Hokkaido Mathematical Journal Vol. 9 (1980) p. 46-58

On a class of pseudo-differential operators and hypoellipticity

By Junichi ARAMAKI

(Received January 9, 1979; Revised April 11, 1979)

0. Introduction

In this paper we shall consider a class of pseudo-differential operators P whose characteristic set Σ is the union of closed conic submanifolds Σ_1, $\Sigma_2, \cdots, \Sigma_n$. Under some transversarity conditions and involutiveness, we shall give the necessary and sufficient condition for hypoellipticity of P.

When $n=1$, our class coincides with $L^{m,M}_{k}(X, \Sigma)$ introduced by Helffer [5] and moreover if $k=2$, it coincides with $L^{m,M}_{c}(X, \Sigma)$ introduced by Sjöstrand [8] (see also [4]). In the case where $n=1$, $M=2$, $k=2$ and Σ is involutive, Boutet de Monvel [1] gives a necessary and sufficient condition for the existence of a parametrix of P in $OPS^{-m,-M}$ (more general class than ours OPL), which is also equivalent to the hypoellipticity of P with loss of 1-derivative. For general M and k, [5] constructs a left parametrix and then proves hypoellipticity with loss of M/k-derivatives, which is a generalization of [1].

In § 1, using the technique developed by [5], we introduce an invariance of P (Theorem 1.3) and state a necessary and sufficient condition for the hypoellipticity of P (Theorem 1.5). In § 2 and § 3, we give their proofs. § 4 is devoted to the study of hypoellipticity for another class of pseudo-differential operators on $\boldsymbol{R}^N$.

1. Notations, Definitions and Statements of the results

Let X be a paracompact C^∞ manifold of dimension N and let $T^*(X)-\{0\}$ be the cotangent bundle minus the zero section.

DEFINITION 1.1. *Let* $\Sigma_1, \Sigma_2, \cdots, \Sigma_n$ *be closed conic submanifolds of codimension* $p_1, p_2, \cdots, p_n$ *respectively in* $T^*(X)-\{0\}$ *and let* $m\in\boldsymbol{R}$, $M_1, M_2, \cdots, M_n\in\boldsymbol{Z}^+$, $k_1, k_2, \cdots, k_n\in\boldsymbol{Z}^+$ *and* $k_j\geqq 2$, $j=1, 2, \cdots, n$. *Then we define* $OPL^{m,M_1,M_2,\cdots,M_n}_{k_1,k_2,\cdots,k_n}(X; \Sigma_1, \Sigma_2, \cdots, \Sigma_n)$ *to be the space of pseudo-differential operators P which, in every local coodinate system* $U\subset X$, *has a symbol of the form*

(1. 1) $p(x,\xi)\sim\sum_{j=0}^{\infty}p_{m-j}(x,\xi)$, *where* $p_{m-j}(x,\xi)$ *are elements of* $C^{\infty}(\boldsymbol{R}^{N}\times(\boldsymbol{R}^{N}-\{0\}))$ *and positively-homogeneous of degree* $m-j$ *and satisfy*:

(1. 2) *For every* $K\Subset U$, *there exists a constant* $C_{K}>0$ *such that*

$$\frac{|p_{m-j}(x,\xi)|}{|\xi|^{m-j}}\leqq C_{K}\prod_{l=1}^{n}d_{l}(x,\xi)^{(M_{l}-k_{l}j)_{+}},\quad (x,\xi)\in K\times\left(\boldsymbol{R}^{N}-\{0\}\right)$$

and $|\xi|\geqq 1$.

(1. 3) *For every* $K\Subset U$, *there exists* $C_{K}'>0$ *such that*

$$\frac{|p_{m}(x,\xi)|}{|\xi|^{m}}\geqq C_{K}'\prod_{l=1}^{n}d_{l}(x,\xi)^{M_{l}},\quad (x,\xi)\in K\times\left(\boldsymbol{R}^{N}-\{0\}\right)$$

and $|\xi|\geqq 1$. *Here*

$$d_{l}(x,\xi)=\inf_{(y,\eta)\in\Sigma_{l}}\left(|x-y|+\left|\eta-\frac{\xi}{|\xi|}\right|\right)$$

and $(s)_{+}=\sup(0,s)$ *for* $s\in\boldsymbol{R}$.

For example, let $P(x,D)=D_{1}^{M_{1}}D_{2}^{M_{2}}\cdots D_{n}^{M_{n}}+\lambda(x,D)$ *in* $\boldsymbol{R}^{N}$ $(n\leqq N)$ where $\lambda(x,D)$ is a pseudo-differential operator of order $M_{1}+M_{2}+\cdots+M_{n}-1$. In this case taking $\Sigma_{i}=\{\xi_{i}=0\}$, $M_{i}=k_{i}$, $i=1,2,\cdots,n$, we find that p belongs to $OPL^{m,M_{1},M_{2},\cdots,M_{n}}_{k_{1},k_{2},\cdots,k_{n}}(X;\Sigma_{1},\Sigma_{2},\cdots,\Sigma_{n})$.

REMARK 1. 2. If $M_{i}=0$ for some i, we have

$$OPL^{m,M_{1},M_{2},\cdots,M_{n}}_{k_{1},k_{2},\cdots,k_{n}}(X;\ \Sigma_{1},\Sigma_{2},\cdots,\Sigma_{n}) = OPL^{m,M_{1},\cdots,M_{i-1},M_{i+1},\cdots,M_{n}}_{k_{1},\cdots,k_{i-1},k_{i+1},\cdots,k_{n}}(X;\ \Sigma_{1},\cdots,\Sigma_{i-1},\Sigma_{i+1},\cdots,\Sigma_{n})$$

and $OPL^{m,M}_{2}(X;\Sigma_{1})$ coincides with $L^{m,M}_{c}(X;\ \Sigma_{1})$ in [5], [8]. (We shall write $OPL^{m,M_{1},\cdots,M_{n}}_{k_{1},\cdots,k_{n}}$ in stead of $OPL^{m,M_{1},\cdots,M_{n}}_{k_{1},\cdots,k_{n}}(X;\ \Sigma_{1},\cdots,\Sigma_{n})$ if this does not lead to confusions.) Moreover note that the characteristic set Σ of P which belongs to $OPL^{m,M_{1},\cdots,M_{n}}_{k_{1},\cdots,k_{n}}$ is the union of $\Sigma_{1},\cdots,\Sigma_{n}$.

For every $\rho\in\Sigma$, we write $I_{\rho}=\{i;\ \rho\in\Sigma_{i}\}$.

Next we assume the transversality condition and involutiveness in the following sense:

$(H. 1)$ For every $\rho\in\Sigma$, if we put $I_{\rho}=(i_{1},i_{2},\cdots,i_{s})$ there exist $p_{i_{1}}+p_{i_{2}}+\cdots+p_{i_{s}}$ C^{∞} real homogeneous functions $u_{i_{j}}^{k}$, $1\leqq k\leqq p_{i_{j}}$, $1\leqq j\leqq s$, defined in a conic neighbourhood of ρ such that

$$\Sigma_{i_{j}}=\{u_{i_{j}}^{1}=u_{i_{j}}^{2}=\cdots=u_{i_{j}}^{p_{i_{j}}}=0\}$$

and the $du_{i_{j}}^{k}$ $(1\leqq k\leqq p_{i_{j}},\ 1\leqq j\leqq s)$ being linearly independent at ρ.

$(H.2)$ Σ_i and $\Sigma_i \cap \Sigma_j$ are involutive, *i.e.* if $u_i^1, u_i^2, \cdots, u_i^{p_i}, u_j^1, u_j^2, u_j^{p_j}$ are as above, then

$$\{u_i^k, u_i^l\} = 0 \text{ at } \Sigma_i$$

$$\{u_i^k, u_j^l\} = 0 \text{ at } \Sigma_i \cap \Sigma_j \qquad (i \neq j)$$

$(H.3)$ The radial vector $\sum_{l=1}^{N} \xi_j \frac{\partial}{\partial \xi_j}$ is linearly independent of $H_{u_{i_j}}^k$, $1 \leqq k \leqq p_{i_j}$, $1 \leqq j \leqq s$, at every point near ρ, where Hamilton-Jacobi field H_f and Poisson bracket $\{f, g\}$ are defined by the following formulas respectively:

$$H_f = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \right),$$

$$\{f, g\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right).$$

If q_1, q_2 are elements in $L^{m, M_1, \cdots, M_n}_{k_1, \cdots, k_n}$, we define the following equivalence relation: $q_1 \equiv q_2$ in a conic neighbourhood U in $T^*(X) - \{0\}$ if and only if $q_1 - q_2 \in L^{m, M_1 + (k_1 - 1), \cdots, M_n + (k_n - 1)}_{k_1, \cdots, k_n}$ in U

We suppose that there exist integers $l_j \geqq 0$ such that $M_j = k_j l_j$, $j = 1, 2, \cdots, n$.

THEOREM 1.3. *Let p be a symbol satisfying* (1.1) *and* (1.2) *and let $\rho \in \Sigma$, $I_\rho = (i_1, i_2, \cdots, i_s)$. Then there exists a conic neighbourhood U of ρ such that in U*

$$q \in L^{m, M_{i_1}, \cdots, M_{i_s}}_{k_{i_1}, \cdots, k_{i_s}} / L^{m, M_{i_1} + (k_{i_1} - 1), \cdots, M_{i_s} + (k_{i_s} - 1)}_{k_{i_1}, \cdots, k_{i_s}}$$

defined by:

$$q \equiv \exp\left(-\frac{1}{2i} \sum_{l=1}^{N} \left(\frac{\partial}{\partial x_l} \frac{\partial}{\partial \xi_l} \right) \right) p = \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \left(\frac{1}{2i} \sum_{l=1}^{N} \frac{\partial}{\partial x_l} \frac{\partial}{\partial \xi_l} \right)^t p \tag{1.4}$$

is invariant under a locally homogeneous canonical transformation: τ; $T^*(X) - \{0\} \to T^*(\boldsymbol{R}^N) - \{0\}$ such that Σ_{i_j} is mapped to Σ'_{i_j}. This means that if F is an elliptic Fourier integral operator associated with τ and p' is a symbol of $P' = FPF^{-1}$ and q' is the symbol associated with P' by the formula (1.4), *then we have $q'(\tau(\rho')) = q(\rho')$ for every ρ' in a conic neighbourhood of ρ.*

Let $q\left(\sim \sum_{j=0}^{\infty} q_{m-j} \right) \in L^{m, M_{i_1}, \cdots, M_{i_s}}_{k_{i_1}, \cdots, k_{i_s}} / L^{m, M_{i_1} + (k_{i_1} - 1), \cdots, M_{i_s} + (k_{i_s} - 1)}_{k_{i_1}, \cdots, k_{i_s}}$

be a symbol associated with p in a conic neighbourhood of $\rho \in \Sigma$, where

$I_\rho=(i_1, \cdots, i_s)$. Then we define $\sum_{l=1}^{s}(M_{i_l}-k_{i_l}\cdot j)$ linear form, denoted by $\tilde{q}_{m-j}$, on

$$\left(T_\rho\left(T^*(X)-\{0\}\right)\right)^{\sum_{l=1}^{s}(M_{i_l}-k_{i_l}\cdot j)}$$

by: For any

$$X^1_{i_1}, X^2_{i_1}, \cdots, X^{M_{i_1}-k_{i_1}\cdot j}_{i_1}, \cdots, X^1_{i_s}, \cdots X^{M_{i_s}-k_{i_s}\cdot j}_{i_s} \in T_\rho\left(T^*(X)-\{0\}\right),$$

$$\tilde{q}_{m-j}(\rho)\,(X^1_{i_1}, \cdots, X^{M_{i_s}-k_{i_s}\cdot j}_{i_s}) =$$

$$=\prod_{l=1}^{s}\frac{1}{(M_{i_l}-k_{i_l}\cdot j)\,!}(\tilde{X}^1_{i_1}\cdots\tilde{X}^{M_{i_s}-k_{i_s}\cdot j}_{i_s}q_{m-j})\,(\rho)$$

where $\tilde{X}$ designs an extension of X to a neighbourhood of ρ.

REMARK 1.4. (1) The above definition of $\tilde{q}_{m-j}$ is independent of the choice of a class of q and $\tilde{q}_{m-j}$ is symmetric.

(2) If $n=1$ and $M_1=k_1$, $q_m(x, \xi)=p_m(x, \xi)$ and $q_{m-1}(x, \xi)=p_{m-1}(x, \xi)-$ $-\dfrac{1}{2i}\sum_{l=1}^{N}\dfrac{\partial}{\partial x_l}\dfrac{\partial}{\partial \xi_l}p_m(x, \xi)$. In this case $\tilde{q}(\rho, X)$ (which is defined below) is the sum of the transversal hessian of p_m and subprincipal symbol of P at ρ.

Next for every $\rho\in\Sigma$, we define

$$\tilde{q}(\rho, X)=\sum_{j=0}^{J_{I_\rho}}\tilde{q}_{m-j}(\rho)\,(X, \cdots, X), \qquad \text{for all} \quad X\in T_\rho\left(T^*(X)-\{0\}\right)$$

where

$$J_{I_\rho}=\max_{1\leq l\leq s}\left(\frac{M_{i_l}}{k_{i_l}}\right) \quad \text{if} \quad I_\rho=(i_1, \cdots, i_s),$$

and also define

$$\Gamma_\rho=\left\{\tilde{q}(\rho, X)\,;\; X\in T_\rho\left(T^*(X)-\{0\}\right)\right\}.$$

Then we obtain the following:

THEOREM 1.5. *Assume that* $(H.1)$, $(H.2)$ *and* $(H.3)$ *are satisfied. Let* $P\in OPL^{m, M_1, M_2, \cdots, M_n}_{\quad k_1, k_2, \cdots, k_n}(X\,;\, \Sigma_1, \Sigma_2, \cdots, \Sigma_n)$. *Then* P *is hypoelliptic at* $\rho\in\Sigma$ *with loss of* M_{I_ρ}*-derivatives if and only if* Γ_ρ *does not meet the origin for* $\rho\in\Sigma$. *Here* $M_{I_\rho}=\dfrac{M_{i_1}+\cdots+M_{i_s}}{k_{i_1}+\cdots+k_{i_s}}$ *if* $I_\rho=(i_1, \cdots, i_s)$ *and we say that* P *is hypoelliptic at* ρ *with loss of* M_{I_ρ}*-derivatives if* $u\in\mathscr{D}'(X)$ *and* $Pu\in H^s$ *at* ρ *implies* $u\in H^{s+m-M_{I_\rho}}$ *at* ρ.

Therefore we obtain a sufficient condition for the usual hypoellipticity:

COROLLARY 1.5′. *Assume that the hypotheses in the above theorem are* *satisfied. If Γ_ρ does not meet the origin for every $\rho\in\Sigma$, then P is hypoelliptic with loss of M-derivatives where $M=\max\{M_{I_\rho}:\ \rho\in\Sigma\}$. (Here we say that P is hypoelliptic with loss of M-derivatives if for all open set O in X, $u\in\mathscr{D}'(X)$ and $Pu\in H^s_{\mathrm{loc}}(O)$ implies $u\in H^{s+m-M}_{\mathrm{loc}}(O)$.))*

REMARK 1. 6. (1) In the proof, we shall construct a left parametrix of P in $L^{M-m}_{\rho,\delta}$ with $\rho=1-\frac{1}{k}$, $\delta=0$ where $k=\min\{k_j;\ 1\leqq j\leqq n\}$. For the definition of $L^{M-m}_{\rho,\delta}$, we refer to [6].

(2) If M_{I_ρ} are constant for $\rho\in\Sigma$, the condition in Corollary 1.5′ is also necessary.

(3) In the case when $n=1$, this theorem is proved by Helffer [5] and moreover for the case when $M=2$ and $k=2$, we refer to [1].

2. Proof of Theorem 1. 3

Let $\rho=(x,\xi)\in\Sigma$, $I_\rho=(i_1, i_2, \cdots, i_s)$ and choose C^∞ real homogeneous functions $u^k_{i_l}$ $(1\leqq l\leqq s,\ 1\leqq k\leqq p_{i_l})$ such that Σ_{i_l}'s are defined locally by $u^1_{i_l}=u^2_{i_l}=\cdots=u^{p_{i_l}}_{i_l}=0$. We may assume that $u^k_{i_l}$ are positively-homogeneous of degree $\frac{1}{k_{i_1}+k_{i_2}+\cdots+k_{i_s}}$. Let $U^k_{i_l}$ $(1\leqq l\leqq s,\ 1\leqq k\leqq p_{i_l})$ be classical pseudo-differential operators with principal symbol $u^k_{i_l}$. Then if $P\in L^{m,M_{i_1},M_{i_2},\cdots,M_{i_s}}_{\ \ k_{i_1},k_{i_2},\cdots,k_{i_s}}$, by using Taylor's formula, P can be written as follows:

$$P=\sum_{j=0}^{J_{I_\rho}}\sum_{\substack{(\alpha)_l\in[1,2,\cdots,p_{i_l}]^{M_{i_l}-k_{i_l}\cdot j}\\ 1\leqq l\leqq s}} A_{\alpha_1,\cdots,\alpha_s,j}(U)^{(\alpha)_1}_{i_1}\cdots(U)^{(\alpha)_s}_{i_s} \tag{2. 1}$$

where

$$J_{I_\rho}=\max\left\{\frac{M_{i_l}}{k_{i_l}};\ 1\leqq l\leqq s\right\},\ (\alpha)_l=(\alpha^1_l,\cdots,\alpha^{M_{i_l}-k_{i_l}\cdot j}_l),\ \alpha^k_l\in\{1,\cdots,p_{i_l}\}$$

and

$$(U)^{(\alpha)_l}_{i_l}=U^{\alpha^1_l}_{i_l}\cdots U^{\alpha^{M_{i_l}-k_{i_l}\cdot j}_l}_{i_l}.$$

Thus $A_{\alpha_1,\cdots,\alpha_s,j}$ are classical pseudo-differential operators of order

$$m-\frac{M_{i_1}+M_{i_2}+\cdots+M_{i_s}}{k_{i_1}+k_{i_2}+\cdots+k_{i_s}}$$

and separately symmetric in the following sense:

$$A_{\alpha_1,\cdots,\alpha_l,\cdots,\alpha_s,j}=A_{\alpha_1,\cdots,\alpha_l',\cdots\alpha_s,j} \quad \text{if}$$
$$(\alpha)_l=(\alpha_l^1,\cdot\cdot,\alpha_l^i,\cdot\cdot,\alpha_l^j,\cdot\cdot,\alpha_l^{M_{i_l}-k_{i_l}\cdot j}) \quad \text{and}$$
$$(\alpha)_l'=(\alpha_l^1,\cdot\cdot,\alpha_l^j,\cdot\cdot,\alpha_l^i,\cdot\cdot,\alpha_l^{M_{i_l}-k_{i_l}\cdot j}).$$

Then we define

$$q_{m-j}=\sigma_{m-j}\Big(\sum_{\substack{(\alpha)_l\in[1,2,\cdots,p_{i_l}]^{M_{i_l}-k_{i_l}\cdot j}\\ 1\leq l\leq s}} A_{\alpha_1,\cdots,\alpha_s,j}(U)_{i_1}^{(\alpha)_1}\cdot\cdot(U)_{i_s}^{(\alpha)_s}\Big)$$

for $j=0, 1, \cdots, J_{I_\rho}$.

REMARK 2.1. The other terms do not affect to the equivalence class of q. Moreover we see that q does not depend on the choice of local co-odinate systems (x, ξ), but needless to say, it depends on the choice of $u_{i_l}^k$, $U_{i_l}^k$ and $A_{\alpha_1,\cdots,\alpha_s,j}$.

In order to prove Theorem 1.3, we need the following

LEMMA 2.2. *Let (for fixed j)*

$$(2.2) \qquad Q=\sum_{\substack{(\alpha)_l\in[1,2,\cdots,p_{i_l}]^{M_{i_l}-k_{i_l}\cdot j}\\ 1\leq l\leq s}} A_{\alpha_1,\cdots\alpha_s,j}(U)_{i_1}^{(\alpha)_1}\cdots(U)_{i_s}^{(\alpha)_s}$$
$$\in L^{m,M_{i_1},\cdots,M_{i_s}}_{k_{i_1},\cdots,k_{i_s}}(X;\ \Sigma_{i_1},\cdots,\Sigma_{i_s})$$

where

$$A_{\alpha_1,\cdots,\alpha_s,j}\in L^{m-\frac{M_{i_1}+\cdots+M_{i_s}}{k_{i_1}+\cdots+k_{i_s}}}$$

is separately symmetric in the above sense. Then the complete symbol q of Q in

$$L^{m,M_{i_1},\cdots,M_{i_s}}_{k_{i_1},\cdots,k_{i_s}}\Big/L^{m,M_{i_1}+(k_{i_1}-1),\cdots,M_{i_s}+(k_{i_s}-1)}_{k_{i_1},\cdots,k_{i_s}}$$

is given by the formula:

$$q\equiv\exp\Big(\frac{1}{2i}\sum_{l=1}^{N}\frac{\partial}{\partial x_l}\frac{\partial}{\partial\xi_l}\Big)\sigma_{m-j}(Q).$$

For the proof we only give an outline here. (cf. [5]). First by multiplication of p by an elliptic symbol and separately symmetricity of $A_{\alpha_1,\cdots,\alpha_s,j}$, it is sufficient to consider the following type:

$$Q=V_{i_1}^{M_{i_1}}\cdots V_{i_s}^{M_{i_s}}$$

where

$$V_{i_j}^{M_{i_j}}\in L^{\frac{1}{k_{i_1}+\cdots+k_{i_s}},1}(X;\ \Sigma_{i_j}).$$

We note the inclusion

$$L^{m,M_{i_1},\dots,M_{i_s}}_{k_{i_1},\dots,k_{i_s}} \subset L^{m+1,M_{i_1}+k_{i_1},\dots,M_{i_s}+k_{i_s}}_{k_{i_1},\dots,k_{i_s}}\,. \tag{2.3}$$

Then we prove by induction on $M_{i_1}+\cdots+M_{i_s}$; it is evident for $M_{i_1}+\cdots+M_{i_s}=1$ by (2.3). Suppose that the lemma is true for $M_{i_1}+\cdots+M_{i_s}=M$, and we prove it for $M_{i_1}+\cdots+M_{i_s}=M+1$. For example

$$V^{M_{i_1}+1}_{i_1} V^{M_{i_2}}_{i_2}\cdots V^{M_{i_s}}_{i_s} \equiv V^{M_{i_1}}_{i_1} V^{M_{i_2}}_{i_2}\cdots V^{M_{i_s}}_{i_s} V_{i_1}$$
$$\mod L^{\frac{M+1}{k_{i_1}+\cdots+k_{i_s}},M_{i_1}+(k_{i_1}-1),\dots,M_{i_s}+(k_{i_s}-1)}_{k_{i_1},\dots,k_{i_s}}\,.$$

If we denote the complete symbol of $V^{M_{i_1}}_{i_1}\cdots V^{M_{i_s}}_{i_s}$ by $q_{M_{i_1},\dots,M_{i_s}}$, thus we see

$$q_{M_{i_1}+1,M_{i_2},\dots,M_{i_s}} \equiv q_{M_{i_1},\dots,M_{i_s}}\cdot v_{i_1} +$$
$$+\frac{1}{2i}\sum_{l=1}^{N}\left(\frac{\partial}{\partial\xi_l} q_{M_{i_1},\dots,M_{i_s}} \frac{\partial}{\partial x_l} v_{i_1} + \frac{\partial}{\partial x_l} q_{M_{i_1},\dots,M_{i_s}} \frac{\partial}{\partial \xi_l} v_{i_1}\right).$$

On the other hand we have

$$\exp\left(\frac{1}{2i}\sum_{l=1}^{N}\frac{\partial}{\partial x_l}\frac{\partial}{\partial\xi_l}\right) v^{M_{i_1}+1}_{i_1}\cdots v^{M_{i_s}}_{i_s}$$
$$\equiv v_{i_1}\exp\left(\frac{1}{2i}\sum_{l=1}^{N}\frac{\partial}{\partial x_l}\frac{\partial}{\partial\xi_l}\right) v^{M_{i_1}}_{i_1}\cdots v^{M_{i_s}}_{i_s} +$$
$$+\frac{1}{2i}\left(\sum_{l=1}^{N}\frac{\partial v_{i_1}}{\partial\xi_l}\frac{\partial}{\partial x_l} + \frac{\partial v_{i_1}}{\partial x_l}\frac{\partial}{\partial\xi_l}\right) q_{M_{i_1},\dots,M_{i_s}}$$

by the hypothesis of induction and (2.3). Thus we obtain the conclusion.

Now using the above lemma, the complete symbol p of P can be written by:

$$\exp\left(\frac{1}{2i}\sum_{l=1}^{N}\frac{\partial}{\partial x_l}\frac{\partial}{\partial\xi_l}\right) q$$
$$\text{in } L^{m,M_{i_1},\dots,M_{i_s}}_{k_{i_1},\dots,k_{i_s}} \big/ L^{m,M_{i_1}+(k_{i_1}-1),\dots,M_{i_s}+(k_{i_s}-1)}_{k_{i_1},\dots,k_{i_s}}\,.$$

Since $\exp\left(\frac{1}{2i}\sum_{l=1}^{N}\frac{\partial}{\partial x_l}\frac{\partial}{\partial\xi_l}\right)$ is bijective on

$$L^{m,M_{i_1},\dots,M_{i_s}}_{k_{i_1},\dots,k_{i_s}} \big/ L^{m,M_{i_1}+(k_{i_1}-1),\dots,M_{i_s}+(k_{i_s}-1)}_{k_{i_1},\dots,k_{i_s}}$$

and its inverse is $\exp\left(-\frac{1}{2i}\sum_{l=1}^{N}\frac{\partial}{\partial x_l}\frac{\partial}{\partial\xi_l}\right)$, this completes the proof of Theorem 1.3.

3. Proof of Theorem 1. 5. (1) *Sufficiency.*

Let $P\in OPL^{m,M_1,M_2,\cdots,M_n}_{k_1,k_2,\cdots,k_n}(X;\Sigma_1,\Sigma_2,\cdots,\Sigma_n)$ and let $\rho\in\Sigma$ where $I_\rho=(i_1,i_2,\cdots,i_s)$. By the Hamilton-Jacobi theory, we see that, under the hypotheses $(H.1)$, $(H.2)$ and $(H.3)$, there exists locally a homogeneous canonical transformation $\tau: T^*(X)-\{0\}\to T^*(\boldsymbol{R}^N)-\{0\}$ which maps Σ_{i_l} into

$$\Sigma'_{i_l}=\left\{(x,\xi)\in T^*(\boldsymbol{R}^N)-\{0\}\ ;\ \xi_{p_1+\cdots+p_{i_{l-1}}+1}=\cdots=\xi_{p_1+\cdots+p_{i_l}}=0\right\}$$

$(l=1,2,\cdots,s)$. (cf. Grigis and Lascar [3]) Since the hypotheses and the conclusion of theorem 1. 5 is invariant under the canonical transformation τ, denoting $\Gamma'_{\tau(\rho)}$ by the set associated to $P'=FPF^{-1}$, we have $\Gamma'_{\tau(\rho)}=\Gamma_\rho$. Thus we are reduced to the case where $X=\boldsymbol{R}^N$ and

$$\Sigma_{i_l}=\left\{(x,\xi)\in T^*(\boldsymbol{R}^N)-\{0\}\ ;\ \xi_{p_1+\cdots+p_{i_{l-1}}+1}=\cdots=\xi_{p_1+\cdots+p_{i_l}}=0\right\}.$$

For brevity, we denote $(\xi_{p_1+\cdots+p_{i_{l-1}}+1},\cdots,\xi_{p_1+\cdots+p_{i_l}})$ by $(\xi)_{i_l}$ similar to the notation for $(U)_{i_l}$. In this case, P can be written by:

$$P=\sum_{j=0}^{J_{I_\rho}}\sum_{\substack{(\alpha)_l\in[1,2,\cdots,p_{i_l}]^{M_{i_l}-k_{i_l}\cdot j}\\ 1\leq l\leq s}}A_{\alpha_1,\cdots,\alpha_s,j}(D_x)_{i_1}^{(\alpha)_1}\cdots(D_x)_{i_s}^{(\alpha)_s}\tag{3. 1}$$

where $A_{\alpha_1,\cdots,\alpha_s,j}$ are classical pseudo-differential operators of order

$$m-\sum_{l=1}^{s}M_{i_l}+j\left(\sum_{l=1}^{s}k_{i_l}-1\right)$$

defined in a conic neighbourhood of $\rho\in\Sigma$, where $I_\rho=(i_1,\cdots,i_s)$. Then the conclusion of theorem 1. 3 gives

$$p'=\sum_{j=0}^{J_{I_\rho}}p_{m-j}\tag{3. 2}$$

does not vanish in some conic neighbourhood of ρ.

Next we shall modify the class of pseudo-differential operators in [1] so as to agree with our situation and list up the fact which we shall need.

$$\text{Let }\ \Sigma_i=\left\{(x,\xi)\in T^*(X)-\{0\}\ ;\ \xi_i^1=\cdots=\xi_i^{p_i}=0\right\}\qquad(i=1,\cdots,n)$$

and let U be an open conic set in $T^*(X)-\{0\}$. If $\xi\in\boldsymbol{R}^N$ and α is a multi-index, we set

$$\xi=\left((\xi)_1,\cdots,(\xi)_n,\xi''\right)\quad\text{where}\quad(\xi)_i=(\xi_i^1,\cdots,\xi_i^{p_i})$$

$$\alpha=\left((\alpha)_1, \cdots, (\alpha)_n, \alpha''\right) \quad \text{where} \quad (\alpha)_i=(\alpha_i^1, \cdots, \alpha_i^{p_i}),\ \alpha_i^j \in Z^+$$

$$\left|(\alpha)_i\right|=\sum_{j=1}^{p_i} \alpha_i^j, \ \left(\frac{\partial}{\partial(\xi)_i}\right)^{\alpha_i}=\left(\frac{\partial}{\partial\xi_i^1}\right)^{\alpha_i^1}\cdots\left(\frac{\partial}{\partial\xi_i^{p_i}}\right)^{\alpha_i^{p_i}}$$

and we set

$$\rho_i(\xi)=\left\{\left(\frac{|(\xi)_i|}{|\xi|}\right)^2+|\xi|^{-\frac{2}{k_1+\cdots+k_n}}\right\}^{1/2}.$$

Then the space $S^{m,M_1,\cdots,M_n}_{k_1,\cdots,k_n}(X;\Sigma_1,\cdots,\Sigma_n)$ where m and M_i are real numbers is the set of all C^∞ functions $a(x,\xi)$ on $T^*(X)-\{0\}$ such that for any compact set $K\subset X$ and for any multi-indices α, β, there exists a constant $C_K>0$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial(\xi)_1}\right)^{(\alpha)_1}\cdots\left(\frac{\partial}{\partial(\xi)_n}\right)^{(\alpha)_n}\left(\frac{\partial}{\partial\xi''}\right)^{\alpha''}a\right|\leqq \tag{3.3}$$

$$\leqq C_K|\xi|^{m-\sum_{i=1}^n|(\alpha)_i|-|\alpha''|}\prod_{i=1}^n\rho_i(\xi)^{M_i-|(\alpha)_i|}$$

for all $(x,\xi)\in K\times(\boldsymbol{R}^N-\{0\})$ and $|\xi|\geqq 1$. Then we have

$$S^{m,M_1,\cdots,M_n}_{k_1,\cdots,k_n}\subset S^{m-\frac{(M_1)_-+\cdots+(M_n)_-}{k_1+\cdots+k_n}}_{\rho,\delta} \tag{3.4}$$

with $\rho=1-\dfrac{1}{k_1+\cdots+k_n}$, $\delta=0$ where $(s)_-=\inf(0,s)$ for $s\in\boldsymbol{R}$. (Here $S^m_{\rho,\delta}$ is the symbol class of Hörmander [6], [7])

In fact, since $|\xi|^{-\frac{1}{k_1+\cdots+k_n}}\leqq\rho_i(\xi)\leqq$ const., the right hand side in (3.3) is estimated by

$$\text{const.}\ |\xi|^{m-\frac{(M_1)_-+\cdots+(M_n)_-}{k_1+\cdots+k_n}-\left(1-\frac{1}{k_1+\cdots+k_n}\right)\left(\sum_{i=1}^n|(\alpha)_i|+|\alpha''|\right)}.$$

(3.5) If $a\in S^{m,M_1,\cdots,M_n}_{k_1,\cdots,k_n}$

and for any compact set $K\Subset X$, there exists a constant $C>0$ such that

$$\left|a(x,\xi)\right|\geqq C|\xi|^m\prod_{i=1}^n\rho_i^{M_i} \quad \text{for} \quad (x,\xi)\in K\times\left(\boldsymbol{R}^N-\{0\}\right)$$

and $|\xi|\geqq 1$, then $a^{-1}\in S^{-m,-M_1,\cdots,-M_n}_{k_1,\cdots,k_n}$.

End of the proof of sufficiency in Theorem 1.5.

Since $d_i(x,\xi)\leqq\rho_i(\xi)$, p' in (3.2) belongs to $S^{m,M_{i_1},\cdots,M_{i_s}}_{k_{i_1},\cdots,k_{i_s}}$ in a conic neibourhood of ρ where $I_\rho=(i_1,\cdots,i_s)$. (In fact we have only to repeat above argument by replacing $(1,2,\cdots,n)$ with $(i_1,\cdots,i_s)$.) Moreover since $p'/|\xi|^m$ has the same semi-homogeneous behavior as $\prod_{l=1}^s\rho_{i_l}^{M_{i_l}}$ and (3.2) is satisfied,

$$q' = p'^{-1} \in S^{-m, -\frac{M_{i_1}}{k_{i_1}}, \cdots, -\frac{M_{i_s}}{k_{i_s}}} \qquad \text{by (3. 5)}$$

Now let $Q' = q'(x, D) \in OPS^{-m, -\frac{M_{i_1}}{k_{i_1}}, \cdots, -\frac{M_{i_s}}{k_{i_s}}}$ (the class of pseudo-differential operators with symbols satisfying (3. 3)). Then we have $Q'P = I - R$, where

$$R \in OPS^{-\frac{1}{k_{i_1} + \cdots + k_{i_s}}, \; 0, \cdots, 0}_{k_{i_1}, \cdots, k_{i_s}} .$$

This proves that P has a left parametrix $Q \in OPS^{-m, -\frac{M_{i_1}}{k_{i_1}}, \cdots, -\frac{M_{i_s}}{k_{i_s}}}$. By (3. 4), we see that P is hypoelliptic at ρ with loss of M_{I_ρ}-derivatives where

$$M_{I_\rho} = \frac{M_{i_1} + \cdots + M_{i_s}}{k_{i_1} + \cdots + k_{i_s}} ,$$

$I_\rho = (i_1, \cdots, i_s)$ and $\rho \in \Sigma$. This completes the proof of sufficiency in Theorem 1. 5 and Corollary 1. 5′.

(2) *Necessity.* We suppose that Γ_ρ contains zero at a point $\rho = (x^0, \xi^0) \in \Sigma$ and let $I_\rho = (i_1, \cdots, i_s)$. Then, in a conic neighbourhood of ρ, P has the form:

$$P = \sum_{j=0}^{J_{I_\rho}} \sum_{\substack{(\alpha)_l \in [1, 2, \cdots, p_{i_l}]^{M_{i_l} - k_{i_l} \cdot j} \\ 1 \leq l \leq s}} A_{\alpha_1, \cdots, \alpha_s, j} (D_x)_{i_1}^{(\alpha)_1} \cdots (D_x)_{i_s}^{(\alpha)_s}$$

where $A_{\alpha_1, \cdots, \alpha_s, j}$ is of degree $m - \sum_{l=1}^{s} M_{i_l} + j\left(\sum_{l=1}^{s} k_{i_l} - 1\right)$. (In the following argument we put $M = \sum_{l=1}^{s} M_{i_l}$, $K = \sum_{l=1}^{s} k_{i_l}$.) Then it is sufficient to prove that there exists a distribution u such that the wave front set $WF(u) \subset \{(x^0, \lambda\xi^0) ; \lambda > 0\}$, $Pu \in H^s$ at ρ, but $u \notin H^{s+m-M_{I_\rho}}$ at ρ.

For brevity, we write $x = ((x)_{i_1}, \cdots, (x)_{i_s}, t)$, the dual variable $\xi = ((\xi)_{i_1}, \cdots, (\xi)_{i_s}, \tau)$ and we may assume $x^0 = 0$, $\xi^0 = ((0)_{i_1}, \cdots, (0)_{i_s}, 0, \cdots, 0, 1)$ $(\tau_N = 1)$. Then our hypothesis on $q(\rho, X)$ means:

$$q(\rho, X) = \sum_{j=0}^{J_{I_\rho}} \sum_{(\alpha)_l} a_{\alpha_1, \cdots, \alpha_s, j}(0, \cdots, 0, \tau_N) \, (\xi)_{i_1}^{(\alpha)_1} \cdots (\xi)_{i_s}^{(\alpha)_s} = 0$$

for some $((\xi)_{i_1}, \cdots, (\xi)_{i_s})$. Here $a_{\alpha_1, \cdots, \alpha_s, j}$ is the homogeneous term of degree $m - M + j(K-1)$ of $A_{\alpha_1, \cdots, \alpha_s, j}$. Therefore if we assign to $((\xi)_{i_1}, \cdots, (\xi)_{i_s})$ the weight 1 and to τ the weight $K/(K-1)$ respectively, we find that $q(\rho, X)$ is quasi-homogeneous of degree $(Km - M)/(K-1)$ of type $(1, K/(K-1))$. For the terms "*quasi-homogeneous symbols*", see Lascar [9]. Then we have

PROPOSITION 3. 1. *Under the above assumptions, we can construct a distribution u such that the wave front set $WF(u) \subset \{(x^0, \lambda\xi^0) ; \lambda > 0\}$, $Pu \in H^s$ at ρ, but $u \notin H^{s+m-M_{I_\rho}}$ at ρ.*

For the proof, if we regard $q(\rho, X)$ as quasi-homogeneous symbol of degree $(Km-M)/(K-1)$ of type $(1, K/(K-1))$, we can apply [9: Lemma 7.1] to P.

This completes the proof of Theorem 1.5 and Corollary 1.5'.

EXAMPLE 3.2. (1) $P(x, D)=D_1^{M_1}D_2^{M_2}\cdots D_n^{M_n}+\lambda(x, D)$ in $\boldsymbol{R}^N (n\leqq N)$ where $\lambda(x, D)$ is a pseudo-differential operator of order $\sum_{i=1}^{n} M_i-1$ $(M_i\geqq 2)$. In this case, taking $k_i=M_i$, $i=1, \cdots, n$, we find that P is hypoelliptic with loss of 1-derivative if and only if $\xi_1^{M_1}\xi_2^{M_2}\cdots\xi_n^{M_n}+\lambda^0(x, \xi)\neq 0$ for all $(\xi_1, \xi_2, \cdots \xi_n)\in\boldsymbol{R}^n$. Here $\lambda^0(x, \xi)$ is the principal symbol of $\lambda(x, D)$.

(2) $P(x, D)=D_1^6(D_2^2+D_3^2)+iD_1^3(D_1^4+D_2^4+D_3^4)+D_1^6+D_2^6+D_3^6$ in $\boldsymbol{R}^3$. In this case, taking $M_1=6$, $k_1=3$, $M_2=k_2=2$, we find P is hypoelliptic with loss of 2-derivatives.

4. Other results

In this section we consider pseudo-differetial operators on an open set Ω in $\boldsymbol{R}^N$. For $(x, \xi)\in T^*(\Omega)=\Omega\times\boldsymbol{R}^N$, we set up the following notations:

$$x=(x', x'', x'''), \qquad \text{dual variable } \xi=(\xi', \xi'', \xi''')$$

where $x'=(x_1', \cdots, x_{\mu'}')$, $\quad \xi'=(\xi_1', \cdots, \xi_{\mu'}')$

$$x''=(x_1'', \cdots, x_{\mu''}''), \qquad \xi''=(\xi_1'', \cdots, \xi_{\mu''}'')$$

$$\Sigma_1'=\{\xi_1'=\cdots=\xi_{i_1}'=0\}, \cdots, \Sigma_{n'}'=\{\xi'_{i_1+\cdots+i_{n'-1}+1}=\cdots=\xi'_{i_1+\cdots+i_{n'}}=0\}$$

where $i_1+\cdots+i_{n'}=\mu'$

$$\Sigma_1''=\{x_1''=\cdots=x_{j_1}''=0\}, \cdots, \Sigma_{n''}''=\{x''_{j_1+\cdots+j_{n''-1}+1}=\cdots=x_{j_1+\cdots+j_{n''}}=0\}$$

where $j_1+\cdots+j_{n''}=\mu''$.

For brevity we use the notations $(x'')_l$, $(\xi')_l$ similar to those in section 3 and define

$$\left|(\xi')_l\right|=(\xi'^2_{i_1+\cdots+i_{l-1}+1}+\cdots+\xi'^2_{i_1+\cdots+i_l})^{1/2}$$

$$\left|(x'')_l\right|=(x''^2_{j_1+\cdots+j_{l-1}+1}+\cdots+x''^2_{j_1+\cdots+j_l})^{1/2}.$$

Then $\quad OPL^{m, M_1', \cdots, M_{n'}': M_1'', \cdots, M_{n''}''}_{k_1', \cdots, k_{n'}': k_1'', \cdots, k_{n''}''}(\Omega\,;\, \Sigma_1', \cdots, \Sigma_{n'}' : \Sigma_1'', \cdots, \Sigma_{n''}'')$

is the space of pseudo-differential operators such that:

(4.1) $\quad p(x, \xi)\sim\sum_{j=0}^{\infty} p_{m-j}(x, \xi)$ where p_{m-j}'s are positively-homogeneous of degree $m-j$ and satisfy:

(4.2) $\quad$ For every $K\Subset\Omega$, there exists a constant $C_K>0$ such that

$$\frac{|p_{m-j}(x,\xi)|}{|\xi|^{m-j}} \leqq C_K \prod_{l=1}^{n''} \left|(x'')_l\right|^{M_l''-k_l''\cdot j} \prod_{l=1}^{n'} \left|(\xi')_l\right|^{M_l'-k_l'\cdot j}$$

$$(x,\xi) \in K \times \left(\boldsymbol{R}^N - \{0\}\right), \qquad |\xi| \geqq 1$$

(4. 3) For every $K \Subset \Omega$, there exists a constant $C_K'>0$ such that

$$\frac{|p_m(x,\xi)|}{|\xi|^m} \geqq C_K' \prod_{l=1}^{n''} \left|(x'')_l\right|^{M_l''} \prod_{l=1}^{n'} \left|(\xi')_l\right|^{M_l'},$$

$$(x,\xi) \in K \times \left(\boldsymbol{R}^N - \{0\}\right), \qquad |\xi| \geqq 1 .$$

Then we note that the characteristic set Σ of P is the union of $\Sigma_1', \cdots, \Sigma_{n'}', \Sigma_1'', \cdots, \Sigma_{n''}''$. As before we denote I_ρ by the set of indices which $\rho \in \Sigma_i'$ or $\rho \in \Sigma_i''$ and we suppose that there exist integers l_j', l_j'' such that $M_j'=k_j' l_j'$ and $M_j''=k_j'' l_j''$. Then for every $\rho \in \Sigma$ (where $I_\rho=(i_1', \cdots, i_{s'}' : i_1'', \cdots, i_{s''}'')$) there exists a conic neighbourhood U of ρ such that in U

$$P \equiv \sum_{j=0}^{J_{I_\rho}} \sum_{\substack{(\alpha)_l \in [1,\cdots,i_l''] M_{i_l''}''-k_{i_l''}'' j \\ (\beta)_l \in [1,\cdots,i_l'] M_{i_l'}'-k_{i_l'}' j}} A_{\alpha_{i_1''},\cdots,\alpha_{i_{s''}''}:\beta_{i_1'},\cdots,\beta_{i_{s'}'},j} (x'')_{i_1''}^{(\alpha)_1} \cdots (x'')_{i_{s''}''}^{(\alpha)_{s''}} (D_x)_{i_1'}^{(\beta)_1} \cdots (D_x)_{i_{s'}'}^{(\beta)_{s'}}$$

$$\operatorname{mod}\ L^{m,\frac{M_{i_1'}'+(k_{i_1'}'-1)}{k_{i_1'}'},\cdots,\frac{M_{i_{s'}'}'+(k_{i_{s'}'}'-1)}{k_{i_{s'}'}'}:\frac{M_{i_1''}''+(k_{i_1''}''-1)}{k_{i_1''}''},\cdots,\frac{M_{i_{s''}''}''+(k_{i_{s''}''}''-1)}{k_{i_{s''}''}''}}$$

where $J_{I_\rho}=\max\limits_{\substack{1\leqq l\leqq s' \\ 1\leqq m\leqq s''}} \{l_{i_l}', l_{i_m}''\}$.

Then we obtain

THEOREM 4. 1. *Let P be a pseudo-differential operator satisfying* (4. 1), (4. 2) *and* (4. 3). *Assume that for every $\rho \in \Sigma$ (where $I_\rho=(i_1', \cdots, i_{s'}' : i_1'', \cdots, i_{s''}'')$),*

$$\sum_{j=0}^{J_{I_\rho}} \sum_{\substack{(\alpha)_l \\ (\beta)_l}} \sigma_{m-\sum_{i=1}^{s''}(M_{i_i''}''-k_{i_l''}'' j)+j}\left(A_{\alpha_{i_1''}'',\cdots,\alpha_{i_{s''}''}'':\beta_{i_1'}',\cdots,\beta_{i_{s'}'}'}\right) \times$$

$$\times (x'')_{i_1''}^{(\alpha)_1} \cdots (x'')_{i_{s''}''}^{(\alpha)_{s''}} (\xi')_{i_1'}^{(\beta)_1} \cdots (\xi')_{i_{s'}'}^{(\beta)_{s'}} \neq 0$$

for all $x'' \in \boldsymbol{R}^{\mu''}$, $\xi' \in \boldsymbol{R}^{\mu'}$. Then P is hypoelliptic with loss of M derivatives where

$$M=\max\left\{\frac{M_{i_1'}'+\cdots+M_{i_{s'}'}'+M_{i_1''}''+\cdots+M_{i_{s''}''}''}{k_{i_1'}'+\cdots+k_{i_{s'}'}'+k_{i_1''}''+\cdots+k_{i_{s''}''}''};\right.$$

$$\left.(i_1', \cdots, i_{s'}') \subset (1, \cdots, \mu')\ (i_1'', \cdots, i_{s''}'' \subset (1, \cdots, \mu'')\right\}.$$

In fact we can construct a left parametrix in $S_{\rho,\delta}^{M-m}$ with $\rho=1-1/k$, $\delta=1/k$

where $k=\min_{\substack{1\leq j'\leq n' \\ 1\leq j''\leq n''}}\{k'_{j'}, k''_{j''}\}$

The proof is similar to that of section 3.

EXAMPLE 4.2. (1) $P(x, D)=x_2^2D_{x_1}^2+\lambda(x, D)$ where λ is a pseudo-differential operator of order 1. In this case, taking $M'_1=k'_1=2$, $M''_1=k''_1=2$, we find that P is hypoelliptic with loss of 1-derivative if $x_2^2\xi_2^2+\lambda^0(x, \xi)\neq 0$ for all $x_2, \xi_1\in \boldsymbol{R}$ where $\lambda^0(x, \xi)$ is the principal symbol of $\lambda(x, D)$.

(2) $P(x, t, D_x, D_t)=t^{2k}|D_x|^4+|D_x|^2+D_t^2$ ($k\geqq 2$, integer) where $(x, t)\in \boldsymbol{R}^n\times \boldsymbol{R}$. In this case, taking $M'_1=2k$, $k'_1=k$, $M''_1=k''_1=4$, we find that P is hypoelliptic with loss of 2-derivatives.

References

[1] BOUTET DE MONVEL, L.: Hypoelliptic operators with double characteristics and related pseudo-differential operators, Comm. Pure and Appl. Math., 27 (1974), 585–639.

[2] DUISTERMAAT, J. J. and HÖRMANDER, L.: Fourier integral operators II, Acta Math., 128 (1972), 183–269.

[3] GRIGIS, M. A. and LASCAR, R.: Équations locales d'un système de sousvariétés involutives, C. R. Acad. Sc. Paris, 283 (1976) 503–506.

[4] HELFFER, B.: Sur une classe d'opérateurs hypoelliptiques à caractérisques multiples, J. Math. pures et appl., 55 (1975), 207–215.

[5] HELFFER, B.: Invariant associés à une classe d'opérateurs pseudo-différentiels et applications à L'hypoellipticité, Ann. Inst. Fourier, Grenoble, 26 (1976), 55–70.

[6] HÖRMANDER, L.: Pseudo-differential operators and hypoelliptic equations, Amer. Math. Soc. Symp. Pure Math. 10 (1966), Singular integrals 138–183.

[7] HÖRMANDER, L.: Fourier integral operators I, Acta Math., 127 (1971), 79–183.

[8] SJÖSTRAND, J.: Parametrices for pseudo-differential operators with multiple characteristics, Arkiv för Mat. 12 (1974), 85–130.

[9] LASCAR, R.: Propagation des singularités des solutions d'équations pseudo-différentielles quasi homogénes, Ann. Inst. Fourier, Grenoble, 27 (1977), 79–123.

Department of Mathematical Science
Faculty of Science and Engineering
Tokyo Denki University
Hatoyama-mura Hiki-gun
Saitama-ken, Japan 350–03