5.2, 5.3 of [10].

(3) Since $R[x]$ is an Asano order in $Q(R[x])$, $S(R[x])$ is an essential overring of $R[x]$ and is a simple ring by Lemma 2.2. Let $A = P_1^{n_1} \cdots P_t^{n_t}$ be any non-zero ideal of $R[x]$, where P_i are maximal ideals of $R[x]$. Then we get $A^{-1} \subseteq Q[x]A^{-1} = (AQ[x])^{-1} = (P_1'^{n_1} \cdots P_t'^{n_t})^{-1} \subseteq S(Q[x])$. Hence $S(R[x]) \subseteq S(Q[x])$. If R is a noetherian and simple ring, then so is $S(R[x])$ by [8, p. 446], because $R[x]$ is a noetherian Asano order.

THEOREM 2.4. *If* R *is a Krull order in* Q, *then* $R[x]$ *is a Krull order in* $Q(R[x])$.

PROOF. Let $R = \cap_i R_{P_i} \cap S$ $(i \in \boldsymbol{I})$, where P_i ranges over all prime v-ideals of R and $S = S(R)$ is the Asano overring of R. Then $R[x] = \cap_i R[x]_{P_i[x]} \cap Q[x] \cap S[x]$ by the proof of Theorem 5.4 of [10]. Since $Q[x]$ and $S[x]$ are both noetherian Asano orders by Example 6.1 of [16], we obtain $Q[x] = \cap_{j \in \boldsymbol{J}} Q_j^* \cap S(Q[x])$ and $S[x] = \cap_{j \in \boldsymbol{J}} S_j^* \cap S(S[x])$ by Theorem 3.1 of [8]. Here $Q_j^* = S_j^*$ are noetherian, local, Asano orders, $S(S[x]) \subseteq S(Q[x])$, and $S(S[x])$ is a noetherian, simple ring and is an essential overring of $R[x]$ by Lemmas 5.2, 5.3 of [10] and Lemma 2.3. Let Q_j' be the unique maximal ideal of Q_j^* $(j \in \boldsymbol{J})$. We consider the following diagram;

$$\begin{array}{ccccccc} R[x] & \subseteq & S[x] & \subseteq & Q[x] & \subseteq & Q_j^* \\ \cup & & \cup & & \cup & & \cup \\ Q_j & \subseteq & Q_j''' & \subseteq & Q_j'' & \subseteq & Q_j' \end{array}$$

where $Q_j = R[x] \cap Q_j'$, $Q_j''' = S[x] \cap Q_j'$ and $Q_j'' = Q[x] \cap Q_j'$. Then $Q_j^* = R[x]_{Q_j}$ by Proposition 1.1 of [10]. Thus we have

$$(*) \qquad R[x] = \cap_{i\in I} R[x]_{P_i[x]} \cap \cap_{j\in J} R[x]_{Q_j} \cap S(S[x]).$$

In the expression (*) of $R[x]$, we get, as in Theorem 5.4 of [10] amd Proposition 2.1, the following:

(i) $R[x]$ satisfies the condition (K 3).

(ii) The integral one-sided v-ideals satisfies the maximum condition.

(iii) $P_i[x]$, $Q_j(i\in I, j\in J)$ are all prime v-ideals of $R[x]$.

To prove that these only are prime v-ideals of $R[x]$, let P be a prime v-ideal of $R[x]$. If $P\cap R\neq 0$, then, since $(P\cap R)^*[x]=((P\cap R)[x])^*\subseteq P^*=P$ by Lemma 1.4, $P\cap R$ is also a prime v-ideal of R so that $P\cap R=P_i$ for some $i\in I$ by Proposition 2.1. Hence $P\supseteq P_i[x]$ and thus $P=P_i[x]$. If $P\cap R=0$, then it follows that $Q[x]PQ[x]\subsetneqq Q[x]$, and so $Q[x]PQ[x]\subseteq Q_j''$ for some $j\in J$. Since $\{Q_j''|j\in J\}$ are the set of maximal ideals of $Q[x]$. Hence $P\subseteq Q_j$ so that $P=Q_j$, as claimed. It remains to prove that $S(S[x])=S(R[x])$. To prove this let A be a non-zero ideal of $R[x]$. We write $A^*=(P_1[x]^{m_1}\cdots P_s[x]^{m_s}\cdot Q_1^{n_1}\cdots Q_t^{n_t})^*$. Then $S[x]\supseteq A^*S[x]\supseteq Q_1^{n_1}\cdots Q_t^{n_t}S[x] =Q_1'''^{n_1}\cdots Q_t'''^{n_t}$ by Proposition 2.1. Thus we have $S(S[x])\supseteq A^*S(S[x])\supseteq Q_1'''^{n_1}\cdots Q_t'''^{n_t}S(S[x])=S(S[x])$ and so $S(S[x])=A^*S(S[x])$. It follows that $A^{-1}\subseteq A^{-1}S(S[x])=A^{-1}A^*S(S[x])\subseteq S(S[x])$. Hence $S(R[x])\subseteq S(S[x])$. To prove the inverse inclusion, let q be any element of $S(S[x])$. We may assume that q is a regular element in $Q(R[x])$ by Lemma 2.2 of [10]. There is a non-zero ideal B' of $S[x]$ such that $qB'\subseteq S[x]$ and so $qB\subseteq S[x]$, where $B=B'\cap R[x]$, Write $B^*=(b_1R[x]+\cdots+b_nR[x])^*$ for some elements b_i of B. Then there exists a non-zero ideal C of R such that $qb_iC\subseteq R[x]$ so that $qb_iC[x]\subseteq R[x]$. It follows that $q(b_1R[x]+\cdots+b_nR[x])C[x]\subseteq R[x]$ and thus we have $R[x]\supseteq(q(b_1R[x]+\cdots+b_nR[x])C[x])^*=q((b_1R[x]+\cdots+b_nR[x])^*C[x])^*=q(B^*C[x])^*=q(BC[x])^*$ by Lemma 2 of [12], which implies $q\in(BC[x])^{-1}\subseteq S(R[x])$. Hence $S(R[x])\supseteq S(S[x])$ and $S(R[x])=S(S[x])$, as desired.

COROLLARY 2.5. *If R is a Krull order in Q, then $R[x_1,\cdots,x_n]$ is a Krull order in $Q(R[x_1,\cdots,x_n])$.*

3. $R[x]$

In the remainder of this paper, $x=\{x_\alpha|\alpha\in A\}$ denotes an arbitrary set of indeterminates over R which commutes with any element of R. We shall study, in this section, the polynomial ring $R[x]$ over Krull order R.

LEMMA 3.1. *Let R be a Krull order in Q and let P be a prime v-ideal of R. Then*

(1) *$R[x]$ satisfies the Ore condition with respect to $C(P[x])$ and $R[x]_{P[x]}=\cup_{x'}R[x']_{P[x']}$, where x' ranges over all finite subsets of x.*

(2) *$R[x]_{P[x]}$ is a neotherian, local and Asano order in $Q(R[x])$.*

PROOF. (1) Let x' and x'' be any finite subsets of x such that $x'\subsetneqq x''$. Since $R[x'']=R[x'][x''-x']$ and $P[x'']=P[x'][x''-x']$, where $x''-x'$ is the complement set of x' in x'', it is evident that $C(P[x'])\subseteq C(P[x''])$. Firstly we shall prove that $C(P[x])=\cup_{x_0'}C(P[x_0'])$, where x_0' ranges over all finite subsets of x. If $c(x')f(x)\in P[x]$, where x' is a finite subset of x, $c(x')\in C(P[x'])$ and $f(x)\in R[x]$, then there exists a finite subset x'' ($\supseteq x'$) of x such that $f(x)\in R[x'']$ and $c(x')f(x)\in P[x'']$. Hence $f(x)\in P[x'']$ and so $C(P[x'])\subseteq C(P[x])$. Conversely, let $c(x)$ be any element of $C(P[x])$ and assume that $c(x)\in R[x']$. If $c(x)g(x)\in P[x']$, where $g(x)\in R[x']$, then $g(x)\in R[x']\cap P[x]=P[x']$. This implies that $c(x)\in C(P[x'])$. Hence $C(P[x])=\cup_{x_0'}C(P[x_0'])$. Next we shall prove that $R[x]$ satisfies the Ore condition with respect to $C(P[x])$. To prove this let $c(x)$ and $a(x)$ be any element of $R[x]$ with $c(x)\in C(P[x])$. Then there is a finite subset x' of x such that $a(x)$, $c(x)\in R[x']$. By Proposition 2.1 and Corollary 2.5, there exist $b(x)$, $d(x)$ in $R[x']$ and $d(x)\in C(P[x'])$ such that $a(x)d(x)=c(x)b(x)$. Hence $R[x]$ satisfies the right Ore condition with respect to $C(P[x])$ and $R[x]_{P[x]}=\cup_{x'}R[x']_{P[x']}$. The other Ore condition is shown to hold by a symmetric proof.

(2) Let P' be the unique maximal ideal of R_P and let x' be any finite subset of x. Since $R[x']_{P[x']}$ is a noetherian, local and Asano order, we obtain that $P[x']R[x']_{P[x']}=R[x']_{P[x']}P[x']$ and that it is the Jacobson radical of $R[x']_{P[x']}$. Let $P'=pR_P=R_Pp$ for some regular element p in P. Then we have $pR[x']_{P[x']}=P[x']R[x']_{P[x']}=R[x']_{P[x']}p$, because $R[x']_{P[x']}=(R_P[x'])_{P'[x']}$. Put $P''=P[x]R[x]_{P[x]}$. Then we obtain that $P''=pR[x]_{P[x]}=\cup_{x''}(pR[x'']_{P[x'']})$ $=\cup(R[x'']_{P[x'']}p)=R[x]_{P[x]}p=R[x]_{P[x]}P[x]$, where x'' ranges over all finite subsets of x. Hence P'' is an ideal of $R[x]_{P[x]}$ and is invertible. It is evident that $P''\cap R[x]_{P[x]}=P[x]$. Since $R[x]/P[x]\simeq R/P[x]$ and $R[x]_{P[x]}/P''$ is the quotient ring of $R[x]/P[x]$, it follows that $R[x]_{P[x]}/P''$ is a simple, artinian ring. So P'' is a maxima ideal of $R[x]_{P[x]}$. To prove that P'' is the

Jacobson radical of $R[\boldsymbol{x}]_{P[\boldsymbol{x}]}$, let V be any maximal right ideal of $R[\boldsymbol{x}]_{P[\boldsymbol{x}]}$. Assume that $V\not\supseteq P''$. Then $R[\boldsymbol{x}]_{P[\boldsymbol{x}]}=V+P''$. Write $1=v+p'$, where $v\in V$ and $p'\in P''$. There is a finite subset $\boldsymbol{x}''$ of $\boldsymbol{x}$ such that $v\in R[\boldsymbol{x}'']_{P[\boldsymbol{x}'']}$ and $p'\in P[\boldsymbol{x}'']\,R[\boldsymbol{x}'']_{P[\boldsymbol{x}'']}$. Then v is a unit in $R[\boldsymbol{x}'']_{P[\boldsymbol{x}'']}$ and so it is a unit in $R[\boldsymbol{x}]_{P[\boldsymbol{x}]}$. Thus we get $V=R[\boldsymbol{x}]_{P[\boldsymbol{x}]}$, which is a contradiction. Hence $V\supseteq P''$ and so P'' is the Jacobson radical of $R[\boldsymbol{x}]_{P[\boldsymbol{x}]}$. Let I be any essential right ideal of $R[\boldsymbol{x}]_{P[\boldsymbol{x}]}$. Then there is a finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$ such that $I\cap R[\boldsymbol{x}']_{P[\boldsymbol{x}']}$ is an essential right ideal of $R[\boldsymbol{x}']_{P[\boldsymbol{x}']}$. It follows that $I\cap R[\boldsymbol{x}']_{P[\boldsymbol{x}']}\supseteq(P[\boldsymbol{x}']\,R[\boldsymbol{x}']_{P[\boldsymbol{x}']})^n$ for some natural number n. Hence we have $I\supseteq P''^n$. this implies that the essential right ideals of $R[\boldsymbol{x}]_{P[\boldsymbol{x}]}$ satisfies the maximum condition, because $R[\boldsymbol{x}]_{P[\boldsymbol{x}]}/P''$ is artinian and P'' is invertible. Further, since dim $R[\boldsymbol{x}]_{P[\boldsymbol{x}]}$ is finite, $R[\boldsymbol{x}]_{P[\boldsymbol{x}]}$ is right noetherian. Similarly, it is left noetherian. Hence $R[\boldsymbol{x}]_{P[\boldsymbol{x}]}$ is a noetherian, local and Asano order in $Q(R[\boldsymbol{x}])$ by Proposition 1. 3 of [8].

Let I be a right $R[\boldsymbol{x}]$-ideal. Then $qI\subseteq I$ for some regular element q in $Q(R[\boldsymbol{x}])$. There is a finite subset $\boldsymbol{x}_0'$ of $\boldsymbol{x}$ such that $q\in Q(R[\boldsymbol{x}_0'])$ and $I\cap Q(R[\boldsymbol{x}_0'])$ is a right $R[\boldsymbol{x}_0']$-ideal, because $I=\cup(I\cap Q(R[\boldsymbol{x}']),$ where $\boldsymbol{x}'$ runs over all finite subsets of $\boldsymbol{x}$. For any finite subset $\boldsymbol{x}''$ of $\boldsymbol{x}$ with $\boldsymbol{x}''\supseteq\boldsymbol{x}_0'$, $I\cap Q(R[\boldsymbol{x}''])$ is a right $R[\boldsymbol{x}'']$-ideal. Thus we have $I=\cup(I\cap Q(R[\boldsymbol{x}']))$. Here $\boldsymbol{x}'$ ranges over all finite subsets of $\boldsymbol{x}$ such that each $I\cap Q(R[\boldsymbol{x}'])$ is a right $R[\boldsymbol{x}']$-ideal. We define $\tilde{I}=\cup(I\cap Q(R[\boldsymbol{x}']))^*$. Clearly $I\subseteq\tilde{I}$ and especially, for right v-ideals, we have

LEMMA 3. 2. *Let R be a maximal order in Q and let I be a right v-ideal of $Q(R[\boldsymbol{x}])$. Then $I=\tilde{I}$.*

PROOF. Let c be a unit in $Q(R[\boldsymbol{x}])$. It is evident that $cR[\boldsymbol{x}]=c\tilde{R}[\boldsymbol{x}]$. So the lemma immediately follows from Proposition 4. 1 of [10].

LEMMA 3. 3. *Let R be a maximal order in Q and let P be a proper ideal of $R[\boldsymbol{x}]$. Then P is a prime v-ideal if and only if $P=P'[\boldsymbol{x}-\boldsymbol{x}']$, where $\boldsymbol{x}'$ is a finite subset of $\boldsymbol{x}$ and P' is a prime v-ideal of $R[\boldsymbol{x}']$.*

PROOF. The sufficiency is clear from Lemma 1. 4. Assume that P is a prime v-ideal. There is a finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$ such that $P\cap R[\boldsymbol{x}']$ is a non-zero. It is a prime ideal of $R[\boldsymbol{x}']$. If $(P\cap R[\boldsymbol{x}'])^*=R[\boldsymbol{x}']$, then $P=R[\boldsymbol{x}]$ by Lemma 3. 2, which is a contradiction. Hence $(P\cap R[\boldsymbol{x}'])^*\subsetneqq R[\boldsymbol{x}']$ so that $P\cap R[\boldsymbol{x}']$ is a prime v-ideal of $R[\boldsymbol{x}']$ by Proposition 1. 7 of [2]. Thus $(P\cap R[\boldsymbol{x}'])[\boldsymbol{x}-\boldsymbol{x}']$ is a prime v-ideal of $R[\boldsymbol{x}]$ contained in P. Therefore $P=(P\cap R[\boldsymbol{x}'])[\boldsymbol{x}-\boldsymbol{x}']$, as desired.

LEMMA 3. 4. *Let R be a Krull order in Q. Then the integral v-ideals of $R[\boldsymbol{x}]$ satisfies the maximum condition.*

PROOF. Let $P_1, \cdots, P_s$ be any prime v-ideals of $R[\boldsymbol{x}]$ and let $n_1, \cdots, n_s$ be any natural numbers. Then we obtain by the same as in Asano orders that the integral v-ideals containing $(P_1^{n_1}\cdots P_s^{n_s})^*$ are the ideals $(P_1^{m_1}\cdots P_s^{m_s})^*$ only $(0\leq m_i\leq n_i)$. So it suffices to prove that any integral v-ideal of $R[\boldsymbol{x}]$ contains an integral v-ideal of such forms. To prove this let A be any proper integral v-ideal of $R[\boldsymbol{x}]$. There exists a finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$ such that $(A\cap R[\boldsymbol{x}'])^*$ is a proper integral v-ideal of $R[\boldsymbol{x}']$. Write $(A\cap R[\boldsymbol{x}'])^*=(P_1^{n_1}\cdots P_t^{n_t})^*$, where P_i are prime v-ideals of $R[\boldsymbol{x}']$. By Lemmas 1.4 and 3.2, we get $A\supseteq(A\cap R[\boldsymbol{x}'])^*[\boldsymbol{x}-\boldsymbol{x}']=(P_1^{n_1}\cdots P_t^{n_t}[\boldsymbol{x}-\boldsymbol{x}'])^*=((P_1[\boldsymbol{x}-\boldsymbol{x}'])^{n_1}\cdots(P_t[\boldsymbol{x}-\boldsymbol{x}'])^{n_t})^*$. Each $P_i[\boldsymbol{x}-\boldsymbol{x}']$ is a prime v-ideal of $R[\boldsymbol{x}]$ by Lemma 3.3.

LEMMA 3.5. *Let R be a Krull order in Q. Then $S(R[\boldsymbol{x}])=\cup_{\boldsymbol{x}'}S(R[\boldsymbol{x}'])$, where $\boldsymbol{x}'$ ranges over all finite subsets of $\boldsymbol{x}'$, it is an essential overring of $R[\boldsymbol{x}]$ and is a simple ring.*

PROOF. Let A be any non-zero ideal of $R[\boldsymbol{x}']$, where $\boldsymbol{x}'$ is a finite subset of $\boldsymbol{x}$. Then we have $A^{-1}\subseteq A^{-1}[\boldsymbol{x}-\boldsymbol{x}']=(A[\boldsymbol{x}-\boldsymbol{x}'])^{-1}$ and $A[\boldsymbol{x}-\boldsymbol{x}']$ is an ideal of $R[\boldsymbol{x}]$. Hence $S(R[\boldsymbol{x}])\supseteq\cup_{\boldsymbol{x}'}S(R[\boldsymbol{x}'])$. Conversely let q be any element of $S(R[\boldsymbol{x}])$. There is an ideal B of $R[\boldsymbol{x}]$ such that $qB\subseteq R[\boldsymbol{x}]$. Since $B^{-1^{-1}-1}=B^{-1}$, we may assume that B is a v-ideal. Write $B=(P_1^{n_1}\cdots P_t^{n_t})^*$, where P_i are prime v-ideals of $R[\boldsymbol{x}]$. There are finite subsets $\boldsymbol{x}'$, $\boldsymbol{x}_i'$ $(1\leq i\leq t)$ of $\boldsymbol{x}$ and prime v-ideals P_i' of $R[\boldsymbol{x}_i']$ such that $q\in Q(R[\boldsymbol{x}'])$, $P_i=P_i'[\boldsymbol{x}-\boldsymbol{x}_i']$ by Lemma 3.3. We set $\boldsymbol{x}''=\boldsymbol{x}'\cup\boldsymbol{x}_1'\cup\cdots\cup\boldsymbol{x}_t'$ and $P_i''=P_i'[\boldsymbol{x}''-\boldsymbol{x}_i']$, which is a prime v-ideal of $R[\boldsymbol{X}'']$. It follows that $q\in Q(R[\boldsymbol{x}''])$ and $P_i=P_i''[\boldsymbol{x}-\boldsymbol{x}'']$. Hence we have $B=((P_1''[\boldsymbol{x}-\boldsymbol{x}''])^{n_1}\cdots(P_t''[\boldsymbol{x}-\boldsymbol{x}''])^{n_t})^*=((P_1''^{n_1}\cdots P_t''^{n_t})[\boldsymbol{x}-\boldsymbol{x}''])^*$ and so $B^{-1}=(P_1''^{n_1}\cdots P_t''^{n_t})^{-1}[\boldsymbol{x}-\boldsymbol{x}'']$. Hence $q\in(P_1''^{n_1}\cdots P_t''^{n_t})^{-1}[\boldsymbol{x}-\boldsymbol{x}'']\cap Q(R[\boldsymbol{x}''])=(P_1''^{n_1}\cdots P_t''^{n_t})^{-1}$, which implies that $S(R[\boldsymbol{x}])\subseteq\cup_{\boldsymbol{x}'}S(R[\boldsymbol{x}'])$. Hence $S(R[\boldsymbol{x}])=\cup_{\boldsymbol{x}'}S(R[\boldsymbol{x}'])$. To prove that $S(R[\boldsymbol{x}])$ is an essential overring of $R[\boldsymbol{x}]$, let C be any non-zero ideal of $R[\boldsymbol{x}]$. Then there is a finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$ such that $0\neq C\cap R[\boldsymbol{x}']$. By Proposition 2.1 and Corollary 2.5, $(C\cap R[\boldsymbol{x}'])S(R[\boldsymbol{x}'])=S(R[\boldsymbol{x}'])$ and hence $CS(R[\boldsymbol{x}])=S(R[\boldsymbol{x}])$ and, by symmetry, $S(R[\boldsymbol{x}])C=S(R[\boldsymbol{x}])$. Hence $S(R[\boldsymbol{x}])$ is an essential overring of $R[\boldsymbol{x}]$ and is a simple ring by Lemma 2.2.

LEMMA 3.6. *Let R be a Krull order in Q and let P be a prime v-ideal of $R[\boldsymbol{x}]$. Then $R[\boldsymbol{x}]=P^{-1}\cap R[\boldsymbol{x}]_P$.*

PROOF. Clearly $R[\boldsymbol{x}]\subseteq P^{-1}\cap R[\boldsymbol{x}]_P$. Since $P^{-1}\cap R[\boldsymbol{x}]_P$ is an $R[x]$-ideal contained in P^{-1}, we get, by Lemma 2 of [12], the following:

$$P^{-1}\cap R[\boldsymbol{x}]_P\subseteq P^{-1}\circ P\circ(P^{-1}\cap R[\boldsymbol{x}]_P)^*=P^{-1}\circ\big(P(P^{-1}\cap R[\boldsymbol{x}]_P)\big)^*$$
$$\subseteq P^{-1}\circ\big(PP^{-1}\cap PR[\boldsymbol{x}]_P\big)^*\subseteq P^{-1}\circ\big(R[\boldsymbol{x}]\cap PR[\boldsymbol{x}]_P\big)^*=P^{-1}\circ P=R[\boldsymbol{x}]\,.$$

Hence $R[\boldsymbol{x}]=P^{-1}\cap R[\boldsymbol{x}]_P$.

THEOREM 3.7. *Let R be a Krull order in Q. Then*

(1) $R[\boldsymbol{x}]=\cap R[\boldsymbol{x}]_P\cap S(R[\boldsymbol{x}])$, *where P ranges over all prime v-ideals of* $R[\boldsymbol{x}]$. $R[\boldsymbol{x}]_P$ *is a noetherian, local, Asano order.* $S(R[\boldsymbol{x}])$ *is a simple ring and is an essential overring of* $R[\boldsymbol{x}]$.

(2) $R[\boldsymbol{x}]$ *satisfies the condition* $(K3)$.

PROOF. (1) Let P be a prime v-ideal of $R[\boldsymbol{x}]$. By Lemma 3.3, there exist a finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$ and a prime v-ideal P' of $R[\boldsymbol{x}']$ such that $P=P'[\boldsymbol{x}-\boldsymbol{x}']$. Hence, by Corollary 2.5 and Lemma 3.1, $R[\boldsymbol{x}]$ satisfies the Ore condition with respect to $C(P)$ and $R[\boldsymbol{x}]_P$ is a noetherian, local, Asano order. The Asano overring $S(R[\boldsymbol{x}])$ is a simple ring and essential overring of $R[\boldsymbol{x}]$ by Lemma 3.5. It remains to prove that $R[\boldsymbol{x}]=\cap R[\boldsymbol{x}]_P\cap S(R[\boldsymbol{x}])$. But, by using Lemmas 3.4 and 3.6, the proof of this proceeds just like that of Theorem 3.1 of [8].

(2) Let $V(P)$ be the set of all prime v-ideals of $R[\boldsymbol{x}]$ and, for any finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$, let $V(P_{\boldsymbol{x}'})$ be the set of all prime v-ideals P such that $P=P'[\boldsymbol{x}-\boldsymbol{x}']$ for some prime v-ideal P' of $R[\boldsymbol{x}']$. If c is a regular element of $R[\boldsymbol{x}]$, then there is a finite subset $\boldsymbol{x}_0$ of $\boldsymbol{x}$ such that $c\in R[\boldsymbol{x}_0]$. By Corollary 2.5, $cR[\boldsymbol{x}_0]_{P_0}\neq R[\boldsymbol{x}_0]_{P_0}$ for only finitely many prime v-ideals P_0 of $R[\boldsymbol{x}_0]$ and so, by Lemma 3.1, $cR[\boldsymbol{x}]_P\neq R[\boldsymbol{x}]_P$ for only finitely many P in $V(P_{\boldsymbol{x}_0})$. Hence it suffices to prove that $cR[\boldsymbol{x}]_P=R[\boldsymbol{x}]_P$ for all P in $V(P)-V(P_{\boldsymbol{x}_0})$. To prove this let P be any element in $V(P)-V(P_{\boldsymbol{x}_0})$. There are a finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$ and a prime v-ideal P' of $R[\boldsymbol{x}']$ such that $P=P'[\boldsymbol{x}-\boldsymbol{x}']$ by Lemma 3.3, *i.e.*, $P\in V(P_{\boldsymbol{x}'})$. Since $P\in V(P_{\boldsymbol{x}'\cup\boldsymbol{x}_0})$ and $P\notin V(P_{\boldsymbol{x}_0})$, we may assume that $\boldsymbol{x}'$ is a minimal element of the set $\{\boldsymbol{x}'\mid\boldsymbol{x}'\supsetneqq\boldsymbol{x}_0 \text{ and } P\in V(P_{\boldsymbol{x}'})\}$. Let x be any element in $\boldsymbol{x}'$ but not in $\boldsymbol{x}_0$ and let $\boldsymbol{x}''=\boldsymbol{x}'-\{x\}$. In case $\boldsymbol{x}''=\boldsymbol{x}_0$, we consider the following;

$$\begin{array}{ccc} Q(T) & \subset & Q(T)[x] \\ \cup & & \cup \\ T=R[\boldsymbol{x}_0] & \subset & T[x] \ (=R[\boldsymbol{x}']) . \end{array}$$

In case $\boldsymbol{x}''\supsetneqq\boldsymbol{x}_0$, we consider the following;

$$\begin{array}{ccccc} Q(R[\boldsymbol{x}_0]) & \subset & Q(T) & \subset & Q(T)[x] \\ \cup & & \cup & & \cup \\ R[\boldsymbol{x}_0] & \subset & T=R[\boldsymbol{x}''] & \subset & T[x] \ (=R[\boldsymbol{x}']) . \end{array}$$

In both cases, there is a prime ideal Q' of $Q(T)[x]$ such that $P'=Q'\cap R[\boldsymbol{x}']$ and $R[\boldsymbol{x}']_{P'}=Q(T)[x]_{Q'}$ by the proof of Theorem 2.4. Since c is a unit in $Q(R[\boldsymbol{x}_0])$, it is a unit in $R[\boldsymbol{x}']_{P'}$. Hence, since $R[\boldsymbol{x}]_P\supseteq R[\boldsymbol{x}']_{P'}$, we have $cR[\boldsymbol{x}]_P = R[\boldsymbol{x}]_P$, as desired. By a symmetric proof, we have $R[\boldsymbol{x}]_P c \neq R[\boldsymbol{x}]_P$ for only finitely many P in $V(P)$.

4. Polynomial and Formal Power Series Extensions

In this section, D will denote a commutative Krull domain with field of quotients K. As is well known, $D[\boldsymbol{x}]$ and $D[[\boldsymbol{x}]]$ are both Krull domains (cf [6, p. 532] and Theorem 2.1 of [5]). Here the formal power series ring $D[[\boldsymbol{x}]]$ is defined to be the union of the rings $D[[\boldsymbol{x}']]$, where $\boldsymbol{x}'$ ranges over all finite subsets of $\boldsymbol{x}$. We denote the fields of quotients of $D[\boldsymbol{x}]$ and $D[[\boldsymbol{x}]]$ by $K(\boldsymbol{x})$ and $K((\boldsymbol{x}))$, respectively.

Let Σ be a central simple K-algebra with finite dimension over K and let Λ be a D-order in Σ in the sense of [4]. Then $\Sigma(\boldsymbol{x}) = \Sigma \otimes_K K(\boldsymbol{x})$ is a central simple $K(\boldsymbol{x})$-algebra and $\Lambda[\boldsymbol{x}]$ ($\cong \Lambda \otimes_D D[\boldsymbol{x}]$) is a $D[\boldsymbol{x}]$-order in $\Sigma(\boldsymbol{x})$. So, from Proposition 4.2 of [11] and Proposition 1.3, we have.

PROPOSITION 4.1. *Let Σ be a central simple K-algebra and let Λ be a maximal D-order in Σ. Then $\Lambda[\boldsymbol{x}]$ is a maximal $D[\boldsymbol{x}]$-order in $\Sigma(\boldsymbol{x})$.*

In case $\boldsymbol{x}$ is a finite set, this result was obtained by Fossum (cf. Theorem 1.11 of [4]).

LEMMA 4.2. *Let Σ be a central simple K-algebra and let Λ be a D-order in Σ. Then*

(1) *The quotient ring $Q(\Lambda[[\boldsymbol{x}]])$ of $\Lambda[[\boldsymbol{x}]]$ is $\Lambda[[\boldsymbol{x}]] \otimes_{D[[\boldsymbol{x}]]} K((\boldsymbol{x}))$ and is a simple artinian ring with finite dimension over $K((\boldsymbol{x}))$.*

(2) *$Q(\Lambda[[\boldsymbol{x}]])$ is central as a $K((\boldsymbol{x}))$-algebra.*

(3) *$\Lambda[[\boldsymbol{x}]]$ is a $D[[\boldsymbol{x}]]$-order in $Q(\Lambda[[\boldsymbol{x}]])$.*

PROOF. First we note that $\Lambda[[\boldsymbol{x}]]$ is a prime ring and that each non-zero element of $D[[\boldsymbol{x}]]$ is regular in $\Lambda[[\boldsymbol{x}]]$.

(1) By Proposition 1.1 of [4], there exists a finitely generated D-free module F in Σ such that $\Lambda \subseteq F$. Then $F[[\boldsymbol{x}]]$ is a finitely generated $D[[\boldsymbol{x}]]$-free module and so $F[[\boldsymbol{x}]] \otimes_{D[[\boldsymbol{x}]]} K((\boldsymbol{x}))$ is a finite dimensional $K((\boldsymbol{x}))$-space. Thus $\Lambda[[\boldsymbol{x}]] \otimes_{D[[\boldsymbol{x}]]} K((\boldsymbol{x}))$ is also a finite dimensional $K((\boldsymbol{x}))$-space, which implies that it is an artinian ring. Further, $\Lambda[[\boldsymbol{x}]] \otimes_{D[[\boldsymbol{x}]]} K((\boldsymbol{x}))$ is an essential extension of $\Lambda[[\boldsymbol{x}]]$ as $D[[\boldsymbol{x}]]$-modules (hence, as $\Lambda[[\boldsymbol{x}]]$-modules). It follows that $\Lambda[[\boldsymbol{x}]] \otimes_{D[[\boldsymbol{x}]]} K((\boldsymbol{x}))$ is a simple artinian ring and is a quotient ring of $\Lambda[[\boldsymbol{x}]]$, since $\Lambda[[\boldsymbol{x}]]$ is a prime ring.

(2) Since $\Lambda[[\boldsymbol{x}]]$ is $D[[\boldsymbol{x}]]$-torsion-free, we may assume that

$$\Lambda\big[[\boldsymbol{x}]\big] \otimes_{D[[\boldsymbol{x}]]} K\big((\boldsymbol{x})\big) = \Lambda\big[[\boldsymbol{x}]\big]\, K\big((\boldsymbol{x})\big)$$

as in [3, p. 1045], and hence it contains Σ. let $\{f_i\}\ q$ be any element of

$\Lambda[[\boldsymbol{x}]]\otimes_{D[[\boldsymbol{x}]]}K((\boldsymbol{x}))$, where $\{f_i\}_{i=1}^{\infty}\in\Lambda[[\boldsymbol{x}']]$ for some finite subset $\boldsymbol{x}'=\{x_1,\cdots,x_s\}$ of $\boldsymbol{x}$, each $f_i\in\Lambda[\boldsymbol{x}']$ and f_i is either 0 or a form of degree i. Suppose that $\{f_i\}\,q$ is an element in the center of $\Lambda[[\boldsymbol{x}]]\otimes K((\boldsymbol{x}))$ and that $\{f_i\}q\neq 0$. Then $\sigma(\{f_i\}\,q)=(\{f_i\}\,q)\,\sigma$ for every $\sigma\in\Sigma$. Since $\{\sigma f_i\}\,q=\{f_i\sigma\}\,q$, we get $\sigma f_i=f_i\sigma$ for all i. Write $f_i=a_{i1}x_1^{n_{11}}\cdots x_s^{n_{1s}}+\cdots+a_{it}x_1^{n_{t1}}\cdots^{n_{ts}}$, where $n_{j1}+\cdots+n_{js}=i$ for $j=1,\cdots,t$ and $a_{ij}\in\Lambda$. Then $a_{ij}\sigma=\sigma a_{ij}$ implies that a_{ij} belongs to the center of Λ and so $a_{ij}\in D$. Hence $\{f_i\}\,q\in K((\boldsymbol{x}))$. This implies that $Q(\Lambda[[\boldsymbol{x}]])$ is central as $K((\boldsymbol{x}))$-algebras.

(3) It only remains to prove that each element of $\Lambda[[\boldsymbol{x}]]$ is integral over $D[[\boldsymbol{x}]]$. To prove this let p be a minimal prime ideal of $D[[\boldsymbol{x}]]$. Then $\Lambda[[\boldsymbol{x}]]\otimes_{D[[\boldsymbol{x}]]}D[[\boldsymbol{x}]]_p\subseteq F[[\boldsymbol{x}]]\otimes_{D[[\boldsymbol{x}]]}D[[\boldsymbol{x}]]_p$, where F is a finitely generated D-free module in Σ such that $F\supseteq\Lambda$, the latter is finitely generated as $D[[\boldsymbol{x}]]_p$-modules and so is the former. Hence each element of $\Lambda[[\boldsymbol{x}]]\otimes_{D[[\boldsymbol{x}]]}D[[\boldsymbol{x}]]_p$ is integral over $D[[\boldsymbol{x}]]_p$ by Theorem 8.6 of [15]. Hence each element of $\Lambda[[\boldsymbol{x}]]$ is integral over $D[[\boldsymbol{x}]]$ by Theorem 1.14 of [15], because $\Lambda[[\boldsymbol{x}]]\subseteq\cap\,\Lambda[[\boldsymbol{x}]]\otimes_{D[[\boldsymbol{x}]]}D[[\boldsymbol{x}]]_p$ and $D[[\boldsymbol{x}]]=\cap\,D[[\boldsymbol{x}]]_p$, where p ranges over all minimal prime ideals of $D[[\boldsymbol{x}]]$.

PROPOSITION 4.3. *Let Σ be a central simple K-algebra and let Λ be a maximal D-order in Σ. Then $\Lambda[[\boldsymbol{x}]]$ is a maximal $D[[\boldsymbol{x}]]$-order in $\Lambda[[\boldsymbol{x}]]\otimes_{D[[\boldsymbol{x}]]}K((\boldsymbol{x}))$.*

PROOF. By Proposition 4.2 of [11] and Lemma 4.2, it suffices to prove that $\Lambda[[\boldsymbol{x}]]$ is a maximal order in $Q(\Lambda[[\boldsymbol{x}]])$ as rings. Firstly we shall prove this in case $\boldsymbol{x}=\{x\}$. Let A be any non-zero ideal of $\Lambda[[x]]$ and q be any element of $O_l(A)$. By the same way as Lemma 2′ of [17], there is a regular element $c(x)=c_nx^n+c_{n+1}x^{n+1}+\cdots$ (c_n: regular) of $\Lambda[[x]]$ such that $c(x)\,q=\lambda(x)\in\Lambda[[x]]$. We get $c(x)^{-1}=x^{-n}d(x)$ for some $d(x)\in\Sigma[[x]]$ by the method of [6, p. 7]. Thus $q=c(x)^{-1}\lambda(x)=x^{-n}d(x)\,\lambda(x)$ and put $e(x)=d(x)\,\lambda(x)=e_0+e_1x+\cdots+e_nx^n+\cdots\in\Sigma[[x]]$. We set $A_i=\{a_i|\,a_ix^i+a_{i+1}x^{i+1}+\cdots\in A\}\cup\{0\}$ for non-negative integers i and set $A^*=\cup_iA_i$. Assume that $A_0=A_1=\cdots=A_{i-1}=0$ and $A_i\neq 0$. Since A_i is an ideal of Λ, there is a regular element a_i in A_i by Goldie's theorem [7] and is an element $a(x)\in A$ such that $a(x)=a_ix^i+a_{i+1}x^{i+1}+\cdots$. Then we get that $qa(x)=x^{-n}e(x)\,a(x)\in A$ and $e(x)\,a(x)\in x^nA$. Hence $e_0=e_1=\cdots=e_{n-1}=0$, because $(x^nA)_0=\cdots=(x^nA)_{n+i-1}=0$ and a_i is regular. Hence $q=x^{-n}e(x)\in\Sigma[[x]]$, and write $q=q_0+q_1x+\cdots+q_nx^n+\cdots$, where $q_i\in\Sigma$. For any non-zero element b_k of A^*, there exists $b(x)=b_kx^k+b_{k+1}x^{k+1}+\cdots$ in A. Then $q_0b_k\in A^*$, because $qb(x)\in A$ and so $q_0\in O_l(A^*)=\Lambda$. Assume that $q_0,\cdots,q_{j-1}\in\Lambda$ and put $q_j(x)=q(x)-(q_0+q_1x+\cdots+q_{j-1}x^{j-1})$. Then since $q_j(x)\,A\subseteq q(x)\,A-(q_0+q_1x+\cdots+q_{j-1}x^{j-1})\,A\subseteq A$, it follows that $q_j\in\Lambda$ by the same way as the above.

Hence $q\in\Lambda[[x]]$ by an induction. Thus $O_l(A)=\Lambda[[x]]$ and, by symmetry, $O_r(A)=\Lambda[[x]]$. Hence $\Lambda[[x]]$ is a maximal order in $Q(\Lambda[[x]])$. In particular if $\boldsymbol{x}$ is finite, then $\Lambda[[\boldsymbol{x}]]$ is a maximal order in $Q(\Lambda[[\boldsymbol{x}]])$. Assume that $\boldsymbol{x}$ is infinite and let B be any non-zero ideal of $\Lambda[[\boldsymbol{x}]]$. If q is any element of $O_l(B)$, then there exists a finite subset $\boldsymbol{x}'$ of $\boldsymbol{x}$ such that $B\cap\Lambda[[\boldsymbol{x}']]$ is non-zero and $q\in Q(\Lambda[[\boldsymbol{x}']])$. It follows that $q(B\cap\Lambda[[\boldsymbol{x}']])\subseteq B\cap Q(\Lambda[[\boldsymbol{x}']])=B\cap Q(\Lambda[[\boldsymbol{x}']])\cap\Lambda[[\boldsymbol{x}']]=B\cap\Lambda[[\boldsymbol{x}']]$. Hence $q\in O_l(B\cap\Lambda[[\boldsymbol{x}']])=\Lambda[[\boldsymbol{x}']]$ and thus $O_l(B)=\Lambda[[\boldsymbol{x}]]$. By symmetric proof, we get $O_r(B)=\Lambda[[\boldsymbol{x}]]$ and therefore $\Lambda[[\boldsymbol{x}]]$ is a maximal order in $Q(\Lambda[[\boldsymbol{x}]])$.

REMARK. (1) In case $\boldsymbol{x}=\{x\}$ and D is a regular local ring, the proposition was proved by Ramras [14].

(2) Let Σ be a central simple K-algebra and let Λ be a D-order in Σ. If Λ is a Krull order in Σ, then $\Lambda[\boldsymbol{x}]$ and $\Lambda[[\boldsymbol{x}]]$ are both Krull orders by Proposition 4.2 of [11] and Propositions 4.1 and 4.3.

(3) Let R be a noetherian prime Goldie ring with quotient ring Q. By [17], $R[[x]]$ is also a noetherian prime Goldie ring with quotient ring $Q(R[[x]])$. The same proof as Proposition 4.3 gives that if R is a maximal order in Q, then $R[[x]]$ is a maximal order in $Q(R[[x]])$.

References

[1] K. ASANO and K. MURATA: Arithemetical ideal theory in semigroups, J. Inst. Poltec. Osaka City Univ. 4 (1953), 9-33.

[2] J. H. COZZENS and F. L. SANDOMIERSKI: Maximal orders and localization I, J. Algebra 44 (1977), 319-338.

[3] E. H. FELLER and E. W. SWOKOWSKI: Prime modules, Can. J. Math. XVII (1965), 1041-1052.

[4] R. M. FOSSUM: Maximal orders over Krull domains, J. Algebra 10 (1968), 321-332.

[5] R. GILMER: Power series rings over a Krull domain, Pacific J. Math. 29 (1969), 543-549.

[6] R. GILMER: Multiplicative Ideal Theory, Pure and Applied Math. 1972.

[7] A. W. GOLDIE: Semi-prime rings with maximum condition, Proc. London Math. Soc. 10 (1960), 201-220.

[8] C. R. HAJARNAVIS and T. H. LENAGAN: Localization in Asano orders, J. Algebra 21 (1972), 441-449.

[9] N. JACOBSON: The Theory of Rings, Amer. Math. Soc., Providence, Rhode, Island, 1943.

[10] H. MARUBAYASHI: Non commutative Krull rings, Osaka J. Math. 12 (1975), 703-714.

[11] H. MARUBAYASHI: On bounded Krull prime rings, Osaka J. Math. 13 (1976), 491-501.

[12] H. MARUBAYASHI: A characterization of bounded Krull prime rings, Osaka J. Math. 15 (1978), 13–20.

[13] H. MARUBAYASHI: Remarks on ideals of bounded Krull prime rings, Proc. Japan Acad. 53 (1977), 27–29.

[14] M. RAMRAS: Maximal orders over regular local rings, Trans. Amer. Math. Soc. 155 (1971), 345–352.

[15] I. REINER: Maximal Orders, Academic Press, 1975.

[16] J. C. ROBSON: Pri-rings and ipri-rings, Quart. J. Math. Oxford 18 (1967), 125–145.

[17] L. W. SMALL: Orders in artinian rings, II, J. Algebra 9 (1968), 266–273.

[18] B. O. STENSTRÖM: Rings and Modules of Quotients, Springer, Berlin, 1971.

College of General Education,
Osaka University,
Toyonaka, Osaka, Japan