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On a parametrix for the hyperbolic mixed problem with diffractive lateral boundary II

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§ 1. Introduction.

In the previous paper ([5]) we proved that there exist parametrices with a loss of 1/3 derivative in a neighborhood of a fixed diffractive point for second order normally hyperbolic mixed problems with boundary conditions more general than Dirichlet's or Neumann's.

The aim of the present paper is to give examples of the mixed problems with such general boundary conditions mentioned above, which may appear in classical Mathematical Physics in a natural way. Namely, we consider the equation with given initial data at $t=0$

$$\partial_t^2 u = \mu \Delta u + (\lambda+\mu) \nabla (\nabla \cdot u) \qquad \text{for } t \geqq 0 \tag{1.1}$$

where $u={}^t(u_1, u_2)$ is the displacement vector and λ, μ the Lamé parameters of the medium which occupies a C^∞-domain Ω of R^2. Denote the stress-strain components by

$$\sigma_{ij} = \lambda(\nabla \cdot u)\, \delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, 2$$

and let

$$\begin{pmatrix} a(X), & -b(X) \\ b(X), & a(X) \end{pmatrix} \in O(2)$$

which is a C^∞-cross-section over $\partial\Omega$. Now we impose the following mixed boundary conditions :

$$\begin{cases} a(X)\, u_1(X) - b(X)\, u_2(X) = 0\,, \\ \sum_{k=1,2} \Big(b(X)\, \sigma_{1k}(X) + a(X)\, \sigma_{2k}(X)\Big)\, n_k(X) = 0 \quad \text{on} \quad \partial\Omega\,. \end{cases} \tag{1.2}$$

Here $(n_1, n_2)(X)$ is the unit inward normal at $X=(x_1, x_2) \in \partial\Omega$ (see [14]). Our boundary conditions mean that for any boundary point X they are rigid in the direction $(a(X), -b(X))$ and are free for the direction $(b(X), a(X))$.

To make (1.1) a normally hyperbolic system we must assume that

(1.3) $\mu>0\,,\qquad 2\mu+\lambda>0\quad\text{and}\quad \mu+\lambda\neq 0\,.$

On the other hand the quantity

$$E(t)=\frac{1}{2}\int\left\{\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}+\sigma_{ij}\frac{\partial u_i}{\partial x_j}\right\}dx$$

is invariant of t, but we can't consider in general it as an energy norm, because in general domains it is not always positive definite for test functions u with the boundary conditions and does not dominate the norm $\|u\|_1$ even if $\frac{\partial u_i}{\partial t}$ is replaced by γu_i for $\gamma\gg 1$. Furthermore it will be not easy to find such a suitable norms, in order to show the existence of solutions to the problem. Fortunately our results in [12], [7] and [8] are applicalbe to the problem and we shall show in section 3 that it is well-posed.

About the propagations of singularities of solutions it seems to be an open problem to decide precisely the lateral wave on the boundary of our mixed problem even in a neighborhood of a gliding point (see [3], also [1] and [11]). In section 4 we only apply the result in [5] to the present mixed problem in order to construct a parametrix for given boundary data in a neighborhood of a diffractive point.

§ 2. The Lopatinskii determinant.

Let (τ, ξ_1, ξ_2) be the covectors with respect to (t, x_1, x_2) and let $Q^2=(\lambda+\mu)^{-1}(\tau^2-(\xi_1^2+\xi_2^2)\,\mu)$. Then the characteristic polynomial of (1.1) is the determinant

(2.1) $$\begin{vmatrix} Q^2-\xi_1^2, & -\xi_1\xi_2 \\ -\xi_1\xi_2, & Q^2-\xi_2^2 \end{vmatrix}=Q^2\left(Q^2-(\xi_1^2+\xi_2^2)\right).$$

Therefore the zeros of (2.1) in τ are

$$\tau^2=\mu|\xi|^2\quad\text{or}\quad(2\mu+\lambda)|\xi|^2$$

corresponding to $Q^2=0$ or $Q^2=|\xi|^2$ respectively and hence by (1.3) we see that (1.1) is normally hyperbolic with respect to t.

For an arbitrary but fixed boundary point X^0 let us consider the following transformations $T(x^0)$:

$$\begin{cases} x=n_1(x_1-x_1^0)+n_2(x_2-x_2^0)\,, \\ y=-n_2(x_1-x_1^0)+n_1(x_2+x_2^0) \end{cases}\quad\text{and}$$

$$\begin{cases} v_1=n_1u_1+n_2u_2\,, \\ v_2=-n_2u_1+n_1u_2 \end{cases}$$

where

$$n=(n_1, n_2)=\left(n_1(X^0),\ n_2(X^0)\right).$$

Then setting $\nabla={}^t\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ we have that (1. 1) is invariant and (1. 2) becomes at $(x, y)=(0, 0)$

$$(2.2)\qquad \begin{cases} Av_1-Bv_2=0\,, \\ B\left\{\lambda\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)+2\mu\frac{\partial v_1}{\partial x}\right\}+A\mu\left(\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x}\right)=0\,. \end{cases}$$

Here

$$A(X^0)=a(X^0)\,n_1(X^0)-b(X^0)\,n_2(X^0) \quad \text{and}$$

$$B(X^0)=a(X^0)\,n_2(X^0)+b(X^0)\,n_1(X^0)\,.$$

Therefore the characteristics of boundary operators for freezing problem is the determinant

$$(2.3)\qquad \begin{vmatrix} A & , & -B \\ B(2\mu+\lambda)\,\xi+A\mu\sigma\,, & & B\lambda\sigma+A\mu\xi \end{vmatrix}$$

which is $A^2\mu+B^2(2\mu+\lambda)\neq 0$ for $(\xi, \sigma)=(1, 0)$. Here (ξ, σ) is the covector with respect to (x, y). This means that the boundary operators are non-characteristic with respect to the boundary surface $\{x=0\}$.

According to Hersh ([4]) we shall calculate the Lopatinskii determinant for the freezing problem (1. 1) in $x\geqq 0$ and boundary conditions (2. 2) on $x=0$.

From now on, in order to use the same notations as in [5], after coordinate transformation let the last component of the coordinates coincide with normal one. Therefore we shall rewrite $(x, y, \xi, \sigma)\in T^*(\{x>0\})$ by (y, x, σ, ξ), but we shall keep the order of (v_1, v_2).

Let $\lambda_{(1)}^+=(\mu^{-1}\tau^2-\sigma^2)^{1/2}$ and $\lambda_{(2)}^+=((2\mu+\lambda)^{-1}\tau^2-\sigma^2)^{1/2}$ be the roots of (2. 1) in ξ, where (σ, ξ) may be identified with (ξ_1, ξ_2), $\mathrm{Im}\,\tau\leqq 0$ and $\mathrm{Im}\,\lambda_{(i)}^+$ $(i=1, 2)$ are non-negative. Now we take two vectors

$$W_1={}^t(\sigma,\ -\lambda_{(1)}^+) \quad \text{and} \quad W_2={}^t(\lambda_{(2)}^+, \sigma)$$

as null ones of the matrix in (2. 1) according as the cases $Q^2=0$ and $Q^2=\sigma^2+\xi^2$ respectively. Then we see that the determinant

$$|W_1, W_2|=\lambda_{(1)}^+\lambda_{(2)}^+ +\sigma^2$$

is zero if and only if $\tau=0$. Denoting by $\boldsymbol{B}(\sigma, \xi)$ the matirx in (2. 3) we see that

$$\boldsymbol{B}\Big(\sigma, \lambda^+_{(1)}(\tau, \sigma)\Big)\, W_1 = {}^t\!\big(A\sigma + B\lambda^+_{(1)},\; A\mu\sigma^2 + 2B\mu\sigma\lambda^+_{(1)} - A\mu\lambda^{+^2}_{(1)}\big)\,,$$

$$\begin{aligned} \boldsymbol{B}\Big(\sigma, \lambda^+_{(2)}(\tau, \sigma)\Big)\, W_2 &= {}^t\Big(-(B\sigma - A\lambda^+_{(2)}), \Big\{B(2\mu+\lambda)\,\lambda^+_{(2)} + A\mu\sigma\Big\}\,\lambda^+_{(2)} \\ &\quad + \{B\lambda\sigma + A\mu\lambda^+_{(2)}\}\,\sigma\Big) \\ &= {}^t\Big(-(B\sigma - A\lambda^+_{(2)}),\; B\lambda\sigma^2 + 2A\mu\sigma\lambda^+_{(2)} \\ &\quad + B(2\mu+\lambda)\,\lambda^{+^2}_{(2)}\Big)\,. \end{aligned} \tag{2.4}$$

Then the Lopatinskii determinant for $\tau \neq 0$ may be considered as

$$|\mathscr{L}| = \begin{vmatrix} A\sigma + B\lambda^+_{(1)} & , & -(B\sigma - A\lambda^+_{(2)}) \\ A\mu\sigma^2 + 2B\mu\sigma\lambda^+_{(1)} - A\mu\lambda^{+^2}_{(1)}, & & B\lambda\sigma^2 + 2A\mu\sigma\lambda^+_{(2)} + B(2\mu+\lambda)\,\lambda^{+^2}_{(2)} \end{vmatrix}.$$

The term containing AB in $|\mathscr{L}|$ is

$$\begin{aligned} &A\sigma\cdot\Big(B\lambda\sigma^2 + B(2\mu+\lambda)\,\lambda^{+^2}_{(2)} + B\lambda^+_{(1)}\cdot 2A\mu\sigma\lambda^+_{(2)} \\ &\qquad + B\sigma\cdot(A\mu\sigma^2 - A\mu\lambda^{+^2}_{(1)}) - A\lambda^+_{(2)}\cdot 2\mu\sigma\lambda^+_{(1)} \\ &= AB\sigma\Big(\lambda\sigma^2 + (2\mu+\lambda)\,\lambda^{+^2}_{(2)} + 2\mu\lambda^+_{(1)}\,\lambda^+_{(2)} \\ &\qquad + (\mu\sigma^2 - \mu\lambda^{+^2}_{(1)}) - 2\mu\lambda^+_{(1)}\,\lambda^+_{(2)}\Big) \\ &= AB\sigma\Big((\lambda+\mu)\,\sigma^2 + (2\mu+\lambda)\,\lambda^{+^2}_{(2)} - \mu\lambda^{+^2}_{(1)}\Big) \\ &= 0\,. \end{aligned}$$

Thus we see that

$$\begin{aligned} |\mathscr{L}| &= A\sigma\cdot 2A\mu\sigma\lambda^+_{(2)} + B\lambda^+_{(1)}\cdot\Big(B\lambda\sigma^2 + B(2\mu+\lambda)\,\lambda^{+^2}_{(2)}\Big) \\ &\qquad + B\sigma\cdot 2B\mu\sigma\lambda^+_{(1)} - A\lambda^+_{(2)}\,(A\mu\sigma^2 - A\mu\lambda^{+^2}_{(1)}) \\ &= A^2\lambda^+_{(2)}\,(2\mu\sigma^2 - \mu\sigma^2 + \mu\lambda^{+^2}_{(1)}) + B^2\lambda^+_{(1)}\Big(\lambda\sigma^2 + 2\mu\sigma^2 + (2\mu+\lambda)\,\lambda^{+^2}_{(2)}\Big) \\ &= (A^2\lambda^+_{(2)} + B^2\lambda^+_{(1)})\,\tau^2\,. \end{aligned}$$

On the other hand, at $\tau=0$ the roots $\lambda^+_{(1)}$ and $\lambda^+_{(2)}$ are equal to $i\,|\sigma|$, therefore we must take another base (W_1, W_2') which are generalized null vectors of the matrix in (2.1) with $\xi_2 = i\,|\sigma|$ and $\xi_1 = \sigma$. Then the resulting Lopatinskii determinant is equivalent to the limit $|\mathscr{L}|\cdot(\lambda^+_{(1)} - \lambda^+_{(2)})^{-1}$ from $\tau \neq 0$. Thus we obtain the following

LEMMA 1. *The Lopatinskii determinant for the freezing problem at the boundary point X^0 is equivalent to*

$$A^2\lambda^+_{(2)} + B^2\lambda^+_{(1)} \tag{2.5}$$

for every τ *with* $\operatorname{Im}\tau\leqq 0$.

By the analogous calculation for the free boundary conditions

$$\sum_{k=1,2}\sigma_{ik}n_k=0 \qquad (i=1,2) \quad \text{on} \quad \partial\Omega\,, \tag{2.6}$$

we have the following

LEMMA 2. *For the problem* (1.1) *and* (2.6) *the Lopatinskii determinant is equivalent to*

$$4\mu^2\sigma^2\lambda_{(1)}^{+}\lambda_{(2)}^{+}+(\tau^2-2\mu\sigma^2)^2 \tag{2.7}$$

for every τ *with* $\operatorname{Im}\tau\leqq 0$.

Here we remark that for $n=3$ and for the free boundary case the Lopatinskii determinant is equivalent to $\lambda_{(1)}^{+}\cdot$(2.7), where $\sigma=(\xi_1, \xi_2)$ are the covectors with respect to tangential components, $\sigma^2=\xi_1^2+\xi_2^2$ and the $\lambda_{(1)}^{\pm}(\tau, \sigma)$ are the solutions of the corresponding characteristic equation in ξ_3.

§ 3. The well-posedness for arbitrary C^∞-domains.

First let Ω be a domain with C^∞-compact boundary. Then for an arbitrary but fixed point X^0 of the boundary and for some neighborhood U of X^0, there is a coordinate transformation T from U to a neighborhood V of the origin such that

$$\begin{aligned} &T(U\cap\Omega)\subset V\cap\{x>0\}\,, \\ &T(U\cap\partial\Omega)\subset V\cap\{x=0\} \quad \text{and} \\ &T(X)=T(X')+(0, s) \end{aligned} \tag{3.1}$$

where $T((x_1, x_2))=(y, x)$, $X=X'+sn(X')$ and $X'\in\partial\Omega$.

By results of [8], to prove the L^2-well-posedness of our problem, we may only consider the freezing problems whose equations are just (1.1) for $x>0$ and the boundary operators at $x=0$ are (2.2) with $A=A(X)$ and $B=B(X)$ uniformly for $X\in\partial\Omega\cap U$. Therefore from Lemma 1 it follows that the Lopatinskii determinants are $L(X, \tau, \sigma)=A(X)^2\lambda_{(2)}^{+}+B(X)^2\lambda_{(2)}^{+}$ for any τ with $\operatorname{Im}\tau\leqq 0$.

Following the previous paper [13], for example we represent $\lambda_{(1)}^{\pm}$ as follows: for τ near $\pm\sqrt{\mu}|\sigma|$ let

$$\zeta=\begin{cases} \tau-\sqrt{\mu}|\sigma| & \text{for} \quad \operatorname{Re}\tau>0 \quad \text{and} \\ \tau+\sqrt{\mu}|\sigma| & \text{for} \quad \operatorname{Re}\tau<0\,, \end{cases}$$

then

$$\lambda^{\pm}_{(1)}=\begin{cases}\mp\zeta^{1/2}\mu^{-1/2}(\tau+\sqrt{\mu}|\sigma|)^{1/2} & \text{for} \quad \operatorname{Re}\tau>0 \text{ and}\\ \pm i\zeta^{1/2}\mu^{-1/2}(-\tau+\sqrt{\mu}|\sigma|)^{1/2} & \text{for} \quad \operatorname{Re}\tau<0\,.\end{cases}$$

Analogously we do also with $\lambda^{\pm}_{(2)}$. Then we see easily that $L(X^0, \tau^0, \sigma^0)$ is zero if and only if

$$\text{i)} \quad B(X^0)=0 \quad \text{and} \quad \lambda^{+}_{(2)}(\tau^0, \sigma^0)=0 \quad (\sigma^0\neq 0)\,,$$

or

$$\text{ii)} \quad A(X^0)=0 \quad \text{and} \quad \lambda^{+}_{(1)}(\tau^0, \sigma^0)=0 \quad (\sigma^0\neq 0)\,.$$

Here we may consider only the first case i). Let X^0 be a point such that $B(X^0)=0$. By the implicity function theorem we see that for $\tau_0>0$, for X near X^0 and for (τ, σ) near (τ^0, σ^0)

$$L=L(X, \sqrt{\zeta}, \sigma)\left(\sqrt{\zeta}-D(X, \sigma)\right),$$

$$L(X, \sqrt{\zeta}, \sigma)\neq 0\,, \qquad D(X^0, \sigma^0)=0\,,$$

(3. 2) $D(X, \sigma)$ is real and non-positive if $2\mu+\lambda>\mu$,

(3. 3) $D(X, \sigma)$ is pure imaginary and has a non-negative imaginary part if $2\mu+\lambda<\mu$.

Thus our conditions in [13] (see also [12]) are satisfied in this case. Analogously we see that in other cases also they are valid. Here we remark that for X near X^0 $B(X)\equiv 0$ if and only if $D(X, \sigma)\equiv 0$.

To prove the well-posedness for our problems we shall consider the reflection coefficients of the problems. Let $v(t, y, x)$ be 2-vector to which u in (1. 1) is transformed by $T(X)$ $(X\in U)$ as in the previous section, $\gamma>0$, let

$$w_1=\left(|D_t-i\gamma|^2+|D_y|^2\right)^{1/2}v, \qquad w_2=D_x v$$

and let

$$w={}^t({}^t w_1, {}^t w_2)\,.$$

Then (1. 1) and (1. 2) are reduced to the following:

$$\text{(3. 4)} \qquad \begin{cases}\boldsymbol{P}w=\left(ED_x-\boldsymbol{A}(x, y, D_t, D_y)\right)w=0 & \text{for} \quad x>0\,,\\ \boldsymbol{B}w=0 & \text{for} \quad x=0\,.\end{cases}$$

where $\boldsymbol{A}$ is a 4×4-matrix of classical pseudo-differential operators of order 1 and $\boldsymbol{B}$ is 2×4-matrix such that

$$A=A\left(T^{-1}(y, x)\right), \qquad B=B\left(T^{-1}(y, x)\right),$$

$$B=\begin{pmatrix} A, & -B, & 0, & 0 \\ A\mu\sigma', & B\lambda\sigma', & B(2\mu+\lambda), & A\mu \end{pmatrix},$$

$$\sigma'=\sigma\cdot\left(|\tau|^2+|\sigma|^2\right)^{-1/2} \quad \text{and}$$

$$\tau=\xi_0-i\gamma \quad \text{with} \quad \gamma\gg 1 .$$

Here we remark that for $x=0$, the principal part of $A(y, x, D_t, D_y)$ is independent of (y, x) and denote its symbol by $A(\tau, \sigma)$. Hereafter we shall use only normalized covector (τ', σ') near $(\tau^{0'}, \sigma^{0'})$ and without confusion we shall also denote them by (τ, σ) in this section.

Furthermore let $h_1^+={}^t({}^tW_1, \lambda_{(1)}^+\cdot{}^tW_1)$, $h_2^+={}^t({}^tW_2, \lambda_{(2)}^+\cdot{}^tW_2)$. Then h_1^+ and h_2^+ are the eigenvectors of $A(\tau, \sigma)$ with the eigenvalues $\lambda_{(1)}^+$ and $\lambda_{(2)}^+$ respectively. Analogouoly if we replace $\lambda_{(i)}^+$ in W_1 and W_2 by $\lambda_{(i)}^-$ respectively, we obtain the eigenvectors h_1^- and h_2^- of $A(\tau, \sigma)$ with the eigenvalues $\lambda_{(1)}^-$ and $\lambda_{(2)}^-$ respectively. Therefore set the 4×4 matrix $(h_1^+, h_2^+, h_1^-, h_2^-)=S$, then we see that

$$S^{-1}A(\tau, \sigma)\,S=\begin{pmatrix} \lambda_{(1)}^+ & & & \\ & \lambda_{(2)}^+ & & \\ & & \lambda_{(1)}^- & \\ & & & \lambda_{(2)}^- \end{pmatrix}$$

$$BS=(V_1^+, V_2^+, V_1^-, V_2^-) \quad \text{and} \quad |V_1^+, V_2^+|=|\mathscr{L}| .$$

For (X, τ, σ) near (X^0, τ^0, σ^0) let

$$\left(\tilde{b}_{ij}(X, \tau, \sigma)\right)=(V_1^+, V_2^+)^{-1}\cdot(V_1^-, V_2^-) .$$

Then from the results of [8] (see also [12]), it follows that our problem is L^2-well-posed if and only if for (X, τ, σ) near (X^0, τ^0, σ^0) $(X\in\partial\Omega)$ and for some positive constant C

$$(\alpha)\quad |\tilde{b}_{11}|=\left||V_1^{-1}, V_2^+|\cdot|\mathscr{L}|^{-1}\right|\leq C\gamma^{-1}|\mathrm{Im}\,\lambda_{(1)}^+|^{1/2}|\mathrm{Im}\,\lambda_{(1)}^-|^{1/2}|\lambda_{(1)}^- -\lambda_{(1)}^+| ,$$

$$(\beta)\quad |\tilde{b}_{12}|=\left||V_2^-, V_2^+|\cdot|\mathscr{L}|^{-1}\right|\leq C\gamma^{-1}|\mathrm{Im}\,\lambda_{(1)}^+|^{1/2}|\mathrm{Im}\,\lambda_{(2)}^-|^{1/2}|\lambda_{(2)}^- -\lambda_{(2)}^+| ,$$

$$(\gamma)\quad |\tilde{b}_{21}|=\left||V_1^+, V_1^-|\cdot|\mathscr{L}|^{-1}\right|\leq C\gamma^{-1}|\mathrm{Im}\,\lambda_{(2)}^+|^{1/2}|\mathrm{Im}\,\lambda_{(1)}^-|^{1/2}|\lambda_{(1)}^- -\lambda_{(1)}^+| \quad \text{and}$$

$$(\delta)\quad |\tilde{b}_{22}|=\left||V_1^+, V_2^-|\cdot|\mathscr{L}|^{-1}\right|\leq C\gamma^{-1}|\mathrm{Im}\,\lambda_{(2)}^+|^{1/2}|\mathrm{Im}\,\lambda_{(2)}^-|^{1/2}|\lambda_{(2)}^- -\lambda_{(2)}^+| ,$$

where $\tau=\xi_0-i\gamma$, $\gamma\geqq 0$ and $\gamma\ll 1$.

To show these inequalities, we first remark that for ζ with $\mathrm{Im}\,\zeta<0$ and for some positive constant C_0

$$
(3.5)\quad
\begin{aligned}
&\operatorname{Im}\lambda^{\pm}_{(2)}\sim\operatorname{Im}\sqrt{\zeta},\\
&|\mathscr{L}|\geqq C_0|\operatorname{Re}\sqrt{\zeta}| \quad \text{and in fact}\\
&\qquad\geqq C_0|\sqrt{\zeta}|,\\
&\gamma=|\operatorname{Im}\zeta|=2|\operatorname{Re}\sqrt{\zeta}|\cdot|\operatorname{Im}\sqrt{\zeta}| \quad \text{and}\\
&\operatorname{Im}\lambda^{\pm}_{(1)}\sim\begin{cases}\gamma & \text{if } 2\mu+\lambda>\mu,\\ 1 & \text{if } 2\mu+\lambda<\mu.\end{cases}
\end{aligned}
$$

(see [13], (3. 2) and (3. 3)).

Now we consider the case where $2\mu+\lambda>\mu$. From (2. 4) it follows

$$
(3.6)\quad
\begin{aligned}
V_1^{\pm}&={}^t(A\sigma+B\lambda^{\pm}_{(1)},\ A\mu\sigma^2+2B\mu\sigma\lambda^{\pm}_{(1)}-A\mu\lambda^{\pm^2}_{(1)}),\\
V_2^{\pm}&={}^t\left(-B\sigma+A\lambda^{\pm}_{(2)},\ B\lambda\sigma^2+2A\mu\sigma\lambda^{\pm}_{(2)}+B(2\mu+\lambda)\,\lambda^{\pm^2}_{(2)}\right).
\end{aligned}
$$

Hence by the same way as one derived (2. 5) we have

$$
\begin{aligned}
|V_1^-, V_2^+|&=(A^2\lambda^+_{(2)}+B^2\lambda^-_{(1)})\,\tau^2 \quad \text{and}\\
|V_1^+, V_2^-|&=(A^2\lambda^-_{(2)}+B^2\lambda^+_{(1)})\,\tau^2,
\end{aligned}
$$

whose absolute values $\leqq|A^2\lambda^+_{(2)}+B^2\lambda^+_{(1)}|\tau^2$, because of arguments of $\lambda^{\pm}_{(1)}$, $\lambda^{\pm}_{(2)}$. Therefore (3. 5) implies (α) and (δ). Furthermore since

$$
|V_2^-, V_2^+|\sim|\lambda^-_{(2)}-\lambda^+_{(2)}|\left(O(\sqrt{\zeta})+O(|B|)\right) \quad \text{and}
$$

considering arguments of $\lambda^+_{(1)}$ and $\lambda^+_{(2)}$, we see that

$$
|B|\sim|B|\ |\lambda^+_{(1)}|^{1/2}\leqq|A^2\lambda^+_{(2)}+B^2\lambda^+_{(1)}|^{1/2},
$$

for $\lambda^+_{(1)}\neq 0$. Hence we have (β).
Finally using the relation $\lambda^+_{(1)}+\lambda^-_{(1)}=0$ we have

$$
|V_1^+, V_1^-|=2\lambda^-_{(1)}AB\tau^2\sim B.
$$

Therefore in the same way as above (3. 5) implies (γ).

In the other case where $2\mu+\lambda<\mu$, analogously we obtain the desired inequalities. Thus we see that freezing problems are uniformly L^2-well-posed. Furthermore in the case of (3. 3) the simple real roots $\lambda_{(i)}$ don't exist for real τ and hence we can construct a tangential symmetrizer of the problem (3. 4).

Furthermore from the forms of (1. 1) and (1. 2) it follows that our problem is reversible and if initial data are of C^∞-class and satisfy the compatibility conditions, there exists a unique C^∞-solution of (1. 1) and (1. 2) for an arbitrary domain Ω with C^∞-bounded boundary.

Finally we say that the boundedness of $\partial\Omega$ in the above considerations may be removed. Here we remark merely about the local uniqueness of solutions to our problem, which is obtained by the usual method using the Holmgren transformation $t'=t+\varepsilon y^2$, $y'=y$, $x'=x$ near the origin. In fact, then we have only to replace (τ, σ) in the above calculations by $(\tau, 2\varepsilon y\tau+\sigma)$ and by the a priori estimate thus obtained we have the local uniqueness near the origin.

§ 4. The existence of a parametrix near a diffractive point.

Let (X^0, τ^0, σ^0) be the point fixed in the section 3 and we assume that $(t^0, X^0, \tau^0, \sigma^0)$ is diffractive with respect to (1. 1) and $\partial\Omega$. By the transformations stated there and giving boundary data we may consider (1. 1) and (1. 2) as follows:

$$\text{(4. 1)} \qquad \begin{cases} \boldsymbol{P}(X, D)\, v=0 & \text{for} \quad x=x_2 \geqq 0\,, \\ \boldsymbol{B}(X, D)\, v=g & \text{for} \quad x_2=0 \quad \text{and} \\ \text{the data } g \text{ and } v=0 \text{ for } t=x_0 \leqq 0\,, \end{cases}$$

where $X=(t, y, x)$ is also denoted by (x_0, x_1, x_2), $\boldsymbol{P}(X, D)$ is the 2×2 matrix transformed from (1. 1) by $T(X)$ in (3. 1) previously and

$$\boldsymbol{B}(X, D)\, v=\begin{cases} \left(|D_{x_0}|^2+|D_{x_1}|^2\right)^{1/2}\cdot\left(A(X)\, v_1-B(X)\, v_2\right), \\ \text{the left hand side of the second equality in (2. 2).} \end{cases}$$

Hereafter a boundary point $(X', 0)$ is often denoted by X'.

Now it may be restricted to the case i) in the previous section, *i. e.*, $2\mu+\lambda>\mu$, $B(X^0)=0$ and $\lambda^{+}_{(2)}(X^0, \xi^{0\prime})=\lambda^{+}_{(2)}(\xi^{0\prime})=0$. Here let $\xi^{0\prime}=(\xi^0_0, \xi^0_1)=(\tau^0, \sigma^0)$ with $\xi^0_0>0$ and let the $\lambda^{\pm}_{(i)}(X, \xi')$ $(i=1, 2)$ be roots of the characteristic polynormial of $\boldsymbol{P}$, which are equal to $\lambda^{\pm}_{(i)}(\xi')$ when $x_2=0$. Hereafter we consider mainly (X, ξ') near the diffractive point $(X^0, \xi^{0\prime})$. Therefore $\lambda^{\pm}_{(1)}(X, \xi')$ are real for real ξ' and $\lambda^{\pm}_{(2)}(X, \xi')$ can be represented as follows: for ζ with $\mathrm{Im}\,\zeta<0$,

$$\begin{aligned} &\lambda^{\pm}_{(2)}(X, \xi')=\lambda(X, \xi')\mp\sqrt{\zeta}\,\mu_3^{1/2}(X, \xi')\,, \qquad (\sqrt{1}=1) \\ &\mu_3(X, \xi')>0\,, \\ &\zeta=\xi_0-\mu_2(X, \xi'')\,(\xi''=\xi_1)\,, \\ &\mu_2(X, \xi'') \text{ is real and} \\ &\lambda(X, \xi')=0 \quad \text{for} \quad x_2=0\,. \end{aligned}$$

Then we see that the characteristic polynomial

$$P_4(X,\xi)=\Big(\big(\xi_2-\lambda(X,\xi')\big)^2-\mu(X,\xi')\Big)\cdot(\text{non-zero factor}),$$
$$\mu(X,\xi')=\zeta(X,\xi')\,\mu_3(X,\xi').$$

Furthermore the diffractiveness at $(X^0,\xi^{0\prime})$ means that

$$\Big\{\xi_2-\lambda(X,\xi'),\ \mu(X,\xi')\Big\}>0$$

if $\xi_2=\lambda(X,\xi')$, $\mu(X,\xi')=0$ and $x_2=0$.

Now let $\varphi_0(X',\eta')$ be the phase function mentioned in the previous paper [5], *i. e.*, for $x_2=0$

$$\varphi_0(X,\eta')=\begin{cases}\theta(X,\eta')-\dfrac{2}{3}\rho(X,\eta')^{3/2}+\dfrac{2}{3}\alpha^{3/2}|\eta'| & \text{for } \alpha\geqq 0,\\ \theta(X,\eta') & \text{for } \alpha<0.\end{cases}$$

Here $\varphi_\pm(X,\eta')=\Big(\theta\pm\dfrac{2}{3}\rho^{3/2}\Big)(X,\eta')$ are the solutions of the eikonal equation

$$\big(\varphi_{x_2}-\lambda(X,\varphi_{x'})\big)^2-\mu(X,\varphi_{x'})=0$$

for $\rho\geqq 0$ and $\alpha=\eta_0/|\eta'|$.

Then we see that if $x_2=0$

$$|\mathscr{L}|(\theta_{x'}-\sqrt{\rho}\,\rho_{x'})=\Big\{A^2(X)\cdot(\theta_{x_2}-\sqrt{\rho}\,\rho_{x_2})+B^2(X)\cdot$$
$$\cdot\lambda^+_{(1)}(\theta_{x'}-\sqrt{\rho}\,\rho_{x'})\Big\}(\theta_{x_0}-\sqrt{\rho}\,\rho_{x_0})^2|\eta'|^{-1}$$

which is by definition the determinant of the matrix

$$\Big((\boldsymbol{B}_1W_1)\big(\theta_{x_1}-k\rho_{x_1},\lambda^+_{(1)}(\theta_{x'}-k\rho_{x'})\big),\ (\boldsymbol{B}_1W_2)\big(\theta_{x_1}-k\rho_{x_1},\theta_{x_2}-k\rho_{x_2}\big)\Big)$$
$$\text{for}\quad k=\sqrt{\rho}\,.$$

Hereafter we shall assume that $W_i\in S^0_{1,0}$ and often abbreviate X in the symbols. Therefore for $x_2=0$ and $\rho=0$ the determinants

$$(4.2)\qquad \begin{aligned}&\Big|(\boldsymbol{B}_1W_1,\boldsymbol{B}_1W_2)\Big|=\Big\{A^2(X)\,\theta_{x_2}+B^2(X)\,\lambda^+_{(1)}(\theta_{x'})\Big\}\,\theta^2_{x_0}|\eta'|^{-1},\\ &\left|(\boldsymbol{B}_1W_1),\frac{\partial}{\partial\xi_2}(\boldsymbol{B}_1W_2)\right|_{k=0}=A^2(X)\,\theta^2_{x_0}|\eta'|^{-1}.\end{aligned}$$

Since $\theta_{x_2}=\lambda_2^\pm(\theta_{x'})$ for $x_2=0$ and $\rho=0$ by Lemma 2.3 of [5] we see that $\theta_{x_2}=0$ there. Thus we have that for $x_2=0$ and $\rho=0$

$$(4.3)\qquad \begin{aligned}&|\boldsymbol{B}_1W_1,\boldsymbol{B}_1W_2|\leqq 0\quad\text{and}\\ &\left|\boldsymbol{B}_1W_1,\frac{\partial}{\partial\xi_2}(\boldsymbol{B}_1W_2)\right|\gneqq C_1>0\,.\end{aligned}$$

Let Σ be a conical neighborhood of $(X^0, \xi^{0'})$ on $T^*(x_2=0)$. Then by the canonical transformation $\chi=\chi(Y', \eta')$ and $\xi=\xi(Y', \eta')$ with the generating function $\varphi_0(X', \eta')-\langle Y', \eta'\rangle$, Σ is transformed to a conical neighborhood Σ' of the point $(Y^0, \eta^{0'})$ such that

$$\varphi_{0x'}(X^0, \eta^{0'})=\xi^{0'}\,, \qquad \varphi_{0\eta'}(X^0, \eta^{0'})=Y^0$$

and

$$|\alpha|\ll 1\,.$$

Now we shall seek a parametrix for (4. 1) in the following form: for a positive $\varepsilon\ll 2/3$ and scalar distributions $F_1\in\mathcal{E}'(\Pi\Sigma)$ and $F_2\in\mathcal{E}'(\Pi\Sigma')$

$$G_1F_1(X)=\int a(X, \xi')\, e^{i\varphi_1(X,\xi')}\, \chi(\xi')^2\hat{F}_1(\xi')\, d\xi' \quad \text{and}$$

$$\begin{aligned} G_2F_2(X)=&\iint_L g(X, \eta', k)\, e^{i\left(\frac{k^3}{3}-\rho k+\theta\right)}dk\cdot \\ &\cdot\left(A(\alpha|\eta'|^{2/3})\right)^{-1}\chi_1\left(\alpha|\eta'|^{\varepsilon}\right)^2\chi(\eta')^2\hat{F}_2(\eta')\, d\eta' \\ &+\int d(X, \eta')\, e^{i(\theta+\theta_1)(X,\eta')}\chi_{-1}\left(\alpha|\eta'|^{\varepsilon}\right)^2\chi(\eta')^2\hat{F}_2(\eta')\, d\eta'\,. \end{aligned}$$

Here $a(X, \xi')$ and $d(X, \eta')$ are 2-vector valued classical symbols such that

$$a(X, \xi')\,, \qquad d(X, \eta')\in S^0_{1,0}\,.$$

Furthermore $\chi(\xi')$ and $\chi(\eta')$ are cut off functions restricting to conical neighborhoods of $\xi^{0'}$ and $\eta^{0'}$ respectively and L is a complex contour

$$k=\begin{cases} |t|e^{-\frac{\pi}{2}i} & \text{for} \quad t\to-\infty\,, \\ te^{\frac{\pi}{6}i} & \text{for} \quad t\to\infty\,. \end{cases}$$

We choose χ_1 and $\chi_{-1}\in C^\infty(R^1)$ such that $\chi_1\geqq 0$, $\chi_{-1}\geqq 0$,

$$\chi_1(t)^2=1 \text{ for } t>-c\,, \qquad \chi_1(t)=0 \text{ for } t<-2c \quad \text{and}$$
$$\chi_{-1}(t)^2=1-\chi_1(t)^2 \text{ for some } c>0\,.$$

Finally the function $(\theta+\theta_1)(X, \eta')$ satisfies the eikonal equation, *i.e.*

$$\left((\theta+\theta_1)_{x_2}-\lambda(X, \theta_{x'})\right)^2-\mu(X, \theta_{x'})=0$$

$$\text{for} \quad x_2=0 \text{ and for } \alpha|\eta'|^{\varepsilon}<-\frac{c}{2}\,,$$

$$\theta_1(X, \eta')=0 \quad \text{for} \quad x_2=0\,.$$

Hereafter using (4. 3) we shall only sketch our consideration to get a parametrix and treat mainly terms containing $\chi_1(\alpha|\eta'|^{\varepsilon})$ (for the detail see

[2] and [5]).

The construction of G_1 is accomplished as that of a parametrix for hyperbolic initial value problem and in fact the principal part $a_0(X, \xi')$ of $a(X, \xi')$ is

$$a_0(X, \xi') = W_1(\xi') \quad \text{for} \quad x_2 = 0 .$$

Since $W_2 = {}^t(\lambda^+_{(2)}, \sigma)$ is a right nullvector and also a left one for $\boldsymbol{P}_2(X, \tau, \sigma, \lambda_2^+)$ if $x_2=0$, extending it for $x_2>0$ we find a null eigenvector

$$W_2(X, \theta_x - \sqrt{\rho}\,\rho_x) = W'(X, \eta') - \sqrt{\rho}\, W''(X, \eta')$$

for $\boldsymbol{P}_2(X, \theta_x - \sqrt{\rho}\,\rho_x)$, where W' and W'' may be considered as elements $\in S^0_{1,0}$ and $\in S^{-1/3}_{1,0}$ respectively.

Setting $g_0(X, \eta', k) = (d_0(X, \eta') - k e_0(X, \eta'))\cdot W_2(X, \theta_x - k\rho_x)$ and solving the scalar solution $d_0 \mp \sqrt{\rho}\, e_0$ of the transport equation in $\rho \geqq 0$ and extending it for the domain $\{\alpha \leqq 0\}$ we get the desired term of order 0. Then for $i = 1, 2, \cdots$ setting

$$g_i(X, \eta', k) = g_i^0(X, \eta') - k h_i^0(X, \eta') + \\ + \Big(d_i(X, \eta') - k e_i(X, \eta')\Big)\, W_2(X, \theta_x - k\rho_x)$$

where $g_i^0(X, \eta') \mp \sqrt{\rho}\, h_i^0(X, \eta')$ are special solutions of $\boldsymbol{P}_2(X, \theta_x \mp \sqrt{\rho}\,\rho_x)\,(g_i^0(X, \eta') \mp \sqrt{\rho}\, h_i^0(X, \eta')) =$ the 2-vectors respectively which are defined by $g_j(X, \eta', k)$ for $j<i$, we solve sucessively the transport equations for $d_i \mp \sqrt{\rho}\, e_i$ in $\rho \geqq 0$ with given inital data $d_i(X, \eta')$ on the surface $\rho(X, \eta')=0$. (see [9]) Here we can take d_0 and e_0 such that for $x_2=0$

$$d_0 \gtreqqless 0 \quad \text{and} \quad e_0 = 0\Big(\alpha^\infty |\eta'|^{-1/3}\Big) \qquad \text{(see [2])} .$$

To seek (F_1, F_2) such that for g with $WF(g) \Subset \Sigma$ and for $x_2=0$

$$\boldsymbol{B}\Big((G_1 F_1 + G_2 F_2)\Big) = g \qquad (\text{mod } C^\infty) ,$$

we shall calculate the symbol of

$$\boldsymbol{B}(X, D) \int_L g_0(X, \eta', k)\, e^{i\left(\frac{k^3}{3} - \rho k + \theta\right)} dk .$$

Since we have for $x_2=0$

$$\boldsymbol{B}_1(X, \theta_x - k\rho_x)\, g_0(X, \eta', k) = C_1(X, \eta') \\ + C_2(X, \eta')\, k + (k^2 - \rho)\, b(X, \eta', k) ,$$

where for $\alpha=0$

$$C_1(X, \eta') = \boldsymbol{B}_1(X, \theta_x)\, d_0(X, \eta')\, W_2(X, \theta_x) \quad \text{and}$$

$$C_2(X, \eta') = (-\rho_{x_2}) \frac{\partial}{\partial \xi_2} (\boldsymbol{B}_1 W_2)(X, \theta_x)\, d_0 - \boldsymbol{B}_1(X, \theta_x)\, W_2(X, \theta_x)\, e_0\,,$$

we can write $\boldsymbol{B}G_2F_2(X)|_{x_2=0}$

$$= \int \left\{ d_1(X, \eta')\, A(\rho)/A\left(\alpha|\eta'|^{2/3}\right) + i d_2(X, \eta')\, A'(\rho)/A\left(\alpha|\eta'|^{2/3}\right) \right\} \times e^{i\theta} \chi_1\left(\alpha|\eta'|^{\varepsilon}\right)^2 \chi(\eta')^2 \hat{F}_2(\eta')\, d\eta'\,.$$

Here

$$d_1(X, \eta') = C_1(X, \eta')\Big|_{\alpha=0} + 0\left(\alpha|\eta'|\right) \qquad \mathrm{mod}\,(S^0_{1,0}) \quad \text{and}$$

$$d_2(X, \eta') = C_2(X, \eta')\Big|_{\alpha=0} + 0\left(\alpha|\eta'|^{2/3}\right) \qquad \mathrm{mod}\,(S^{-1/3}_{1,0})\,.$$

Obviously

$$\boldsymbol{B}G_1F_1(X)|_{x_2=0} = \int (\boldsymbol{B}_1 W_1)\left((\xi_1, \lambda_1^+(\xi_0, \xi_1)\right) e^{i\langle x', \xi'\rangle} \chi(\xi')^2 \hat{F}_1(\xi')\, d\xi' \qquad \mathrm{mod}\,(L^0_{1,0})\,.$$

Now for $x_2=0$ let Φ be an elliptic Fourier integral operator

$$(\Phi V)(X) = \int e^{i(\varphi_0(x', \eta') - \langle y', \eta'\rangle)} c(X', \eta')\, V(Y')\, dY'\, d\eta'$$

where $c(X', \eta') \in S^0_{1,0}(R^2 \times R^2|0)$ and is positive and let Φ^{-1} be the inverse elliptic Fourier integral operator such that

$$\Phi\Phi^{-1} V = V \qquad (\mathrm{mod}\, \boldsymbol{C}^\infty)$$

for any $V \in \mathcal{E}'(\Pi\Sigma)$ such that $WF(V) \subset \Sigma$.
Then we shall consider the following problem: for $x_2=0$

$$(4.4) \qquad \Phi^{-1}\boldsymbol{B}\left(G_1\Phi(\Phi^{-1}F_1) + G_2F_2\right) = \Phi^{-1}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \qquad \mathrm{mod}\,(\boldsymbol{C}^\infty)\,.$$

Here if the cut off functions are neglected, $\Phi^{-1}\boldsymbol{B}G_1\Phi$ and $\Phi^{-1}\boldsymbol{B}G_2$ are pseudo-differential operators with principal symbols such that for $X = X(Y', \eta')$, $x_2=0$ and for $\alpha|\eta'|^{\varepsilon} > -2c$

$$(\boldsymbol{B}_1 W_1)\left(\theta_{x''}, \lambda^+_{(1)}(\theta_{x'})\right) + 0\left(\alpha|\eta'|\right)\,,$$

$$(\boldsymbol{B}_1 W_2)(\theta_x)\, d_0(X, \eta') + 0\left(\alpha|\eta'|\right)$$

$$+(-i)\left\{\rho_{x_2}\frac{\partial}{\partial\xi_2}(\boldsymbol{B}_1W_2)(\theta_x)\,d_0(X,\eta')\right.$$
$$\left.+(\boldsymbol{B}_1W_2)(\theta_x)\,e_0(X,\eta')+0\left(\alpha|\eta'|^{2/3}\right)\right\}\frac{A'(\alpha|\eta'|^{2/3})}{A(\alpha|\eta'|^{2/3})}$$

whose determinant is just

$$\left(|\boldsymbol{B}_1W_1,\boldsymbol{B}_1W_2|d_0+0\left(\alpha|\eta'|^2\right)\right)-i\left(\rho_{x_2}\left|\boldsymbol{B}_1W_1,\frac{\partial}{\partial\xi_2}(\boldsymbol{B}_1W_2)\right|d_0\right.$$
$$\left.+0\left(\alpha|\eta'|^{5/3}\right)\right)\cdot\frac{A'(\alpha|\eta'|^{2/3})}{A(\alpha|\eta'|^{2/3})}$$

since e_0 can be taken as $0(\alpha^{\infty}|\eta'|^{-1/3})$.

Next we extend the above symbols except those $0(\alpha|\eta'|)$ and $0(\alpha|\eta'|^{2/3})$ over the whole space $R^2\times R^2\backslash 0$ such that (4.3) is valid and for sufficient large X' they are independent of X' and $|\boldsymbol{B}_1W_1,\boldsymbol{B}_1W_2|<0$. Here and in (4.6) let us regard the symbols $0(\alpha|\eta'|^k)$ as those multiplied by a cut off function $\delta(X')\,\chi_1'(\alpha|\eta'|^{\varepsilon})\,\chi'(\eta')$ such that $\delta(X')=1$ on O, $WF(g)\subset O\subset\pi(\Sigma)$ and

$$\chi_1'(t)\,\chi'(\eta')=1 \quad \text{on} \quad \operatorname{supp}\left(\chi_1(t)\,\chi(\eta')\right).$$

Then (4.4) with extended symbols defined above are rewritten with obvious notations in the following form:

$$(4.5)\qquad \begin{pmatrix} g_1^{(1)}, & g_2^{(1)}+g_3^{(1)}K \\ g_1^{(2)}, & g_2^{(2)}+g_3^{(2)}K\end{pmatrix}\begin{pmatrix}\tilde{F}_1\\ \tilde{F}_2\end{pmatrix}=\begin{pmatrix}\tilde{g}_1\\ \tilde{g}_2\end{pmatrix}.$$

Here $(g_1^{(1)}g_3^{(2)}-g_1^{(2)}g_3^{(1)})|D_y'|^{1/3}$ is elliptic, $g_1^{(1)}(g_2^{(2)}+g_3^{(2)}K)-g_1^{(2)}(g_1^{(1)}+g_3^{(1)}K)$ has the property mainly considered in [5] which we denote by Q_1 and Q_2 respectively, $\sigma(K)=A'(\alpha|\eta'|^{2/3})/A(\alpha|\eta'|^{2/3})$ modified suitably outside $\operatorname{supp}(\chi')$ and all of other symbols $\in S_{1-\varepsilon,\varepsilon}$.

Let, for example, $\mathring{g}_2^{(1)}$be the symbol of $g_2^{(1)}$ when $\alpha=0$. Setting $g_2^{(1)}=\mathring{g}_2^{(1)}+0(\alpha|\eta'|)$ near Σ', we shall consider the operator

$$(4.6)\qquad \begin{pmatrix} g_3^{(2)}|D_y'|^{1/3}, & -g_3^{(1)}|D_y'|^{1/3} \\ -g_1^{(2)} & , \quad g_1^{(1)}\end{pmatrix}\cdot\left(\text{the left hand side of } (4.5)\right)$$

whose principal symbol near Σ' is contained in the matrix

$$\begin{pmatrix} \mathring{Q}+0(\alpha|\eta'|^2), & (\mathring{g}_3^{(2)}\mathring{g}_2^{(1)}-\mathring{g}_3^{(1)}\mathring{g}_2^{(2)})|\eta''|^{1/3}+0(\alpha|\eta'|^2)+0(\alpha|\eta'|^{5/3})\,\sigma(K) \\ 0(\alpha|\eta'|^2) & , \quad \mathring{Q}_2+(0(\alpha|\eta'|^2)+0(\alpha|\eta'|^{5/3})\,\sigma(K))\end{pmatrix}.$$

From (3.6) and (4.2) it follows that $(\mathring{g}_3^{(2)}\mathring{g}_2^{(1)}-\mathring{g}_3^{(1)}\mathring{g}_2^{(2)})|\eta''|^{1/3}$ is

$$\left|(\boldsymbol{B}_1 W_2)(\theta_x), \frac{\partial}{\partial \xi_2}(\boldsymbol{B}_1 W_2)(\theta_x)\right| d_0^2 \rho_{x_2}(-i)|\eta''|^{1/3}$$
$$= \begin{vmatrix} -B\sigma, & A \\ B\lambda\sigma' \sigma, & 2A\mu\sigma' \end{vmatrix} (\theta_x)\cdot d_0^2\cdot \rho_{x_2}(-i)|\eta''|^{1/3} \bmod (S^1_{1-\varepsilon,\varepsilon}),$$

whose absolute value

$$\leqq C_2 d_0 |B|\,|\eta''|\cdot d_0|\eta''| \quad \text{and} \quad |B|^2|\eta''|^2 \leqq C_3 \big| |\boldsymbol{B}_1 W_1, \boldsymbol{B}_1 W_2| \big|$$
$$\leqq C_4 |\mathring{g}_1^{(1)} \mathring{g}_2^{(2)} - \mathring{g}_1^{(2)} \mathring{g}_2^{(1)}| \quad \text{for} \quad |\eta''| \gg 1 .$$

Moreover we see that $\sigma(K)=0(|\eta'|^{1/3})$. Hence by the same way as those in [5] we have that for some $C_5>0$ and for some $\theta\left(0<\theta<\frac{\pi}{2}\right)$

$$\text{(4. 7)} \qquad \operatorname{Re}\left(-e^{i\theta}(4.6),\ {}^t(\tilde{F}_1, \tilde{F}_2)\right) \geqq C_5\left(\|\tilde{F}_1\|^2_{5/6} + \|\tilde{F}_2\|^2_{5/6}\right)$$

if $|\eta'|\gg 1$, $\varepsilon \ll 1$ in all of symbols contained in (4. 7), $0<\delta<d_0\ll 1$ and $(\tilde{F}_1, \tilde{F}_2)=(\chi_1 \Phi^{-1} F_1, \chi_1 F_2)$.

To derive the same a priori estimate as (4. 7) in the region where $\alpha|\eta'|^{\varepsilon}<-c$ we first remark that it can be taken as follows: for $x_2=0$

$$d(X, \eta') = d_0 W_2(\theta_x + \theta_{1x}),$$
$$\theta_{x_2}(X, \eta') = a_1(X, \eta')\,\alpha,$$
$$\mu(X, \theta_{x'}) = a_2(X, \eta')\,\alpha, \qquad a_2>0 \quad \text{and}$$
$$\theta_1(X, \eta') = i x_2\left(a_2(X, \eta')|\alpha|\right)^{1/2} - x_2 a_1(X, \eta')\,\alpha$$

(see [2] and [5]).

From these remarks it follows that for $\alpha<0$ and for $x_2=0$

$$\theta_{x_2} + \theta_{1,x_2} = i\left(a_2(X, \eta')|\alpha|\right)^{1/2} = i\left(\mu(X, \theta_{x'})\right)^{1/2}$$
$$= i\sqrt{|\rho|}\,\rho_{x_2} + 0(\alpha)|\eta'|,$$

for $\theta_{x_2} - \sqrt{\rho}\,\rho_{x_2} = (\mu(X, \theta_{x'}))^{1/2}$ if $\rho \geqq 0$.

Hence we have that for $\alpha \lesseqgtr 0$ and for $x_2=0$

$$\boldsymbol{B}_1 W_2(\theta_x + \theta_{1x})$$
$$= \boldsymbol{B}_1(\theta_x)\, W_2(\theta_x) + 0\left(\alpha|\eta'|\right) + \left\{\left(\frac{\partial}{\partial \xi_2}(\boldsymbol{B}_1 W_2)(\theta_x)\right)(-i\rho_{x_2}) + \right.$$
$$\left. + 0\left(\alpha|\eta'|^{2/3}\right)\right\}(-\sqrt{|\rho|}).$$

Now we replace $\Phi^{-1}F_1$ by $\chi_{-1}\Phi^{-1}F_1$ in (4. 4) and $\Phi^{-1}\boldsymbol{B}G_2F_2$ by

$$\Phi^{-1}\boldsymbol{B}\int d(X, \eta')\, e^{i(\theta+\theta_1)}\, \chi_{-1}^2 \cdot \chi^2 \hat{F}_2(\eta')\, d\eta'$$

and extend symbols as in the case $\alpha|\eta'|^{\varepsilon}>-2c$. Then we obtain the similar equation to (4. 5) where

$$\sigma(K)=-\sqrt{|\rho|}\in S_{1-\varepsilon,\varepsilon}^{1/3} \text{ and } (\tilde{F}_1, \tilde{F}_2)=(\chi_{-1}\Phi^{-1}F_1, \chi_{-1}F_2) .$$

Therefore by the same way as the previous case we obtain also (4. 7).

Adding up the bilinear forms (4. 7) obtained above where the $(\tilde{F}_1, \tilde{F}_2)$ are $(\chi_1\Phi^{-1}F_1, \chi_1F_2)$ and $(\chi_{-1}\Phi^{-1}F_1, \chi_{-1}F_2)$ respectively we see by the usual method that for (4. 4) with symbols suitably modified and without $\chi(\xi')^2$ and $\chi(\eta')^2$

$$C_6\left(||F_1||_{5/6}+||F_2||_{5/6}\right) \leqq \left(||g_1||_{1/6}+||g_2||_{1/6}\right)$$

for some $C_6>0$ and considering the conjugate form of the bilinear one obtained above and using the ellipticity of Q_1 we have the solution of the extended equation (4. 4) without $\chi(\xi')^2$ and $\chi(\eta')^2$. Furthermore we see that the extended operator in the left hand side of (4. 4) is hypoelliptic (see [5]). Therefore we have the desired solution (4. 4) mod (C^∞). Thus we get the desired parametrix and hence obtain that near the diffractive point there are no lateral waves propagating on $\partial\Omega$ from it.

Finally we say that under the free boundary consitions the construction of a parametrix is more direct, because in such a case even if the Lopatinskii determinant is zero, the corresponding $D(X, \xi')\equiv 0$ or this determinant has only simple real zeros for elliptic region, *i. e.*, $|\xi''|\gg|\xi_0|$ (see Lemma 2 and [16]).

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