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Boundary behavior of Dirichlet solutions at regular boundary points

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Let G be a bounded domain in the complex plane. Let f be an extended real-valued continuous function on the boundary ∂G of G. If f is bounded, there exists the Dirichlet solution H_f^G ([2]) and if p_0 is a regular boundary point, then

$$\lim_{G\ni z\to p_0} H_f^G(z)=f(p_0)\,. \tag{1}$$

From M. Brelot's example ([1]) we can make an example which violates $\lim_{G\ni z\to p_0} H_f^G(z)=f(p_0)$ for an unbounded continuous resolutive f at a regular boundary point p_0. Here we show that under a certain condition (1) holds for an unbounded continuous resolutive f at a regular boundary point p_0. Set $f^+=\max\{f, 0\}$ and $f^-=\max\{-f, 0\}$. Our result is the following.

THEOREM. *Let G be a bounded domain in the complex plane and p_0 be a regular boundary point of G. Let f be an extended real-valued continuous and resolutive boundary function on ∂G.*

(i) *The case where $f(p_0)$ is finite.* $\lim_{G\leftarrow\ni pz} H_f^G(z)=f(p_0)$ *holds if* $\iint_{G\cap V} H_{|f|}^G(z)\,dxdy<\infty$ *for a neighborhood V of p_0.*

(ii) *The case where $f(p_0)=+\infty$.* $\lim_{G\ni z\to p_0} H_f^G(z)=f(p_0)$ *holds if* $\iint_{G\cap V} H_{f^-}^G(z)\,dxdy<\infty$ *for a neighborhood V of p_0.*

(iii) *The case where $f(p_0)=-\infty$.* $\lim_{G\ni z\to p_0} H_f^G(z)=f(p_0)$ *holds if* $\iint_{G\cap V} H_{f^+}^G(z)\,dxdy<\infty$ *for a neighborhood V of p_0.*

PROOF. We first suppose f is non-negative. Let $f_n=\min\{f, n\}$, $u_n=H_{f_n}^G$ and $u=H_f^G$, then $u_n\uparrow u$ $(n\to\infty)$. If $f(p_0)=+\infty$, $n=\lim_{G\ni z\to p_0} u_n(z)\leqq \varliminf_{G\ni z\to p_0} u(z)$. Letting $n\to\infty$, we obtain $\lim_{G\ni z\to p_0} u(z)=+\infty=f(p_0)$. If $f(p_0)$ is finite, we denote by α the component of ∂G containing p_0.

Case 1. Suppose $\alpha=\{p_0\}$. Let Γ be a Jordan curve in G surrounding p_0 which is sufficiently near p_0 and does not meet ∂G. Let Ω be a domain in the complex plane bounded by Γ. There is a positive constant M such that $\max_{\partial\Omega\ni z} u(z)\leqq M$. Since f is continuous and $f(p_0)$ is finite, $u_n-u_{n_0}=0$ on $\partial G\cap\Omega-\gamma$ ($n>n_0$, cap $\gamma=0$) for a sufficiently large n_0. Hence

$$u_n(z)-u_{n_0}(z)\leqq M\Big(1-\omega(z,\partial G\cap\Omega, G\cap\Omega)\Big) \qquad (z\in G\cap\Omega).$$

$\omega(z, E, D)$ is the harmonic measure of E at the point z with respect to D. Letting $n\to\infty$, then

$$u(z)-u_{n_0}(z)\leqq M\Big(1-\omega(z,\partial G\cap\Omega, G\cap\Omega)\Big) \qquad (z\in G\cap\Omega).$$

Therefore

$$\begin{aligned} 0 &\leqq \overline{\lim_{G\cap\Omega\ni z\to p_0}}\Big(u(z)-u_{n_0}(z)\Big)\\ &\leqq \overline{\lim_{G\cap\Omega\ni z\to p_0}} M\Big(1-\omega(z,\partial G\cap\Omega, G\cap\Omega)\Big)=0. \end{aligned}$$

This shows that $\lim_{G\ni z\to p_0} u(z)=\lim_{G\cap\Omega\ni z\to p_0} u_{n_0}(z)=f(p_0)$.

Case 2. Suppose that α is a continuum. Since $\iint_{G\cap D(p_0,r_0)} u(p_0+re^{i\theta})\, rdrd\theta\leqq\iint_{G\cap V} u(z)\, dxdy<\infty$ for a sufficiently small positive r_0, where $D(\zeta, r)=\{z;\ |z-\zeta|<r\}$, $\int_{G\cap\partial D(p_0,r)} u(p_0+re^{i\theta})\, d\theta$ is finite for almost all r ($0<r\leqq r_0$). We shall show that $\int_{G\cap\partial D(p_0,r)} u(p_0+re^{i\theta})\, d\theta<\infty$ for every r smaller than some r_1 ($0<r_1\leqq r_0$) in the proposition. Let g be equal to u on $\partial D(p_0,r)\cap G$ ($0<r\leqq r_1$) and equal to zero on $\partial D(p_0,r)-\partial D(p_0,r)\cap G$. Since $\int_{\partial D(p_0,r)} g(p_0+re^{i\theta})\, d\theta=\int_{\partial D(p_0,r)\cap G} u(p_0+re^{i\theta})\, d\theta<\infty$, then $v=H_g^{D(p_0,r)}$ is finite. Let w be a harmonic function on $D'=D\left(p_0,\frac{2}{3}r\right)-\partial G\cap\bar{D}\left(p_0,\frac{r}{3}\right)$ with the boundary value v on $\partial D\left(p_0,\frac{2}{3}r\right)$ and the value 0 on $\partial G\cap\bar{D}\left(p_0,\frac{r}{3}\right)$. Then $0\leqq w\leqq v<\infty$ on D'. Since w is bounded harmonic on D', $\lim_{D'\ni z\to p_0} w(z)=0$. Since f is continuous and $f(p_0)$ is finite, $0\leqq u_n-u_{n_0}\leqq w(n>n_0)$ on $G\cap D\left(p_0,\frac{2}{3}r\right)$ for a sufficiently large n_0. We let $n\to\infty$ to obtain $0\leqq u-u_{n_0}\leqq w$ on $G\cap D\left(p_0,\frac{2}{3}r\right)$. Hence

$$0 \leqq \overline{\lim_{G\cap D\left(p_0,\frac{2}{3}r\right)\ni z\to p_0}} \left(u(z)-u_{n_0}(z)\right) \leqq \overline{\lim_{G\cap D\left(p_0,\frac{2}{3}r\right)\ni z\to p_0}} w(z)=0\,.$$

This shows that $\lim_{G\ni z\to p_0} u(z) = \lim_{G\cap D\left(p_0,\frac{2}{3}r\right)\ni z\to p_0} u_{n_0}(z) = f(p_0)$.

Proof of (i). Since $H^G_{|f|}=H^G_{f^+}+H^G_{f^-}$, both $\iint_{G\cap V} H^G_{f^+}(z)\,dxdy$ and $\iint_{G\cap V} H^G_{f^-}(z)\,dxdy$ are finite. The above argument shows that $\lim_{G\ni z\to p_0} H^G_{f^+}(z)=f^+(p_0)$ and $\lim_{G\ni z\to p_0} H^G_{f^-}(z)=f^-(p_0)$. Note that $f^+(p_0)$ and $f^-(p_0)$ are finite. Thus

$$\begin{aligned}\lim_{G\ni z\to p_0} H^G_f(z) &= \lim_{G\ni z\to p_0} H^G_{f^+}(z)-\lim_{G\ni z\to p_0} H^G_{f^-}(z)\\ &= f^+(p_0)-f^-(p_0)=f(p_0)\,.\end{aligned}$$

Proof of (ii) and (iii). We prove (ii). Part (iii) is proved similarly. Note that f^- is non-negative and $f^-(p_0)=0$. By the hypothesis of f^- we obtain $\lim_{G\ni z\to p_0} H^G_{f^-}(z)=0$. Since f^+ is non-negative and $f^+(p_0)=+\infty$, $\lim_{G\ni z\to p_0} H^G_{f^+}(z)=+\infty$. Thus $\lim_{G\ni z\to p_0} H^G_f(z)=\lim_{G\ni z\to p_0} H^G_{f^+}(z)-\lim_{G\ni z\to p_0} H^G_{f^-}(z)=f(p_0)$.

COROLLARY 1. *Let G and p_0 be the same as in Theorem. Let f be a non-negative continuous resolutive boundary function of G.*

(i) *The case where $f(p_0)$ is finite.* $\lim_{G\ni z\to p_0} H^G_f(z)=f(p_0)$ *holds if and only if* $\iint_{G\cap V} H^G_f(z)\,dxdy<\infty$ *for a neighborhood V of p_0.*

(ii) *The case where $f(p_0)=+\infty$.* $\lim_{G\ni z\to p_0} H^G_f(p_0)=f(p_0)$ *holds.*

PROOF. "Only if" part in (i) is clear. The rest is already proved.

COROLLARY 2. *Let G and f be the same as in Theorem and let $\{p\,;\,f(p)=\pm\infty\}$ be of capacity zero. If* $\iint_{G\cap V(p)} H^G_{|f|}(z)\,dxdy<\infty$ *holds at every regular boundary point p with finite $f(p)$, where $V(p)$ is a neighborhood of p, then* $\lim_{G\ni z\to p} H^G_f(z)=f(p)$ *holds at every regular boundary point.*

PROOF. If $f(p)$ is finite, the assertion is already proved. Let $f(p)=+\infty$. Note first that $\lim_{G\ni z\to p} H^G_{f^+}(z)=+\infty$. Choose $r>0$ such that f^+ is finite on $\partial G\cap\partial D(p,r)$, f^- is zero on $\partial G\cap\bar{D}(p,r)$ and $\partial D(p,r)$ meets no irregular boundary point of G. Let g be a boundary function of $G_r=G\cap D(p,r)$ which is equal to f^- on $\partial G\cap\bar{D}(p,r)$ and equal to $H^G_{f^-}$ on $G\cap\partial D(p,r)$. Then g is resolutive and $H^{G_r}_g=H^G_{f^-}$ on G_r. Take $q\in\partial G\cap\partial D(p,r)$, then

$\lim_{G\ni z\to q} H_{f^-}^G(z)=f^-(q)=0$ by the hypothesis. This implies that g is bounded on ∂G_r. From this fact we know that $\lim_{G\ni z\to p} H_{f^-}^G(z)=\lim_{G_r\ni z\to p} H_g^{G_r}(z)=g(p)=0$. Thus

$$\lim_{G\ni z\to p} H_f^G(z)=\lim_{G\ni z\to p} H_{f^+}^G(z)-\lim_{G\ni z\to p} H_{f^-}^G(z)=f(p).$$

If $f(p)=-\infty$, the assertion is proved similarly.

An existence of a finite $\int_{G\cap\partial D(p_0,r)} u(p_0+re^{i\theta})\,d\theta$ is most important in the proof of Theorem. So we furthermore remark the following proposition.

PROPOSITION. *With the notation of Theorem we obtain the following.*

(i) $\int_{G\cap\partial D(p_0,r)} \left(u_n(p_0+re^{i\theta})-u_{n_0}(p_0+re^{i\theta})\right)d\theta$ $(n>n_0)$ *decreases as* $r\to 0$ *provided that* r *is smaller than some small positive number and* n_0 *is sufficiently large.*

(ii) *If* $\int_{G\cap\partial D(p_0,r_1)} u(p_0+r_1e^{i\theta})\,d\theta$ *is finite, where* r_1 *is sufficiently small, then* $\int_{G\cap\partial D(p_0,r)} u(p_0+re^{i\theta})\,d\theta$ *is finite for every* r $(0<r\leqq r_1)$.

PROOF. Let v_r be equal to $u_n-u_{n_0}$ $(n>n_0)$ on $G\cap\partial D(p_0,r)$ and equal to zero on $\partial D(p_0,r)-G\cap\partial D(p_0,r)$. Then

$$H_{v_r}^{D(p_0,r)}(p_0)=\frac{1}{2\pi}\int_{G\cap\partial D(p_0,r)}(u_n-u_{n_0})(p_0+re^{i\theta})\,d\theta.$$

Since we can assume that $u_n-u_{n_0}=0$ on $\partial G\cap D(p_0,r)-\gamma$ (cap $\gamma=0$) provided that r is smaller than some small positive number and n_0 is sufficiently large, we have $H_{v_r}^{D(p_0,r)}\geqq u_n-u_{n_0}$ on $G\cap\partial D(p_0,t)$ for each t $(0<t\leqq r)$. It follows that $H_{v_r}^{D(p_0,r)}\geqq H_{v_t}^{D(p_0,t)}$ on $D(p_0,t)$. In particular $H_{v_r}^{D(p_0,r)}(p_0)\geqq H_{v_t}^{D(p_0,t)}(p_0)$. This shows that

$$\begin{aligned}&\int_{G\cap\partial D(p_0,r)}\left(u_n(p_0+re^{i\theta})-u_{n_0}(p_0+re^{i\theta})\right)d\theta\\ &\geqq\int_{G\cap\partial D(p_0,t)}\left(u_n(p_0+te^{i\theta})-u_{n_0}(p_0+te^{i\theta})\right)d\theta.\end{aligned}\tag{2}$$

To prove (ii), we let $r=r_1$, $t=r$ and $n\to\infty$ in (2) to obtain

$$\begin{aligned}\infty&>\int_{G\cap\partial D(p_0,r_1)}\left(u(p_0+r_1e^{i\theta})-u_{n_0}(p_0+r_1e^{i\theta})\right)d\theta\\ &\geqq\int_{G\cap\partial D(p_0,r)}\left(u(p_0+re^{i\theta})-u_{n_0}(p_0+re^{i\theta})\right)d\theta.\end{aligned}$$

This shows that $\int_{G\cap\partial D(p_0,r)} u(p_0+re^{i\theta})\,d\theta<\infty$ for every r $(0<r\leqq r_1)$.

REMARK. We cannot remove the hypotheses of (ii) and (iii) in Theorem. By modifying M. Brelot's example ([1]), we obtain a Dirichlet solution H_f^G such that f is an unbounded continuous and resolutive boundary function and $\overline{\lim}_{G\ni z\to p_0} H_f^G(z)=+\infty$ at a regular boundary point p_0, where $f(p_0)=-\infty$. For another sufficient condition, we refer to Z. Kuramochi and Y. Nagasaka [3].

References

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