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On the Hall-Higman and Shult theorems (II)

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We use the same notation as in the preceding paper [5], Theorem 1 (a), (b), (c). In that paper, to prove Hall-Higman's and Shult's theorems, we noticed that the proof is reduced to the case where V is a kQ-irreducible ([5], Hypothesis 1 (3)) But the proof of this fact is not trivial. The most well-known method to show this is to use the fact that the projective representations of cyclic groups are of degree one ([4], p. 704). See also [2], p. 363. We will give an easy proof of the following well-known theorem.

THEOREM A. *Let G be a finite group, Q a normal subgroup of G, k an algebraic closed field, and V an irreducible kG-module of finite dimensional such that V_Q is a direct sum of isomorphic irreducible kQ-modules. Assume that G/Q is cyclic. Then V_Q is kQ-irreducible.*

REMARK. The conclusion holds even if G/Q is a p-group and $char(k)=p$. The proof is similar as cyclic case.

LEMMA. *Every k-algebra automorphism of $M(n, k)$, the k-algebra of all $n\times n$ matrices over a field k, is inner.*

This lemma is a particular case of a well-known theorem of Skolem-Nother. This theorem and its proof are found in [3], Cor. of Th. 4. 3. 1 and [1], § 10. 1. In the present case, the proof of this lemma is easy. For example, use the fact that if U is a vector space over k of dimensional n and f is a k-algebra automorphism of $E=End_k(U)\cong M(n, k)$, then $U\cong Uf$ as E-modules.

We can now prove the theorem. Assume that V_Q is the direct sum of n isomorphic irreducible kQ-modules $W_1, \cdots, W_n$. Set $E=End_{kQ}(V_Q)$. Then $E\cong M(n, k)$ as k-algebras, because $\mathrm{Hom}_{kQ}(W_i, W_j)\cong k$ for any i, j by Schur's lemma. Remember that k is algebraic closed. Each element x of G induces a k-algebra automorphism of E by $(v)f^x=(vx^{-1})fx$ for $v\in V$, $f\in E$, and so E is a kG-module. Let Z be the center of E, so that Z consists of all scalar transformations, and so $Z\cong k$. Let x be an element of G which, together with Q, generates G. Since Q acts trivially on the kG-module E and $End_{kG}(V)=Z$ by Schur's lemma, we have that

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$$\{f \in E \mid f^{x} = f\} = Z \cong k .$$

Thus by the above lemma, there is an invertible element $\bar{x}$ of E such that

$$C_{E}(\bar{x}) = \{f \in E \mid f\bar{x} = \bar{x}f\} = Z \cong k .$$

Clearly $C_{E}(\bar{x})$ contains $\bar{x}$ and $Z \cong k$. Thus $\bar{x} \in Z$, so $C_{E}(\bar{x}) = E = Z \cong k$. Hence $n=1$. The theorem is proved.

References

[1] BOURBAKI, N.: "Algèbre," Éléments de Math. 23, Hermann, Paris (1958).

[2] GORENSTEIN, D.: "Finite Groups," Harper and Row, New York (1968).

[3] HERSTEIN, I. N.: "Noncommutative Rings," Math. Assoc. of America (1968).

[4] SHULT, E.: On groups admitting fixed point free abelian operators, Ill. J. Math., 9 (1965), 701–720.

[5] YOSHIDA, T.: On the Hall-Higman and Shult theorems, Hokkaido Math. J., 7 (1978), 336–338.

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