Hokkaido Mathematical Journal Vol. 9 (1980) p. 222-257

On G-functors (I): transfer theorems for cohomological G-functors

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(Received June 5, 1979)

1. Introduction

The purpose of this seuqence of papers is to study and to apply G-functors. The concept of G-functors was introduced by Green [10] during the study of modular representation, particularly a relation between Brauer's theory of blocks and Green's theory of indecomposable modules [9], and it was quite useful for the arrangement of many concepts about representation theory, cohomology theory, etc. Some interesting examples are found in [10], § 5. Afterward, Dress defined the concept Mackey functors which are generalizations of G-functors and applied them to some fields (Dress [3], [4], [5]). Lam's theory also seems to contribute to their theories. These concepts are frequently used rather in equivariant topology, theory of bilinear forms, etc. than in finite group theory itself.

Now, let's observe first the character ring of a finite group G. It is well known that the following theorems about induced characters play important parts in representation theory.

(M) If $H, K\leq G$ and $\alpha\in ch(H)$, then

$$\alpha^{G}{}_{K}=\sum\alpha^{g}{}_{H^{g}\cap K}{}^{K},$$

where g runs over a complete set of representatives of $H\backslash G/K$.

(F) If $H\leq G$, $\alpha\in ch(H)$, $\beta\in ch(G)$, then

$$\alpha^{G}\cdot\beta=(\alpha\cdot\beta_{H})^{G}.$$

The first formula follows from the *Mackey subgroup theorem*. The second means essentially the same fact as the usual *Frobeniud reciprocity*. See, for example, [16], Th. 2.1, A, B. It is surprising that the formulas (M) and (F) appear also in cohomology theory of finite groups ([2], Prop, 12.9.1; [19], Prop. 4.3.7). The formula (M) is usually called the *double coset for-- mula*. Furthermore the following holds in this case.

* Partially supported by an NSF grant in Princeton I. A. S.

(C) If $H \leq G$, A is a G-module and $\alpha \in H^*(G, A)$, then

$$\alpha_H{}^G = |G : H| \alpha .$$

See [2], § 12. 8 (6), [19], Cor. 2. 4. 9. The character rings do not satisfy (C).

Abstracting these formulas, Green defined G-functors and proved a transfer theorem ([10], Th. 2 and Th. 3. 5 in this paper). His papers show how to use the formulas (F) and (M). But there is still room for growth in his theory. Using his theorem and the isomorphism $H^{-2}(G, \boldsymbol{Z}) \cong G/G'$, we have some transfer theorems for finite groups, for example, it is proved that if a finite group G has an elementary abelian Sylow p-subgroup P, then $P \cap G' = P \cap N_G(P)'$. But by his theorem, we cannot prove not only Wielandt's theorem ([11], Th. 14. 4. 2, [12], Satz 4. 8. 1) but also D. Higman's focal subgroup theorem ([8], Th. 7. 3. 4) and Burnside's theorem ([8], Th. 7. 4. 3). See Example 6. 4, Remark 6. 2, and [20] § 1.

In the present paper, we try to generalize these theorems to cohomological G-functors. It is lucky that we can apply the method which has been developed to prove many transfer theorems in finite group theory. In Section 2, we define G-functors and give some examples. In Section 3, we prove a generalization (Theorem 3. 2) of the focal subgroup theorem. In Section 4, we generalize the concept of singularities which was introduced in [20] § 3. In Section 5, we treat conjugation families and prove an analogue of [20], Th. 4. 9. In Section 6, we give some easy examples.

Notation and terminology are standard and taken from Gorenstein's book [8] for finite groups. The letter G denotes always a finite group, p a prime, k a commutative ring with unit. The notation $H < G$ means that H is a *proper* subgroup of G (that is, $H \neq G$). We shall partially use RPN (the reverse Polish notation) for maps and functors which is usually used in finite group theory. The composition $f: X \to Y$ and $g: Y \to Z$ is denoted by $fg: X \to Z$. The image of an element or a subset or an object A of X by $f: X \to Y$ is denoted by Af, $(A)f$, Af, or $f(A)$. For a ring (or a group) R, the category of right R-modules and R-homomorphisms is denoted by $\mathscr{M}_R$. For $H, K \leq G$, a complete set of representatives of $H \backslash G / K$ is denoted simply $H \backslash G / K$ if there is no damger of confusion. This set is not uniquely defined, but it shall be used in the independent cases to the choice of representatives. The commutator subgroup of G is denoted by G'. The subgroup generated by G' and all p'-elements of G is denoted by $G'(p)$. The abelian groups of linear characters of G is denoted by $\hat{G}$ or $G^\wedge$. If X is an element or a subset of G and g is an element of G, then we set $X^g = g^{-1} X g$. When G acts on a set V (on the right), $C_V(G)$

denotes the set of elements that all elements of G fix. If V is a G-module, then this set is denoted by $C_V(G)$ or $c_V(G)$. The socle of an R-module V is the submodule of V generated by all minimal R-submodules of V and is denoted by $Soc(V)$. The Jacobson radical of a ring R is denoted by $J(R)$.

2. G-functors

In this section, we give some definitions about Green's G-functors and some examples of them. After this, G is always a finite group, k is a commutative ring with unit, and p is a prime.

DEFINITION 2.1 (Green [10], Def. 1.3 and Prop. 1.83). A *G-functor* $a=(a, \tau, \rho, \sigma)$ over k consists of k-modules $a(H)$ $(H\leq G)$ and k-maps

$$\tau^K=\tau_H^K:\ a(H)\longrightarrow a(K):\ \alpha\longmapsto\alpha^K,$$

$$\rho_H=\rho_H^K:\ a(K)\longrightarrow a(H):\ \beta\longmapsto\beta_H,$$

$$\sigma^g=\sigma_H^g:\ a(H)\longrightarrow a(H^g):\ \alpha\longmapsto\alpha^g,$$

for all $H\leq K\leq G$, $g\in G$. These families must satisfy the following axioms:

Axioms for G-functors (In these aximos, D, H, K, $L\leq G$; g, $g'\in G$; $\alpha\in a(H)$, $\beta\in a(K)$).

(G. 1) $\alpha^H=\alpha$, $(\alpha^K)^L=\alpha^L$ if $H\leq K\leq L$,

(G. 2) $\beta_K=\beta$, $(\beta_H)_D=\beta_D$ if $D\leq H\leq K$,

(G. 3) $(\alpha^g)^{g'}=\alpha^{gg'}$, $\alpha^h=\alpha$ if $h\in H$,

(G. 4) $(\alpha^K)^g=(\alpha^g)^{K^g}$, $(\beta_H)^g=\beta^g{}_{H^g}$,

(G. 5) (*Mackey axiom*) If H, $K\leq L$, then

$$\alpha^L{}_K=\sum_{g\in H\backslash L/K}\alpha^g{}_{H^g\cap K}{}^K,$$

(where g runs over a complete set of representatives of $H\backslash L/K$. The sum does not depend on the choice of representatives).

DEFINITION 2.2 ([10], 1.4). A G-functor a is called to be *cohomological* if it satisfies Axiom C:

(C) *Whenever $H\leq K\leq G$ and $\beta\in a(K)$, $\beta_H{}^K=|K:H|\beta$.*

LEMMA 2.1. *Let a be a cohomological G-functor over k. Let $H\leq K\leq G$ and $\alpha\in a(H)$. Then*

$$\alpha^K{}_H-|K:H|\alpha=\sum_{g\in H\backslash K/H}(\alpha^g{}_{H^g\cap H}-\alpha_{H^g\cap H})^H.$$

PROOF. This follows easily from the Mackey axiom and Axiom C. Use the formula

$$\sum_{g\in H\backslash K/H} |H: H^g \cap H| = |K: H| .$$

DEFINITION 2.3 ([10], 1.3 (G.5) and 1.4 (M); Dress [4], p. 195). Let a, b and c be G-functors over k. Then a *pairing* $a\times b\to c$ is defined to be a family of k-linear maps

$$a(H)\times b(H) \longrightarrow c(H): (\alpha, \beta) \longmapsto \alpha\cdot\beta (H\leq G),$$

which satisfy the following axioms: If $H\leq K\leq G$ and $g\in G$, then

$$\text{(P. 1)} \quad (\alpha'\cdot\beta')_H = \alpha'_H\cdot\beta'_H \quad \big(\alpha'\in a(K),\ \beta'\in b(K)\big),$$

$$\text{(P. 2)} \quad (\alpha\cdot\beta)^g = \alpha^g\cdot\beta^g \quad \big(\alpha\in a(H),\ \beta\in b(H)\big),$$

$$\text{(P. 3)} \quad \alpha^K\cdot\beta' = (\alpha\cdot\beta'_H)^K \quad \big(\alpha\in a(H),\ \beta'\in b(K)\big),$$

$$\text{(P. 4)} \quad \alpha'\cdot\beta^K = (\alpha'_H\cdot\beta)^K \quad \big(\alpha'\in a(K),\ \beta\in b(H)\big).$$

The axioms (P. 3) and (P. 4) are called the *Frobenius axioms* in Green's paper [10].

DEFINITION 2.4 ([4], p. 198). Let a be a G-functor over k with pairing $a\times a\to a$. Then a is called a *multiplicative* G-functor. Furthermore, the G-functor a is called a *ring* provided the bilinear map

$$a(H)\times a(H) \longrightarrow a(H): (\alpha, \beta') \longmapsto \alpha\cdot\beta$$

makes $a(H)$ into a ring with unity for each $H\leq G$. Let a and m be G-functors over k with pairing $m\times a\to m$. Then the G-functor m is called a (right) *a-module* provided a is a ring and each $m(H)$ becomes a right unitary $a(H)$-module by the bilinear map

$$m(H)\times a(H) \longrightarrow m(H): (\mu, \alpha) \longmapsto \mu\cdot\alpha .$$

LEMMA 2.2. *Let a be a G-functor over k and H a subgroup of G. Then the k-module $a(H)$ is a $N_G(H)/H$-module by*

$$\alpha\cdot\bar{g} := \alpha^g \big(\alpha\in a(H),\ g\in N_G(H),\ \bar{g} = gH\in N_G(H)/H\big).$$

If furthermore a is a ring, then the action of $N_G(H)/H$ on $a(H)$ preserves the multiplication.

This lemma is clear by the definitions. *After this, $a(H)$ is regarded as a $N_G(H)/H$-module by this action* if there is no special attention.

DEFINITION 2.5 (Green [10], 1.5). Let $a=(a, \tau, \rho, \sigma)$ and $a'=(a', \tau', \rho', \sigma')$

be G-functors over k. Then a *morphism* of G-functors $\theta: a \rightarrow b$ is a family $(\theta(H))_{H \leq G}$ of k-maps $\theta(H): a(H) \rightarrow b(H)$ such that

$$\theta(H)\,\tau'^{K}_{H} = \tau^{K}_{H}\theta(K)\,,$$
$$\theta(K)\,\rho'^{K}_{H} = \rho^{K}_{H}\theta(H)\,,$$
$$\theta(H)\,\sigma'^{g}_{H} = \sigma^{g}_{H}\theta(H^{g})$$

for all $H \leq K \leq G$ and $g \in G$. Assume further that a and b are rings. Then a *ring homomorphism* $\theta = (\theta(H))_{H \leq G}: a \rightarrow b$ is a morphism between G-functors such that each $\theta(H)$ is a homomorphism of rings (preserving the units). Furthermore if a is a ring and if m and n are a-modules, then an a-homomorphism $\theta = (\theta(H))_{H \leq G}: a \rightarrow b$ is a morphism of G-functors such that each $\theta(H): m(H) \rightarrow n(H)$ is an $a(H)$-homomorphism.

We denote by $\mathcal{M}_k(G)$ the category whoose objects are all G-functors over k and with morphisms as just defined. Let $\mathcal{M}_k(G)^c$ be the full subcategory of $\mathcal{M}_k(G)$ whose objects are cohomological. Similarly, the category of rings $\mathcal{A}_k(G)$ and the category of a-modules $\mathcal{M}_a$ (and furthermore, $\mathcal{A}_k(G)^c$ and $\mathcal{M}^c_a$) are defined. The category $\mathcal{M}_k(G)$, $\mathcal{M}_k(G)^c$, $\mathcal{M}_a$, $\mathcal{M}^c_a$ are all abelian categories. See Green [10], 1.5.

DEFINITION 2.6. Let a be a G-functor over k. Then a *subfunctor* b of a is a map $H \rightarrow b(H)$ $(H \leq G)$ such that each $b(H)$ is a k-submodule of $a(H)$ and

$$\beta^{L} \in b(L),\ \beta_{H} \in b(H),\ \beta^{g} \in b(K^{g})$$

for all $H \leq K \leq L \leq G$, $g \in G$, $\beta \in b(K)$.

Let b be a subfunctor of a G-functor a. For $H \leq K \leq L \leq G$ and $g \in G$, let τ'^{L}_{K}, ρ'^{K}_{H}, σ'^{g}_{K} be the restrictions of τ^{L}_{K}, ρ^{K}_{H}, σ^{g}_{H}, respectively, to $b(K)$. Then $(b, \tau', \rho', \sigma')$ is a G-functor.

DEFINITION 2.7. Let a be a G-functor over k. Then a *quotient functor* c of a is a map $H \mapsto c(H)$ $(H \leq G)$ such that there is a subfunctor b of a such that $c(H) = a(H)/b(H)$ for all $H \leq G$.

If c is a quotient functor of a G-functor a, then τ, ρ, σ induce maps $\bar{\tau}$, $\bar{\rho}$, $\bar{\sigma}$ which makes $(c, \bar{\tau}, \bar{\rho}, \bar{\sigma})$ into a G-functor over k.

The concepts of subfunctors and quotient functors are equivalent to ones of subobjects and quotient objects in the category $\mathcal{M}_k(G)$. We shall give some examples of these concepts.

EXAMPLE 2.1. Let $\theta: a \rightarrow b$ be a morphism of G-functors over k. Define three maps as follows:

$$Ker\,\theta:\ H \longmapsto Ker\,\theta(H) \subseteq a(H)\,,$$

$$Im\,\theta:\ H\longmapsto Im\,\theta(H)\subseteq b(H),$$

$$Coker\,\theta:\ H\longmapsto Coker\,\theta(H)=b(H)/Im\,\theta(H).$$

Then $Ker\,\theta$ is a subfunctor of a, and $Im\,\theta$ is a subfunctor of b, and $Coker\,\theta$ is a quotient functor of b. These G-functors $Ker\,\theta$, $Im\,\theta$, $Coker\,\theta$ represent the kernel, the image, the cokernel, respectively, of θ in the category $\mathcal{M}_k(G)$.

EXAMPLE 2.2. Let a be a G-functor over k and $\mathfrak{X}$ a family of subgroups of G. Let $a^{\mathfrak{X}}(H)$ be the k-submodule of $a(H)$ generated by all α^H, where $\alpha\in a(X^g\cap H)$, $g\in G$, $X\in\mathfrak{X}$. Let ${}^{\mathfrak{X}}a(H)$ be the set of all elements α of H such that $\alpha_Y=0$ for all $Y=X^g\cap H$, where $g\in G$, $X\in\mathfrak{X}$. Then $a^{\mathfrak{X}}$ and ${}^{\mathfrak{X}}a$ are both subfunctors of a. If $\mathfrak{X}=1=\{1\}$, then a^1 and $a/{}^1a$ are cohomological. See Green [10], 5.2. (This example was given by T. Okuyama.)

DEFINITION 2.8. Let $a=(a,\tau,\rho,\sigma)$ be a G-functor over k and M a subgroup of G. Then a M-functor $a_{|M}=(a_{|M},\tau',\rho',\sigma')$ is defined by

$$(a_{|M})(H)=a(H)\qquad(H\leq M)$$

$$\tau'^K_H=\tau^K_H,\ \rho'^K_H=\rho^K_H,\ \sigma'^m_H=\sigma^m_H\quad(H\leq K\leq L\leq M,\ m\in M).$$

We shall give some examples of G-functors. Many of them are taken from Green's paper [10], § 5 and Dress' lecture note [3].

EXAMPLE 2.3 ([10], 5.1). The *character ring functor ch* is defined as follows:

$ch(H)$; the character ring of H;

$\tau^K_H:\ \alpha\longmapsto\alpha^K$: the induced character;

$\rho^K_H:\ \beta\longmapsto\beta_H$: the restriction to H;

$\sigma^g_H:\ \alpha\longmapsto\alpha^g$: the conjugation by g

(*i.e.*, $\alpha^g(y)=\alpha(gyg^{-1})$ for $y\in H^g$). This G-functor belongs to $\mathcal{A}_{\boldsymbol{Z}}(G)$. The Mackey axiom and the Frobenius axiom are well known as the Mackey decomposition theorem and the Frobenius reciprocity.

EXAMPLE 2.4 ([10], 5.3). Let V be a kG-module. The *(Tate) cohomology ring functor* $\hat{h}^*_V=\coprod_{n\in\boldsymbol{Z}}\hat{h}^n_V$ is defined as follows.

$\hat{h}^*_V(H):=\hat{h}^*(H,V)=\coprod_{n\in\boldsymbol{Z}}\hat{h}^n(H,V)$: the Tate cohomology group of H;

$\tau^K_H:=cor_{H,K}$: the corestriction (transfer);

$\rho^K_H:=res_{K,H}$: the restriction;

$\sigma^g_H:=con_{H,g}$: the conjugation.

Then h_V^* is a cohomological G-functor over k. The Mackey axiom is called the double coset formula in [2], 12. 9. 2. Since τ, ρ, σ preserve the graduation, we get G-functors $\hat{h}_V^n$, $n \in \boldsymbol{Z}$. Let U, V, W be kG-modules and let $\theta: U \times V \to W$ be a G-pairing, that is, θ is k-bilinear and $\theta(u, v) g = \theta(ug, vg)$ for all $u \in U$, $v \in V$, $g \in G$. Then θ induces a pairing $\cup_\theta$: $h_U^* \times h_V^* \to h_W^*$ which is called a cup product with respect to θ. By this pairing, $\hat{h}_k^*$ is a ring and if V is a kG-module, then $\hat{h}_V^*$ is an $\hat{h}_k^*$-module. Next, let $0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$ be an exact sequence of kG-modules. Then there exist three $\hat{h}_k^*$-homomorphisms f_*, g_*, δ_* between $\hat{h}_k^*$-modules such that the following sequence is exact:

$$\hat{h}_U^* \xrightarrow{f_*} \hat{h}_V^* \xrightarrow{g_*} \hat{h}_W^* \xrightarrow{\delta_*} \hat{h}_U^* \xrightarrow{f_*} \hat{h}_V^* ,$$

where f_* and g_* are of degree 0, but δ_* is of degree 1. See [19], 4. 2. 2, 4. 3. 7, 2. 1. 9.

Similarly, the cohomology ring functor $h_V^* = \coprod_{n \geq 0} h_V^n$ is defined by $h_V^*(H) = H^*(H, V)$. If $n > 0$, then $h_V^n = \hat{h}_V^n$. This G-functor h_V^* has similar properties as $\hat{h}_V^*$. The cohomologies of groups of two kinds are found, for example, in Cartan-Eilenberg [2], Chapter XII.

We get many cohomological G-functors which are subfunctors or quotient functors of cohomology ring functors as stated below.

EXAMPLE 2. 5. ([10], 5. 5). Let V be a kG-module. Then the *centralizer functor* c_V is defined as follows:

$$c_V(H) = \{v \in V \mid vh = v \text{ for all } h \in H\};$$

$$\tau_H^K: \alpha \longmapsto \alpha^K = \sum_{g \in H \backslash K} \alpha g ;$$

$$\rho_H^K: \beta \longmapsto \beta \text{ (the inclusion)};$$

$$\sigma_H^g: \alpha \longmapsto \alpha g .$$

Then c_V is a cohomological G-functor over k and c_V is isomorphic to h_V^0.

Similarly, the Tate centralizer functor $\hat{c}_V$ which is a quotient functor of c_V is defined by

$$\hat{c}_V(H) = c_V(H)/Vt_H, \text{ where } t_H = \sum_{h \in H} h \in kH .$$

Then $\hat{c}_V$ is a cohomological G-functor over k and $\hat{c}_V = \hat{h}_V^0$. If V is a G-algebra, then c_V and $\hat{c}_V$ are rings.

By the way, we give explicit representations of two G-functors related to cohomology ring functors. Define $d_V := h_0^V := (d_V, \tau, \rho, \sigma)$ as follows:

$d_V(H)=h_0^V(H)=V/[V, H]$, where $[V, H]$ is the k-subspace generated by $vh-v$ for all $v\in V$, $h\in H$;

$$\tau_H^K: v+[V, H]\longmapsto v+[V, K] \text{(the natural map)};$$

$$\rho_H^K: v+[V, K]\longmapsto \sum_{g\in H\backslash K} vg+[V, H];$$

$$\sigma_H^g: v+[V, H]\longmapsto vg+[V, H^g].$$

The G-functor $\hat{d}_V:=\hat{h}_V^{-1}$ is defined to be a subfunctor as follows:

$$\hat{d}_V(H)=\hat{h}_V^{-1}(H)=ann_V(t_H)/[V, H], \text{ where}$$

$$ann_V(t_H)=\{v\in V\mid vt_H=0\},\ t_H=\sum_{h\in H} h\in kH.$$

(Our notation: $c_V(G)$, Vt_G, $[V, G]$, $ann_V(t_G)$ are different from usual one. See [2], p. 236 and [19], p. 27.)

EXAMPLE 2.6. The *abelian factor functor ab* is defined as follows:

$ab(H)=H/H'$: the maximal abelian factor group, H' is the commutator subgroup of H;

$\tau_H^K: xH'\longmapsto xK'$: the natural map;

$\rho_H^K: yK'\longmapsto T(y)H'$: the group-theoretic transfer.

$\sigma_H^g: xH'\longmapsto x^g(H^g)'$: the conjugation.

This G-functor over $\boldsymbol{Z}$ is cohomological and isomorphic to the cohomology ring functor $h_{\boldsymbol{Z}}^{-2}$. The Mackey axiom is proved by the pure group-theoretic method. See [6], Prop. 1.6.2, [8], Th. 7.3.3, [18], Prop. 2.3, etc. The quotient functors ab_p, el_p is defined by $ab_p(H):=H/H'(p)$, $el_p(H):=H/H^pH'$ for a prime p. The functors ab_p is regarded as a G-functor over the ring $\boldsymbol{Z}/p^e\boldsymbol{Z}$ for a large e, and el_p is one over the field $\boldsymbol{Z}/p\boldsymbol{Z}$.

Next, the *dual group functor* ab' is defined as follows:

$$ab'(H)=\hat{H}:=Hom(H, \boldsymbol{C}^*);$$

$$\rho_H^K: \beta\longmapsto\beta_{|H}: \text{the restriction};$$

$$\sigma_H^g: \alpha\longmapsto\alpha^g: \text{the conjugation};$$

$$\tau_H^K: \alpha\longmapsto det(\alpha^K+1_H^K), \text{ where}$$

$\alpha^K+1_H^K$ is the induced character and det is the determinant of the character. This G-functor is isomorphic to $\hat{h}_T^1(T:=\boldsymbol{R}/\boldsymbol{Z})$ and $h_{\boldsymbol{Z}}^2$. It is remarkable that Axiom C follows from the Frobenius axiom for characters. For details, see [20], § 2. The subfunctors ab'_p and el'_p are defined as follows:

$ab'_p(H)=\hat{H}_p$ is a unique Sylow p-subgroup of $ab'(H)$.
$el'_p(H)$ is the subgroup of $ab'(H)$ generated by all elements of order p.

EXAMPLE 2.7. The *multiplier functor* M is defined to be $H^2_{C^*}$. The group $M(H)$ is the Schur multiplier of the group H. Let $M_p(H)$ be a unique Sylow p-subgroup of $M(H)$. Then M_p is also a cohomological G-functor.

EXAMPLE 2.8. Assume that G acts as an automorphism group on a p-group P with a descending central series $P=P_0\geq P_1\geq \cdots$. Let $L(P)=\coprod_i(P_i/P_{i+1})$ be the associated Lie ring of P ([8], § 5.6). Then G acts on $L(P)$ as a group of automorphisms of the Lie ring, and so we have the centralizer functor $c_{L(P)}$ (Example 2.3). The G-functor is cohomological and multiplicative. Each $c_{L(P)}(H)$ is a Lie ring.

EXAMPLE 2.9. The *class function ring functor* cl_k is defined as follows:

$$cl_k(H): \text{ the set of class functions of } H \text{ ro } k,$$

$$\tau_H^K: \alpha \longmapsto \alpha^K: x \longmapsto \sum_{u\in H\backslash K}' \alpha(uxu^{-1}),$$

where the sum is taken over representatives u such that $uxu^{-1}\in H$;

$$\rho_H^K: \beta \longmapsto \beta_H: \text{ the restriction};$$

$$\sigma_H^g: \alpha \longmapsto \alpha^g: \text{ the conjugation}.$$

By the element-wise sum and product, $cl_k(H)$ is a ring and cl_k is a multiplicative G-functor over k. If $k=\boldsymbol{C}$, then ch is a subfunctor of $cl_{\boldsymbol{C}}$ (Example 2.3).

EXAMPLE 2.10. The G-functor z is defined as follows:

$$z(H)=Z(kH): \text{ the center of the group algebra.}$$

For each subset X of G, we denote by $\bar{X}$ the sum of the elements of X in kG. Then $Z(kH)$ has a basis $\{\bar{C}|C \text{ is a conjugate class of } H\}$.

$$\tau_H^K: \alpha \longmapsto \sum_{g\in H\backslash K} g^{-1}\alpha g;$$

$$\rho_H^K: \bar{D} \longmapsto \overline{D\cap H}, \text{ where } D \text{ is a conjugate class of } K;$$

$$\sigma_H^g: \alpha \longmapsto g^{-1}\alpha g.$$

If C is a conjugate class of H, and D is a conjugate class of K containing C, and $y\in C$, then

$$\tau_H^K: \bar{C} \longmapsto |C_K(y): C_H(y)|\bar{D}.$$

This G-functor z is neither multiplicative nor cohomological, but it is isomorphic to cl_k in the category $\mathscr{M}_k(G)$ (see Example 2.9). This isomorphism is given by

$$Z(kH) \longrightarrow cl_k(H)$$

$$\sum_{x \in H} \alpha_x x \longmapsto \alpha : x \longmapsto \alpha_x .$$

EXAMPLE 2.11 (Dress [4], § 5). The *Burnside ring functor* Ω is defined as follows: Let $c(H)$ be the free k-module with basis $\{D^H | D \leq H\}$. Define

$$\tau_H^K : c(H) \longrightarrow c(K) : D^H \longmapsto D^K ;$$

$$\rho_H^K : c(K) \longrightarrow c(H) : E^K \longmapsto \sum_{g \in E \backslash K/H} (E^g \cap H)^H$$

$$\sigma_H^g : c(H) \longrightarrow c(H^g) : D^H \longmapsto (D^g)^{H^g} .$$

Then $c=(c, \tau, \rho, \sigma)$ is a G-functior over k. Let $c'(H)$ be the k-submodule of $c(H)$ generated by $D^H-(D^h)^H$ for all $D \leq H$ and $h \in H$. Then c' is a subfunctor of c. The Burnside ring functor Ω is defined to be c/c'. Each $\Omega(H)$ is the Grothendieck ring of the category of finite H-sets. Define the multiplication of elements D^H and E^H of $c(H)$ by

$$D^H \cdot E^H = \sum_{h \in D \backslash H/E} (D^h \cap E)^H ,$$

so Ω is a ring. (This product in $c(H)$ depends on the choice of the representatives of $D \backslash H/E$, but doesn't in $\Omega(H)$). The important fact is that each G-functor a is an Ω-module. The pairing $a \times \Omega \to a$ is given by

$$(\alpha, D^H) \longmapsto \alpha_D{}^H \left(\alpha \in a(H),\ D \leq H\right) .$$

Define a ring homomorphism $\varepsilon : \Omega \to c_k$ by

$$\varepsilon(H) : \Omega(H) \longrightarrow c_k(H) = k : D^H \longmapsto |H : D| .$$

Then a G-functor a is cohomological if and only if $Ker\, \varepsilon$ annihilates a. The Burnside ring functor is constructed also as a quotient functor of the relative free G-functor of the constant G-semifunctor ([10], 5.6).

3. The focal subgroup theorem

In this section, we give a transfer theorem which is a generalization of the focal subgroup theorem ([8], Th. 7.3.4) of D. Higman.

LEMMA 3.1. *Let (a, τ, ρ, σ) be a cohomological G-functor over k and H a subgroup of G. Assume that k has an inverse of $|G : H|$. Then the* *following hold:*

(a) $\rho_H^G: a(G)\to a(H)$ *is a monomorphism,*

(b) $\tau_H^G: a(H)\to a(G)$ *is an epimorphism,*

(c) $a(H)=Im\ \rho_H^G \oplus Ker\ \tau_H^G$, *and so*

$a(G)$ *is isomorphic to a direct summand of* $a(H)$.

PROOF. Set $n=|G:H|$. By Axiom C and the assumption, the k-map

$$\rho\tau = n\cdot id:\ a(G)\longrightarrow a(G):\ \beta\longmapsto \beta_H{}^G = n\beta$$

is an isomorphism. Thus ρ is a mono and τ is an epi. Let α be an element of $a(H)$. Then by Axiom C,

$$\begin{aligned}(\alpha - n^{-1}\alpha^G{}_H)^G &= \alpha^G - n^{-1}\alpha^G{}_H{}^G \\ &= \alpha^G - n^{-1}|G:\ H|\alpha^G \\ &= 0\,,\end{aligned}$$

and so $\alpha - n^{-1}\alpha^G{}_H \in Ker\ \tau_H^G$. Since

$$\alpha = (n^{-1}\alpha^G)_H + (\alpha - n^{-1}\alpha^G{}_H)\,,$$

we have that $\alpha\in Im\ \rho_H^G + Ker\ \tau_H^G$. Thus $a(H)=Im\ \rho_H^G+Ker\ \tau_H^G$. Finally let α be an element of $Im\ \rho_H^G\cap Ker\ \tau_H^G$, so that there is $\beta\in a(G)$ such that $\alpha=\beta_H$. But then $0=\alpha^G=\beta_H{}^G=n\beta$, and so $\beta=0$ by the assumption. Thus $Im\ \rho_H^G\cap Ker\ \tau_H^G=0$. Hence $a(H)=Im\ \rho_H^G\oplus Ker\ \tau_H^G$. Then lemma is proved.

THEOREM 3.2 (Generalized focal subgroup theorem). *Let* (a,τ,ρ,σ) *be a cohomological G-functor over k and H a subgroup of G. Assume that k has an inverse of* $|G:\ H|$. *Then the following hold:*

(a) $Im\ \rho_H^G=\{\alpha\in a(H)|\alpha^g{}_{H\cap H^g}=\alpha_{H\cap H^g}$ *for all* $g\in G\}$.

(b) $Ker\ \tau_H^G$ *is the k-submodule of* $a(H)$ *generated by* $\beta^{gH}-\beta^H$, *where* $g\in G$ *and* $\beta\in a(H\cap gHg^{-1})$.

PROOF. (a) Let α be an element of $a(H)$ such that

(1) $\alpha^g{}_{H\cap H^g} = \alpha_{H\cap H^g}$ for all $g\in G$.

Then by Lemma 2.1,

$$\begin{aligned}\alpha^G{}_H - |G:\ H|\alpha &= \sum_{\in H\backslash G/H} (\alpha^g{}_{H\cap H} - \alpha_{H\cap H^g})^H \\ &= 0\,.\end{aligned}$$

Thus $\alpha=(|G:\ H|^{-1}\alpha^G)_H\in Im\ \rho_H^G$. Conversely, let α be an element of $Im\ \rho_H^G$. Take an element β of $a(G)$ such that $\alpha=\beta_H$. Let $g\in G$ and set $D=H\cap H^g$. Then by the axioms for G-functors, we have that

$$\alpha^g{}_D = \beta_H{}^g{}_D = (\beta^g{}_{H^g})_D = \beta^g{}_D = \beta_D = \alpha_D\,.$$

Thus α satisfies (1). Hence the statement (a) is proved.

(b) let M be the k-submodule of $a(H)$ generated by $\beta^{gH}-\beta^{H}$, where $g\in G$ and $\beta\in a(H\cap gHg^{-1})$. We must show $Ker\,\tau_H^G=M$. It follows easily from the axioms for G-functors that $Ker\,\tau_H^G$ contains M. Let α be an element of $Ker\,\tau_H^G$. Then by Lemma 2.1,

$$(2)\qquad \begin{aligned}|G:H|\alpha &= |G:H|\alpha-\alpha^{G}{}_{H}\\ &= \sum_{g\in H\backslash G/H}(\alpha_{H\cap H^g}-\alpha^{g}{}_{H\cap H^g})^{H}.\end{aligned}$$

Let g be any element of G and set $D=H\cap H^g$, $\beta=\alpha^{g}{}_{D}$. By the axioms for G-functors,

$$\beta^{g^{-1}H}=|H:gDg^{-1}|\alpha=\alpha_D{}^{H},$$

and so

$$(\alpha_D-\alpha^{g}{}_{D})^{H}=\beta^{g^{-1}H}-\beta^{H}.$$

Thus each term of the sum (2) is contained in M. Since $|G:H|^{-1}\in k$, we have that α is also in M. Thus $Ker\,\tau_H^G$ is contained in M. Hence (b) is proved.

EXAMPLE 3.1 (Focal subgroup theorem). Let P be a Sylow p-subgroup of G. Then $P\cap G'$ is generated by $x^{-1}gxg^{-1}$, where $g\in G$ and $x\in P\cap P^g$.

PROOF. Apply Theorem 3.2 (b) to ab_p (Example 2.6). Since $Ker\,\tau_P^G=(P\cap G')/P'$ the statement follows directly from the theorem. See also Example 6.5.

EXAMPLE 3.2 (Cartan-Eilenberg [2], Th. 12.10.1). Let A be a kG-module. Then $\hat{H}^*(G,A)_p$ is isomorphic to the submodule:

$$\left\{\alpha\in\hat{H}^*(P,A)\,\middle|\,\alpha^{g}{}_{P\cap P^g}=\alpha_{P\cap P^g}\ for\ g\in G\right\}.$$

PROOF. This follows directly from Theorem 3.2 (a) and Lemma 3.1 (a).

LEMMA 3.3. *Let (a,τ,ρ,σ) be a cohomological G-functor over k and let H be a subgroup of G. Assume that $|G:H|^{-1}\in k$.*

(a) *Define a quotient functors $\bar{a}=a/J(k)\,a=(\bar{a},\bar{\tau},\bar{\rho},\bar{\sigma})$ of a by*

$$\bar{a}(K)=a(K)/J(k)\,a(K)\qquad for\ all\ K\leq G.$$

Assume that $a(H)$ is finite generated as k-module and that $\bar{\rho}_H^G$: $\bar{a}(G)\to\bar{a}(H)$ is an isomorphism. Then ρ_H^G: $a(G)\to a(H)$ is also an isomorphism.

(b) *Define a subfunctor $Soc\,(a)$ of a by*

$$Soc\,(a)\,(K)=Soc\,\big(a(K)\big)\qquad for\ all\ K\leq G.$$

Assume that $a(H)$ is Artinean and the ρ_H^G: $Soc\,(a)\,(G)\to Soc\,(a)\,(H)$ is an *isomorphism. Then* ρ_H^G: $a(G)\to a(H)$ *is also an isomorphism.* (*Remark*: *J and Soc are defined in the Introduction.*)

PROOF. (a) Set $A=a(H)$, $j=J(k)$, $I=Im\,\rho_H^G$. Then we have that $Im\,\bar{\rho}_H^G=I+jA/jA$. Since $\bar{\rho}_H^G$ is an isomorphism, $A=I+jA$. Thus $A=I$ by the well-known Nakayama's lemma. By Lemma 3.1 (a), we have that ρ_H^G is an isomorphism, as required.

(b) It is easily proved that any k-map $A\to B$ induces a k-map $Soc(A)\to Soc(B)$. Thus $Soc(a)$ is surely a G-functor. By Lemma 3.1, $a(H)=Im\,\rho_H^G \oplus Ker\,\tau_H^G$. Furthermore, by the assumption, we have that

$$\begin{aligned} Soc\big(a(H)\big) &= \big(Soc\big(a(G)\big)\big)\rho_H^G \\ &\subseteq Soc\,(Im\,\rho_H^G)\,. \end{aligned}$$

Thus $Soc(Ker\,\tau_H^G)=0$, and so $Ker\,\tau_H^G=0$, because $a(H)$ is Artinean. By Lemma 3.1, ρ_H^G is an isomorphism. The lemma is proved.

EXAMPLE 3.3. (Tate's theorem, e.g., [10], L 2.5 (1)). Let H be a subgroup of G of index prime to p. Assume that $G/G^pG'\cong H/H^pH'$. Then $G/G'(p)\cong H/H'(p)$. (Here $H^p=\langle x^p|x\in H\rangle$)

PROOF. Take the G-functor $a:=ab_p$ over k, where $k=\boldsymbol{Z}/p^e\boldsymbol{Z}$ for a large integer e (Example 2.6). Then $J(k)=pk$, and so we have that $(a/pa)(K)=K/K^pK'=el_p(K)$ for $K\leq G$. By the assumption, $\rho:(a/pa)(G)\to(a/pa)(H)$ is an isomorphism. Thus by Lemma 3.3 (a), $a(G)$ is isomorphic to $a(H)$, and so $G/G'(p)\cong H/H'(p)$, as required.

We shall argue about the *duality principle* on G-functors. This concept is very useful, but this technique is not used in the present paper and we introduce only the outline here. Observing the definition of G-functors, we know that we can define G-functors into any abelian category $\mathscr{C}$. So if (a,τ,ρ,σ) is a G-functor into $\mathscr{C}$, then each $a(H)$ is an object of $\mathscr{C}$, and τ_H^K, etc. are morphisms in $\mathscr{C}$, and the axioms for G-functors are represented as commutativity of diagrams.

LEMMA 3.4. *Let* (a,τ,ρ,σ) *be a G-functor into an abelian category* $\mathscr{C}$. *Let* $F:\mathscr{C}\to\mathscr{D}$ *be a contravariant additive functor between abelian categories. Then we have a G-functor* $(a^F,\tau',\rho',\sigma')$ *into* $\mathscr{D}$ *which is defined as follows*:

$$\begin{aligned} (a^F)(H) &= F\big(a(H)\big)\,; \\ \tau'^K_H &= F(\rho_H^K)\,; \\ \rho'^K_H &= F(\tau_H^K)\,; \\ \sigma'^g_H &= F(\sigma^{g^{-1}}{}_{H^g})\,. \end{aligned}$$

We omit the proof of this lemma. By this lemma, we know that there is the dual statement of every statement on G-functors and that using the duality principle in category theory, we do not need to prove the dual statements, provided the ordinary statements are already proved in any abelian category. For example, the dualization of $Im\ \rho_H^G$ is $Coim\ \tau_H^G$ by Lemma 3. 4 and $Ker(Coim\ \tau_H^G)=Ker\ \tau_H^G$. Furthermore, Theorem 3. 2 (b) and its proof are the dualization of (a). We give some other examples. First Lemma 3. 1 (b) is the dual of Lemma 3. 2 (a), and Lemma 3. 1 (c) is self-dual. Applying Lemma 3. 4 to the contravariant functor $\mathcal{M}_{kG}\rightarrow\mathcal{M}_{kG}$: $V\mapsto V^*=Hom_k(V, k)$, we have that h_V^0, $\hat{h}_V^0$ in Example 2. 5 are the dual of h_0^V, $\hat{h}_V^{-1}$, respectively. Axion C and the Mackey axiom are self-dual, but the axiom for pairings is not. See [14], p. 32.

Finally, we state Green's transfer theorem. Note that it gives only weak results but it is powerful to not only cohomological but also non-cohomological G-functors. See [9], § 5.

THEOREM 3. 5 (Green [10], Theorem 2). *Let (a, τ, ρ, σ) be a G-functor over k, and let $D\leq H\leq G$. Set*

$$\mathfrak{X}=\{D\cap D^g \mid g\in G-H\}\,, \qquad \text{and}$$

$$\mathfrak{Y}=\{H\cap D^g \mid g\in G-H\}\,.$$

Let $a(\mathfrak{X})^H$, etc. be the k-submodule of $a(H)$ generated by $Im\ \tau_X^H$ for $X\in\mathfrak{X}$, etc. Assume that

$$a(\mathfrak{X})^H=a(D)^H\cap a(\mathfrak{Y})^H\,.$$

(This assumption is satisfied if, for example, the subgroup D is a Sylow subgroup of G and $H=N_G(D)$.) Then τ_H^G and ρ_H^G induce isomorphisms

$$t:\ a(D)^H/a(\mathfrak{X})^H\longrightarrow a(D)^G/a(\mathfrak{X})^G\,,$$

$$r:\ a(D)^G/a(\mathfrak{X})^G\longrightarrow a(D)^H/a(\mathfrak{X})^H\,,$$

which are the inverse of each other. If, furthermore, a is multiplicative, then t and r are multiplicative k-maps.

4. Singularities

In this section, we study transfer theorems for cohomological G-functors which are generalizations of Wielandt type theorems for finite groups (Huppert [12], Satz 4. 8. 1). The concept of singularities is essential for transfer theorems of this type. This concept was introduced in Yoshida [20], § 3 for the dual group functor (Example 2. 4) and was very effective to prove some transfer theorems for finite group theory (for example, [20], Th. 4.2). See also Glauberman [6] and Suzuki [18]. The definition of singularities for cohomological G-functors is imitation of [20] as is shown below, but it seems to be very difficult to study singularities for general cohomological G-functors, for example, even for Schur multiplier functors, and there remain many unsolved questions. The results in this section are very basic.

DEFINITION. Let H be a finite group and (a, τ, ρ, σ) be a cohomological H-functor. Let S and X be subgroups of H and let α be an element of $a(S)$. Then the triplet (S, α, X) is called a *singularity* for a (or in H) provided

(S. 1) $\alpha^H{}_X \neq 0$, and

(S. 2) if $Y \leq S$ and $|Y| < |X|$, then $\alpha_Y = 0$.

The subgroup S is called the *singular subgroup* of the singularity (S, α, X). Furthermore, a subgroup which is the singularity of a singularity also is called a singular subgroup. If the singular subgroup of a singularity in H is a proper subgroup of H, then the singularity is called to be *proper*.

REMARK 1. There is another definition of singularities. It is given by replacing the above condition (S. 2) by the following:

(S. 2′) If Y is a subgroup which is conjugate to a proper subgroup of X in H, then $\alpha_Y = 0$.

Clearly, if (S. 2) holds, then so does (S. 2′). The triplet (S, α, X) which satisfies (S. 1) and (S. 2′) is called also a *singularity*. The second definition is a direct generalization of singularities for the dual group functors. See [20], Def. 3.1. In this paper, we adopt the first definition (S. 1+S. 2). The results and their proofs are almost the same in either case.

REMARK 2. The concept of singularities is not self-dual and thus cosingularities are defined by the dualization as follows: Let S, $X \leq H$ and a be a cohomological H-functor over k. Then (S, B, X) is called a *cosingularity* provided B is a k-submodule of $a(S)$ and

(C S. 1) $a(X)\tau^H \rho_S \not\subseteq B$, and

(C S. 2) if $Y \leq S$ and $|Y| < |X|$, then

$a(Y)\tau^S \subseteq B$.

Similarly, (C S. 2′) is stated. This concept is useful to study $Ker\, \tau^G_H$. When a is the abelian factor functor of H, this concept is the same as one in Suzuki [18], Def. 2.4. But in the present paper, we shall use singularities rather than cosingularities.

LEMMA 4.1. *Let (a, τ, ρ, σ) be a cohomological G-functor over k and*

H a subgroup of G such that k has the inverse of $|G: H|$. Let B be a k-submodule of $a(H)$ containing $Im\,\rho_H^G$. Assume that $B\neq Im\,\rho_H^G$. Then the following hold:

(a) *$B=Im\,\rho_H^G\oplus(B\cap Ker\,\tau_H^G)$, and so there is a nonzero element β of $B\cap Ker\,\tau_H^G$.*

(b) *Let β be any nonzero element of $Ker\,\tau_H^G$. Then there exist $g\in G-H$ and $X\leq H$ such that (S,α,X) is a singularity for $a_{|H}$, where $\alpha=\beta^g{}_S-\beta_S$ and $S=H\cap H^g$. (Note: Let $\mathfrak{X}$ be the set of all subgroups Y of H with $\beta_Y\neq 0$. Then as the above subgroup X, we can take a member of $\mathfrak{X}$ of minimal order.)*

PROOF. The statement (a) follows from Lemma 3.1 (c). Take a subgroup X as in the Note. By the assumption and Lemma 2.1,

$$\begin{aligned}0&\neq-|G:H|\beta_X\\&=(\beta^G{}_H-|G:H|\beta)_X\\&=\sum_{g\in H\backslash G/H}(\beta^g{}_{H\cap H^g}-\beta_{H\cap H^g})^H{}_X,\end{aligned}$$

and so there is an element g of G such that

(1) $\quad \alpha^H{}_X\neq 0,$

where $\alpha=\beta^g{}_S-\beta_S$ and $S=H\cap H^g$. Clearly g is in $G-H$. Let Y be a subgroup of S with $|Y|<|X|$. Then Y, $gYg^{-1}\notin\mathfrak{X}$ by the minimality of the order of X. Thus

(2) $\quad \alpha_Y=(\beta_{gYg^{-1}})^g-\beta_Y=0$

(1) and (2) mean that (S,α,X) is a singularity for $a_{|H}$. The lemma is proved.

The following lemma is basic to study singularities, which corresponds to [20], L 3.2. The same results were independently proved by Sasaki in his paper [17], Lemma 2. For singularities which are defined by (S. 1)+(S. 2), the corresponding results hold.

LEMMA 4.2. *Let (a,τ,ρ,σ) be a cohomological H-functor over k for a finite group H. Let (S,α,X) be a singularity for a. Then the following hold:*

(a) *Let u, $v\in H$. Then (S^u,α^u,X^v) is also a singularity for a.*

(b) *X is contained in a conjugate of S.*

(c) *Let T be a subgroup of S such that k has an inverse of $|S:T|$. Then (T,α_T,X) is also a singularity for a.*

(d) *Let $S\leq R\leq H$. Then (R,α^R,X) is also a singularity for a.*

(e) *Let $S\leq D\leq H$. Then there is a conjugate Z of X such that (S,α,Z) is a singularity for $a_{|D}$.*

(f) *Let* $X \leq D \leq H$. *Then there is an element* h *of* H *such that* $(T, \alpha^h{}_T, X)$ *is a singularity for* $a_{|D}$, *where* $T = S^h \cap D$.

(g) *Let* $S \leq R \leq H$. *If there is an element* β *of* $a(R)$ *such that* $\alpha = \beta_S$, *then* $|R: S| \neq 0$ *in* k.

(h) *Let* β *be an element of* $a(S)$ *such that* $\beta_Y = 0$ *for each* $Y \leq S$ *with* $|Y| < |X|$. *Then* (S, β, X) *or* $(S, \alpha - \beta, X)$ *is a singularity in* H.

PROOF. We will prepare an easy result before starting to prove the lemma.

(1) *If* $h \in H$, *and* $Z \leq S^h$, *and* $\alpha^h{}_Z \neq 0$, *then* $|Z| \geq |X|$.

Set $Y = hZh^{-1} \leq S$. Then $0 \neq \alpha^h{}_Z = (\alpha_Y)^h$, and so $\alpha_Y \neq 0$. By the condition (S. 2) in the definition of singularities, we have that $|Y| = |Z| \geq |X|$, proving (1).

Now, we start the proof of the lemma.

(a) Clear.

(b) By the Mackey axiom, we have

$$0 \neq \alpha^H{}_X = \sum_{h \in S \backslash H / X} (\alpha^h{}_{X \cap S^h})^X$$

Thus there is an element h of H such that $\alpha^h{}_T \neq 0$, where $T = X \cap S^h$. By (1), $|T| \geq |X|$. Thus $T = X \leq S^h$, as required.

(c) Since a is cohomological, $\alpha_T{}^H = (\alpha_T{}^S)^H = |S: T| \alpha^H$. Since $\alpha^H{}_X \neq 0$ and $|S: T|^{-1} \in k$, we have that $\alpha_T{}^H{}_X = 0$, so (T, α_T, X) satisfies (S. 1). Let Y be a subgroup of T $(\leq S)$ with $|Y| < |X|$. Then $(\alpha_T)_Y = \alpha_Y = 0$ by (S. 2). Thus (T, α_T, X) satisfies (S. 1) and (S. 2), and so it is a singularity for a.

(d) Clearly $(\alpha^R)^H{}_X = \alpha^H{}_X \neq 0$, so (R, α^R, X) satisfies (S. 1). Let Y be a subgroup of R with $|Y| < |X|$. Then

$$\alpha^R{}_Y = \sum_{t \in S \backslash R / Y} \alpha^r{}_{S^r \cap Y}{}^Y$$

by the Mackey axiom. By (1), we have that $\alpha^R{}_Y = 0$. Hence (R, α^R, X) satisfies also (S. 2) and it is a singularity for a, as required.

(e) By the Mackey axiom,

$$\begin{aligned} 0 \neq \alpha^H{}_X &= (\alpha^D)^H{}_X \\ &= \sum_{h \in D \backslash H / X} (\alpha^D)^h{}_{D^h \cap X}{}^X . \end{aligned}$$

Thus there is an element h of H such that

$$(\alpha^D)^h{}_{D^h \cap X} \neq 0 .$$

Set $Z = D \cap hXh^{-1}$, so that $\alpha^D{}_Z \neq 0$. Again by the Mackey axiom,

$$0 \neq \alpha^D{}_Z = \sum_{d \in S \backslash D / Z} \alpha^d{}_{S^d \cap Z}{}^Z .$$

Thus by (1), there is an element d of D such that $|X|\leq|S^d\cap Z|\leq|Z|$. Since $Z=D\cap hXh^{-1}$, this implies that $Z=hXh^{-1}$. It is already shown that (S, α, Z) satisfies (S. 1) for $a_{|D}$ (that is, $\alpha^D{}_Z\neq 0$). Furthermore, it is clear that it satisfies also (S. 2) for $a_{|D}$ (that is if $Y\leq S$ and $|Y|<|Z|$, then $\alpha_Y=0$). Hence (S, α, X) is a singularity, as required.

(f) By the Mackey axiom,

$$0\neq\alpha^H{}_X=(\alpha^H{}_D)_X$$
$$=\sum_{h\in S\backslash H/D}\alpha^h{}_{S^h\cap D}{}^D{}_X\,.$$

Thus there is an element h of H such that $(\alpha^h{}_T)^D{}_X\neq 0$, where $T=D\cap S^h$. Let Y be a subgroup of $T=D\cap S^h$ such that $|Y|<|X|$. Then (1) yields $(\alpha^h{}_T)_Y=\alpha^h{}_Y=0$. Hence $(T, \alpha^h{}_T, X)$ is a singularity, as required.

(g) By Axiom C, have $0\neq\alpha^H{}_X=(\beta_S{}^R)^H=|R: S|\beta^H{}_X$, and so $|R:S|\neq 0$ in k.

(h) Set $\gamma=\alpha-\beta$. If $\beta^H{}_X=\gamma^H{}_X=0$, then $\alpha^H{}_X=(\beta+\gamma)^H{}_X=0$, a contradiction. Thus $\beta^H{}_X\neq 0$ or $\gamma^H{}_X\neq 0$. Since $\beta_Y=\gamma_Y=0$ for a subgroup Y of S with $|Y|<|X|$, we have that (S, β, X) or (S, γ, X) is a singularity according to $\beta^H{}_X\neq 0$ or $\gamma^H{}_X\neq 0$. The proof of the lemma is completed.

LEMMA 4.3. *Let H be a finite group and (a, τ, ρ, σ) be a cohomological H-functor over k. Let D and S be subgroups such that $H=DS$. Assume that there are $X\leq D$ and $\alpha\in a(S)$ such that $(D, \alpha_{S\cap D}, X)$ satisfies the conditions* (S. 1) *and* (S. 2′) *for $a_{|D}$. Then (S, α, X) also satisfies* (S. 1) *and* (S. 2′) *for a.*

PROOF. Set $T=D\cap S$. We must show that (S. 1) $\alpha^H{}_X\neq 0$ and (S. 2) if Y is a subgroup of S which is conjugate to a proper subgroup of X, then $\alpha_Y=0$. First since $H=DS$, it follows from the Mackey axiom that $\alpha^H{}_D=\alpha_T{}^D$. Thus $\alpha^H{}_X=(\alpha_T)^D{}_X\neq 0$ by (S. 1) for (T, α_T, X), and so (S, α, X) also satisfies (S. 1). Next, take any subgroup Y of S such that $Y^h<X$ for an element h of H. We must show $\alpha_Y=0$. Choose $d\in D$ and $s\in S$ with $h=sd$. Set $Z=Y^s$. Then $Z<dXd^{-1}$, and so $Y^s=Z\leq S\cap D=T$. Thus Z is a subgroup of T which is conjugate to a proper subgroup of X in D. By (S. 2′) for (T, α_T, X), we have that $\alpha_Z=0$. Thus $(\alpha_Y)^s=\alpha^s{}_Z=\alpha_Z=0$, so $\alpha_Y=0$, as required. Hence (S, α, X) satisfies (S. 1) and (S. 2′). The lemma is proved.

REMARK 3. This lemma doesn't hold for singularities adopted in this paper (S. 1+S. 2). This lemma bridges the gap in [20], L 3. 4.

When we try to apply transfer theorems for cohomological G-functors, we have to solve first the following problem:

PROBLEM 1. Let (a, τ, ρ, σ) be an H-functor over k. For what kinds of H and a does there exist no proper singularity?

The importance of this problem is obvious by the following lemma.

LEMMA 4.4. *Let P be a Sylow p-subgroup of G and (a, τ, ρ, σ) be a cohomological G-functor over k. Assume that $|G: P|^{-1}\in k$ and that P has no proper singularity for $a_{|P}$. Then $a(G)$ is isomorphic to $a(N_G(P))$.*

This lemma follows directly from Lemma 4.1 and Lemma 3.1. See also [17], Th. 1.

In general, it is very hard to solve Problem 1 for any given H-functor. In practice, it suffices to consider Problem 1 only in the case where H is a p-group and k is a field of characteristic p. As an example, we take cohomology ring functor (Example 2.4). Let P be a Sylow p-subgroup of G and $T=\boldsymbol{Q}/\boldsymbol{Z}$. Set $N=N_G(P)$. For what kind of P can we say that $H^n(G, T)_p\cong H^n(N, T)_p$? Define a subfunctor h^n of H_T^n by $h^n(H)=\{\bar{\alpha}\in H^n(H, T)|p\alpha=0\}$. By Lemma 3.3 and Lemma 4.4, we know that this problem solves if the following solves:

PROBLEM 2. Characterize p-groups without proper singularities for h^n.

When $n=1$, this problem was happily solved, that is, a p-group P has no proper singularity for $h^1(=el'_p)$ if and only if P has no epimorphism onto the wreath product $\boldsymbol{Z}_p \wr \boldsymbol{Z}_p$. Thus if G has such a p-group P as a Sylow p-subgroup, the $P\cap G'=P\cap N_G(P)'$. Refined proofs are found in [6] and [18].

Sasaki is studying Problem 2 in the case $n=2$. He made clear the machinery of the proof of Holt's theorem ([17]) and he proved that if $cl(P)< p/2$, then P has no proper singularity for h^2. He furthermore announced that some p-groups, for example, 2-groups of maximal class has no proper singularity for h^2. The problem is more complicated than the case of $n=1$. When $n=2$, the above problem is completely rewritten by the language of pure group theory. Nevertheless, it is still hard to solve the problem.

Finally, we add another easy result about singularities which we need later.

LEMMA 4.5. *Let P be a p-group and (a, τ, ρ, σ) be a cohomological P-functor over k, where k is a field of characteristic p. Let $S\trianglelefteq P$ and set $\bar{P}=P/S$. If S is a singular subgroup for a, then the $k\bar{P}$-module $a(S)$ (Lemma 2.2) contains a regular $k\bar{P}$-submodule $k\bar{P}$. When $S=1$, the converse also holds.*

PROOF. Assume first that (S, α, X) is a singularity for a. By Lemma 4.2 (b), X is contained in S. Thus $\alpha^P{}_X=(\alpha^P{}_S)_X\neq 0$, and so by the Mackey axiom,

$$0\neq \alpha^P{}_S=\sum_{x\in P/S}\alpha^x\,.$$

Set $\beta\bar{x}=\beta^x$ for $\beta\in a(S)$, $x\in P$, $\bar{x}=xS\in\bar{P}$. Set $\bar{t}=\sum_{\bar{x}\in\bar{P}}\bar{x}\in k\bar{P}$, so that $\alpha\bar{t}\neq 0$. Set $I=\{y\in k\bar{P}|\alpha y=0\}$. Then I is an ideal of $k\bar{P}$ which does not contain $\bar{t}$, and $\alpha\cdot k\bar{P}\cong k\bar{P}/I$. The minimal right ideal of $k\bar{P}$ is only $k\bar{t}$. This fact follows from the fact that only irreducible $k\bar{P}$-module is a trivial one. But since $\bar{t}\notin I$, we have that $I=0$, and so $a(S)\supseteq\alpha\cdot k\bar{P}\cong k\bar{P}$, as required. Assume next that $a(1)$ contains a regular kP-module. Then there is $\alpha\in a(1)$ such that $\alpha\cdot kP\cong kP$, and so $\alpha\cdot\bar{t}=\alpha^P{}_1\neq 0$. Thus $(1,\alpha,1)$ is a singularity. The lemma is proved.

COROLLARY 4.6. *Let P be a p-group, and k be a field of characteristic p, and V be a finite generated kP-module. Assume that $\hat{c}_V(P)=0$ (Example 2.5). Then V is a free kP-module.*

PROOF. We shall argue by the induction on $dim_k V$. We may assume that V is indecomposable. We shall consider the P-functor (c_V,τ,ρ,σ) in Example 2.5. Then the assumption $\hat{c}_V(P)=0$ means that $\tau:=\tau_1^P: V\to c_V(P)$ is an epimorphism. Let v be a nonzero element of $c_V(P)$, so that v is written as the form $v=u\tau$ for an element u of V. Thus we have that $(1,u,1)$ is a singularity for c_V. By the lemma, V has a regular kP-submodule U. Since $U(\cong kP)$ is an injective kP-module, U is a direct summand of V, and so $U=V\cong kP$, for V is indecomposable, as required. (The injectivity of kP follows, for example, from the fact that the dual module $(kP)^*$ is isomorphic to kP.)

5. Conjugation families

In this section, we treat the relations between transfer theorems for G-functors and conjugation families. The purpose of this section is to generalize Alperin [1], Th. 4.2 and Yoshida [20], Th. 4.9. In order to make sure, we begin with the definition of conjugation families.

DEFINITION 5.1. Let P be a Sylow p-subgroup of G. A family $\mathscr{F}$ for P is a set of pairs (F,N), where $F\leq P$ and $F\trianglelefteq N\leq G$. A family $\mathscr{F}$ for P is called a *conjugation family* (for P in G) provided whenever A and B are subsets of P and $A^g=B$ for an element g of G, then there exist members $(F_1,N_1)\cdots$, (F_m,N_m) of $\mathscr{F}$ and elements $g_1,\cdots,g_m$ of G such that

$$g_i\in N_i,\ g=g_1,\cdots,g_m\,,$$
$$A^{g_1\cdots g_i}\subseteq F_i\quad(i=1,\cdots,m)\,.$$

By the well known Alperin's theorem ([1], Main theorem), there exists a conjugation family. In practice, we desire conjugation families of which members satisfy especial properties. For the purpose, some kinds of conjugation families are known (for example, Goldschmidt [7], Puig [15], etc.).

DEFINITION 5. 2. Let $\mathscr{F}$ be a family for a Sylow p-subgroup P of G and let $a=(a, \tau, \rho, \sigma)$ be a cohomological G-functor over k. Then we say that $\mathscr{F}$ *controls* transfer for a provided $Im\ \rho_P^G$ equals the set of all $\alpha\in a(P)$ such that

$$\alpha_F{}^n=\alpha_F \text{ for all } (F, N)\in\mathscr{F} \text{ and } n\in N.$$

THEOREM 5. 1. *Let (a, τ, ρ, σ) be a cohomological G-functor over a field k of characteristic p and $\mathscr{F}$ be a conjugation family for a Sylow p-subgroup P of G. Let $\mathscr{F}^*$ be the set of all (F, N) in $\mathscr{F}$ such that for some $y\in G$, $P\cap P^y=P\cap F^y$ is a singular subgroup for $a_{|P}$. Then $\mathscr{F}^*$ controls transfer for a.*

PROOF. Let B be the set of all element α of $a(P)$ such that $\alpha_F{}^n=\alpha_F$ for all $(F, N)\in\mathscr{F}^*$ and $n\in N$. Then it follows easily from the axioms for G-functors that B contains $Im\ \rho_P^G$. Suppose the theorem is false, By Lemma 4. 1, there are $\beta\in B$, $X\leq P$, $g\in G$ such that $(S, \beta^g{}_S-\beta, X)$, where $S=P\cap P^g$ is a singularity for $a_{|P}$ and X is of minimal order under the condition that $\beta_X\neq 0$. (See also Note in Lemma 4. 1.) Set $R=gSg^{-1}$. Take $(F_i, N_i)\in\mathscr{F}$ and $g_i\in N_i$ for $i=1, \cdots, m$ which satisfy

$$R^{g_1\cdots g_i}\subseteq F_i \quad \text{for} \quad i=1, \cdots, m\,,$$
$$g=g_1\cdots g_m\,.$$

For each i, set

$$g_i'=g_{i+1}\cdots g_m,\ g_0'=g,\ g_m'=1\,,$$
$$\beta_i=\beta^{g_i'}{}_S\,.$$

Then each β_i is well-defined. We have that

$$0\neq(\beta^g{}_s-\beta_S)^P{}_X=\sum_i(\beta_{i-1}-\beta_i)^P{}_X\,.$$

Thus there exists i such that $(\beta_{i-1}-\beta_i)^P{}_X\neq 0$. Furthermore, if Y is a subgroup of S with $|Y|<|X|$, then $(\beta_{i-1})_Y=(\beta_i)_Y=0$. This follows from the minimality of X. Thus $(S, \beta_{i-1}-\beta_i, X)$ is a singularity for $a_{|P}$. Set $F=F_i$, $h=g_i'$, $x=g_i$, and $T=P\cap F^x$. Then

$$\beta_{i-1}-\beta_i=\beta^{nx}{}_S-\beta^x{}_S$$
$$=\left((\beta_F{}^n-\beta_F)^x{}_T\right)_S\,.$$

Since T contains S and $|T:S|$ is a power of p $(=ch\ k)$, Lemma 4. 2 (g) yields that $S=P\cap F^x=T$. Next by Lemma 4. 2 (h), we have that (S, β^y, X)

is a singularity for $y=x$ or nx. Since $P\cap P^y \supseteq P\cap F^y = P\cap F^x = S$, it follows again from Lemma 4. 2 (g) that $P\cap F^y = S$. The theorem is proved.

DEFINITION 5. 3. Let M be a proper subgroup of G. Then M is called a *strongly p-embedded subgroup* of G provided M has a nontrivial Sylow p-subgroup and $M\cap M^g$ is a p'-group for each $g\in G-M$.

Let M be a proper subgroup of G with nontrivial Sylow p-subgroup P. Then M is strongly p-embedded in G if and only if M contains the subgroup

$$\left\langle N_G(Q) \middle| 1\neq Q\leq P \right\rangle .$$

It is well known that if M is a strongly p-embedded subgroup of G and M does not contain $O_{P'}(G)$ (for example, G is p-solvable), then a Sylow p-subgroup of G is cyclic or generalized quaternion.

DEFINITION 5. 4. Let P be a Sylow p-subgroup of G. A subgroup A of P is called *tame* (for P) if $N_P(A)$ is a Sylow p-subgroup of $N_G(A)$.

THEOREM 5. 2. *Let (a, τ, ρ, σ) be a cohomological G-functor over a field k of characteristic p and let P be a Sylow p-subgroup of G. let $\mathscr{F}$ be the family of pairs (F, N), where $F<P$ and $N_P(F)\leq N\leq N_G(F)$, which satisfy the following conditions (a) to (g):*

(a) *F is tame, that is, $T=N_P(F)$ is a Sylow p-subgroup of $L:=N_G(F)$.*

(b) *There is an element β of $a(T)$ such that $\bar{M}:=M/F$ is strongly p-embedded in $\bar{L}:=L/F$, where $M:=\{g\in L|\beta_{F^g}=\beta_F\}\leq L$. In particular, $F=O_p(L)$.*

(c) *$N\not\leq M$ and $F=O_p(N)$. Furthermore, if $\bar{K}\trianglelefteq\bar{N}$, then either $\bar{K}\leq O_{p'}(\bar{N})$ or $\bar{K}\geq O^p(\bar{N})$. In particular, $O^{p'}(N)=N$ and $O^p(N/O_{pp'}(N))$ is 1 or noncyclic simple.*

(d) *Assume that F is not a Sylow p-subgroup of $O_{p'p}(L)$. Then $N=O_{pp'p}(N)$ and $\bar{T}:=T/F$ is cyclic or generalized quaternion. If $p\neq 2$, then L is p-solvable.*

(e) *If $C_T(F)\not\leq F$, for example, if N is not p-constrained, then $O^p(N)\leq C_G(F)$.*

(f) *The subgroup F is a singular subgroup for $a_{|M}$. The $k\bar{T}$-module $a(F)$ (see Lemma 2. 2) contains a submodule isomorphic to $k\bar{T}$ (that is, a regular $k\bar{T}$-submodule).*

(g) *F contains a conjugate of a singular subgroup for $a_{|P}$.*

Now, let $\mathscr{F}_0$ be a family for P which contains $(P, N_G(P))$ and satisfies the follows:

(*) *For each $(F, N)\in\mathscr{F}$, there are $g\in G$ and $N_0\leq N_G(F^g)$ such that $(F^g, N_0)\in\mathscr{F}_0$, $N^g\leq N_0$ and F^g is tame for P.*

Then $\mathscr{F}_0$ *controls transfer for a.*

To prove this theorem, we need the following lemma:

LEMMA 5.3. *Let a be a cohomological G-functor over a field of characteristic p, and P be a Sylow p-subgroup of G. Let* α *be an element of* $a(P)-Im\,\rho_P^G$ *such that* $\alpha^n=\alpha$ *for all* $n\in N_G(P)$. *Then there are* $F<P$ *and* $N\leq G$ *satisying the following:*

(i) *The conditions (a) to (g) in Theorem 5.2 hold. We can take* $\beta=\alpha_T$ *as the element* β *in (b), and so* $\alpha_F{}^n\neq\alpha_F$ *for some element n of N.*

(ii) *Let E be a tame subgroup of P conjugate to F in G. Then there is a G-conjugate* N_1 *of N such that* (E, N_1) *also satisfies the conditions (a) to (g) in the theorem. We can take* α_U *as* β *in (b), where* $U=N_P(E)$, *so* $\alpha_E{}^m\neq\alpha_E$ *for some* $m\in N_1$.

PROOF. For each subgroup F of P, we set

$$M(F)=\left\{n\in N_G(F)\,\middle|\,\alpha_F{}^n=\alpha_F\right\}.$$

Then $N_P(F)\leq M(F)\leq N_G(F)$. Let $\mathscr{F}'$ be the set of all subgroups F of P satisfying the following conditions:

(1) F contains a conjugate of a singular subgroup for $a_{|P}$;

(2) $M(F)\neq N_G(F)$, that is, there is $n\in N_G(F)$ such that $\alpha_F{}^n\neq\alpha_F$.

By Alperin's theorem, the family of all subgroup of P together with the normalizers is a conjugation family for P. Thus $\mathscr{F}'\neq\phi$ by Theorem 5.1. Since $\alpha_P{}^n=\alpha_P$ for all $n\in N_G(P)$, we have that $P\notin\mathscr{F}'$ and so if $F\in\mathscr{F}'$, then $F<N_P(F)$. Let $\mathscr{F}^*$ be the set of elements of $\mathscr{F}'$ of maximal order. We shall first show the following assertion:

(3) *If a subgroup E is G-conjugate to a member F of* $\mathscr{F}^*$, *then there is an element g of G such that* $M(E)=M(F)^g$ *and* $E=F^g$. *In particular,* $E\in\mathscr{F}^*$

By Alperin's theorem, we can take $D_i\leq P$ and $g_i\in N_G(D_i)$, $i=1,\cdots,m$, such that

$$F^{g_1\cdots g_i}\subseteq D_i\,,\qquad i=1,\cdots,m\,,$$

$$E=F^g\,,\qquad g=g_1,\cdots,g_m\,.$$

Assume that $|D_j|=|F|$ for a number j. Then

$$F^{g_1\cdots g_j}=D_J=D_j^{g_j}\,,$$

and so $F^{g'}=E$, where $g'=g_1\cdots g_{j-1}g_{j+1}\cdots g_m$. Thus we can exclude such (D_j, g_j) from the series $(D_1, g_1),\cdots,(D_m, g_m)$. We may now assume that $|D_i|>|F|$ for each i. We will show that for each i,

$$M(F)^{g_1\cdots g_i}=M(F^{g_1\cdots g_i})\,.$$

Set $x=g_1$, $A=F^x$, $D=D_1$. By the induction argument, it will suffice to show that $M(F)^x=M(A)$. Let $n\in N_G(F)$ and set $n'=n^x$. Then the axioms for G-functors yield the following:

(4) $(\alpha_F{}^n-\alpha_F)^x=(\alpha^x{}_D-\alpha_D)_A{}^{n'}-(\alpha^x{}_D-\alpha_D)_A+(\alpha_A{}^{n'}-\alpha_A)\,.$

Suppose $\alpha^x{}_D\neq\alpha_D$. Then D satisfies (1) and (2), and so $F\leq D\in\mathscr{F}$. Since F is an element of $\mathscr{F}$ of maximal order, we have that $F=D$, a contradiction. Thus $\alpha_D{}^x=\alpha_D$. By (4),

$$(\alpha_F{}^n-\alpha_F)^x=(\alpha_A{}^{n'}-\alpha_A),\ A=F^x,\ n'=n^x\,.$$

Thus we have that $n\in M(F)$ if and if and only if $n^x\in M(F^x)$, and hence $M(F)^x=M(F^x)$, as required. (3) is proved.

Let $\mathscr{F}^{**}$ be the set of pairs (F,N) satisfying the following conditions:

(5) $F\in\mathscr{F}^*$ and F is tame for P;

(6) $N_P(F)\leq N\leq N_G(F)$ and N is not contained in $M(F)$;

(7) N is a minimal subject to (6).

By Sylow's theorem and (3), there is a subgroup F satisfying (5). Furthermore, for such a subgroup F, since $M(F)<N_G(F)$ by (2), there is a subgroup N satisfying (6) and (7). Thus $\mathscr{F}^{**}\neq\phi$. We claim the following:

(8) *If $(F,N)\in\mathscr{F}^{**}$ and a tame subgroup E of P is G-conjugate to F, then there is $g\in G$ such that $E=F^g$, $M(E)=M(F)^g$, $N_P(E)=N_P(F)^g$ and $(E,N^g)\in\mathscr{F}^{**}$.*

To prove this, first take an element g of G such that $E=F^g$ and $M(E)=M(F)^g$. By (3), there is such an element g. Set $T=N_P(F)$ and $S=N_P(E)$. Then T is a Sylow p-subgroup of $M(F)$ and S is a Sylow p-subgroup of $M(E)=M(F)^g$. Since S and T^g are both Sylow p-subgroups of $M(E)=M(F)^g$, there is $m\in M(E)\leq N_G(E)$ such that $T^{gm}=S$. Exchanging gm with g, we may assume that $T^g=S$. Finally, by these facts, we can easily check that $(E,N^g)\in\mathscr{F}^{**}$. Thus (8) is proved.

We shall next show the following:

(9) *If $(F,N)\in\mathscr{F}^{**}$ and we set $\beta=\alpha_T$, $T=N_P(F)$. Then (F,N) together with β satisfies the conditions (a) to (g) in Theorem* 5.2.

To prove this, set $T=N_P(F)$, $L=N_G(F)$, $M=M(F)=\{n\in L\,|\,\alpha_F{}^n=\alpha_F\}$, $\bar{L}=L/F$, $\bar{M}=M/F$, $\bar{T}=T/F$. By (1), the condition (g) holds. By (5), T is a Sylow p-subgroup of L, so (a) holds. By (2), $M<L$. Let $F<E\leq T$ and $y\in N_L(E)$. Then $\alpha_E{}^y=\alpha_E$ by the maximality of $|F|$, and so $\alpha_F{}^y=\alpha_F$. Thus $N_L(E)\leq M$. This means that $\bar{M}=M/F$ is a strongly p-embedded subgroup of $\bar{L}=L/F$, proving (b). By (6), we have that N is not contained in M.

Since $\bar{M}$ is strongly p-embedded in $\bar{L}$, we have that $O_p(\bar{N})=1$, and so $O_p(N)=F$, so the first statement of (c) holds. Assume that $F<K\trianglelefteq N$ and $\bar{K}=K/F$ is not a p'-group. Then by the Frattini argument, $N=N_N(K\cap T)K\leq MK$, because $F<K\cap T$ and $\bar{M}$ is strongly p-embedded in $\bar{L}$. Thus K is not contained in M. By the minimality of N, we have that $N=KT$, and so K contains $O_p(N)$. Thus $\bar{K}\geq O^p(\bar{N})$. To complete the proof of (c), we need to show that $O^p(\bar{N}O_{p'}(\bar{N}))$ is simple or 1. But this follows from the fact that this group also has strongly p-embedded subgroup or it is covered by M. (c) is proved.

Let E be a Sylow p-subgroup of $O_{p'p}(L)$. Clearly, $F\leq E$. Assume that $F\neq E$. By the Frattini argument and (b), $L=O_{p'}(L)N_L(E)=O_{p'}(L)M$. Thus $O_{p'}(L)$ is not contained in M, and so

$$O_{p'}(L)\neq\left\langle C_L(t)\cap O_{p'}(L)\middle|t\in T-F\right\rangle \qquad (\leq M).$$

Thus T/F is of p-rank 1 ([8], Th. 5.3.16).

Assume that $\bar{T}$ is cyclic. This assumption holds if $p\neq 2$. Since

$$\begin{aligned} L &= O_{p'}(L)\,N_L(E) \\ &= O_{p'}(L)\,C_L(E/F)\,N_L(T) \end{aligned}$$

by the Frattini argument and since $C_L(E/F)/F$ has a normal p-complement by Burnside's theorem, we have that $L=O_{p'pp'}(L)$. By (c), $N=O^{p'}(N)=O_{p'p}(N)=O_{pp'p}(N)$. In the cyclic case, (d) holds. Next assume that $\bar{T}$ is generalized quaternion. Set $\bar{Z}=Z(T/F)$. By Brauer-Suzuki's theorem, $O_{p'}(\bar{N})\bar{Z}\lhd\bar{N}$, and so $\bar{N}=O_{p'}(\bar{N})(\bar{M}\cap\bar{N})$. By the minimality of N, and (6), and (7), we have that $\bar{N}=O_{p'}(\bar{N})\bar{T}$, so $N=O_{pp'p}(N)$, proving (d). Assume that $C_T(F)$ is not contained in F. Since $C_N(F)F\trianglelefteq N$, (c) implies that $O^p(N)\leq C_N(F)F$. Thus $O^p(N)\leq C_N(F)$, proving (e).

We shall finally prove that (F, N) satisfies (f). Set

$$\gamma=\beta^L{}_M-|L:\ M|\beta^M, \qquad \text{where } \beta=\alpha_T.$$

Then $\gamma^L=0$. Claim:

(10) $M=\left\{g\in L\middle|\gamma_F{}^g=\gamma_F\right\}$ *and* $\gamma_F\neq 0$.

Indeed since $M=\{g\in L|\beta_F{}^g=\beta_F\}$ and $F\lhd L$,

$$\begin{aligned} \beta^M{}_F &= \sum_{m\in T\backslash M/F}\beta^m{}_{T^m\cap F}{}^F \\ &= \sum_{m\in T\backslash M}\beta_F{}^m \\ &= |M:\ T|\beta_F \end{aligned}$$

by the Mackey axiom. Thus we have that

$$\begin{aligned}\gamma_F &= \beta^L{}_F - |L:\ M|\beta^M{}_F \\ &= \beta^L{}_F - |L:\ T|\beta_F .\end{aligned}$$

If $g \in L$, then $\beta^L{}_F{}^g = \beta^{Lg}{}_F = \beta^L{}_F$. Thus for each $g \in L$,

$$\gamma_F{}^g - \gamma_F = -|L:\ T|(\beta_F{}^g - \beta_F).$$

Since $|L:\ T|$ is prime to p $(=chk)$,

$$\begin{aligned}M &= \left\{g \in L \,\middle|\, \beta_F{}^g = \beta_F\right\} \\ &= \left\{g \in L \,\middle|\, \gamma_F{}^g = \gamma_F\right\}.\end{aligned}$$

Furthermore, since $M < L$, we have that $\gamma_F{}^g \neq \gamma_F$ for an element g of L, and so $\gamma_F \neq 0$. Hence (10) is proved. Next take a subgroup X of F of minimal order such that $\gamma_X \neq 0$. By the Mackey axiom and Axiom C, we have that

$$\begin{aligned}0 &\neq -|L:\ T|\gamma_X \\ &= |M:\ T|\gamma^L{}_X - |L:\ T|\gamma_X \\ &= \gamma_T{}^L{}_X - |L:\ M|\gamma_T{}^M{}_X \\ &= \sum_{g \in T\backslash L/M} (\gamma^g{}_{T^g \cap M} - \gamma_{T^g \cap M})^M{}_X .\end{aligned}$$

Thus there is an element g of L such that

$$\delta^M{}_X \neq 0, \qquad \textit{where } \delta = \gamma^g{}_S - \gamma_S,\ S = T^g \cap M.$$

Claim:

(11) $g \in L - M$ *and* $S = T^g \cap M = F$.

Indeed by the Mackey axiom,

$$\begin{aligned}0 \neq \delta^M{}_X &= \sum_{m \in S\backslash M/F} \delta^m{}_{S^m \cap F}{}^F{}_X \\ &= \sum_{m \in S\backslash M} \delta_F{}^m{}_X .\end{aligned}$$

Thus $\delta_F = \gamma^g{}_F - \gamma_F \neq 0$. By (10), $g \notin M$, and so $S = T^g \cap M = F$, because M is strongly p-embedded in L, proving (11). Claim:

(12) (F, δ, X) *is a singularity for* $a_{|M}$.

Since $\delta^M{}_X \neq 0$, it will suffice to show that $\delta_Y = 0$ for each subgroup Y of F with $|Y| < |X|$. Indeed, set $Z = gYg^{-1}$ for such a subgroup Y. Then

$$\delta_Y = \gamma^g{}_Y - \gamma_Y = \gamma_Z{}^g - \gamma_Y .$$

Since $|Y|=|Z|<|X|$, it follows from the minimality of X that $\gamma_Z=\gamma_Y=0$, and so $\delta_Y=0$, as required. Thus (12) is proved and the first statement of (f) holds. By Lemma 4. 2 (e), F is a singular subgroup of $a_{|T}$. Thus the $k\bar{T}$-module $a(F)$ (Lemma 2. 2) contains a regular $k\bar{T}$-submodule by Lemma 4. 5, proving (f). Hence the proof of (9) is completed.

Take a member (F, N) of $\mathscr{F}^{**}$. Then it satisfies (i) of the lemma by (9). We shall prove that it satisfies also (ii). Let E be a subgroup of P conjugate to F in G such that $U:=N_P(E)$ is a Sylow p-subgroup of $N_G(E)$. By (8), there is an element g of G such that $E=F^g$, $N_P(E)=N_P(F)^g$, $(E, N^g)\in\mathscr{F}^{**}$. By (9), (E, N^g) also satisfies the condition (a) to (g) in Theorem 5. 2. By the definition of $M(F)$, we can take α_U as the element β in (b). The lemma is proved.

Proof of Theorem 5. 2. Suppose the theorem false. Then there is an element α of $a(P)-Im\,\rho_P^G$ satisfying

(1) $\alpha_{F^y}=\alpha_F$ for all $(F, N_0)\in\mathscr{F}_0$, $y\in N_0$.

By Lemma 5. 3, there is $(E, N_1)\in\mathscr{F}$ which satisfies the conditions (i), (ii) in the lemma. By (*) in the theorem, there is $g\in G$ and $N_0\leq G$ such that

(2) $F:=E^g\lhd N_0$, $(F, N_0)\in\mathscr{F}_0$.

F is tame for P and $N_1^g\leq N_0$. Since (E, N_1) satisfies (ii) of Lemma 5. 3, there is a G-conjugate $N:=N_1^x$ of N_1 such that $(F, N)\in\mathscr{F}$ and β in (b) is taken to be α_T, where $T=N_P(F)$. Set $L=N_G(F)$ and $M=\{n\in L|\alpha_{F^n}=\alpha_F\}$ as in the theorem. Then $N_0\leq M$ by (1). Set $n=x^{-1}g$. By (2), we have that

(3) $T^n\leq N^n=N\leq N_0\leq M$ and $n\in L$.

Thus T and T^n are both Sylow p-subgroups of M, and so $n\in N_L(T)\,M$ by Sylow's theorem. But M is strongly p-embedded in L by the condition (b), and so $n\in N_L(T)\,M=M$. Since $N^n\leq M$ by (3), we have that $N\leq nMn^{-1}=M$. This contradicts the fact that $(F, N)\in\mathscr{F}$ and so N is not contained in M by (c). The theorem is proved.

Remark. Let $\mathscr{F}$ be the family given in the theorem and set $\mathscr{F}_1=\mathscr{F}\cup\{(P, N_G(P))\}$. Then $\mathscr{F}_1$ controls transfer. Define an equivalent relation $\approx$ on $\mathscr{F}_1$ by $(F, N)\approx(F_1, N_1)$ if and only if N_1 is conjugate in G to N. Let $\mathscr{F}_2$ be a complete set of representatives of equivalent classes of $\mathscr{F}_1$. Then $\mathscr{F}_2$ controls transfer. Next, for any subgroup H of P, we set

$$P_1=N_P(H),\ N_1=N_G(H)$$

and define recursively

$$P_{i+1}=N_P(P_i),\ N_{i+1}=N_G(P_i)\,.$$

We say that H is *well-placed* in P provided each P_i is a Sylow p-subgroup of N_i. It is known that any subgroup of the Sylow p-subgroup P is conjugate in G to a well-placed subgroup of P ([8], Th. 8.4.6). Let $\mathscr{F}_3$ be the set of members $(F, N) \in \mathscr{F}_1$ such that F is well-placed in P. Then $\mathscr{F}_3$ controls transfer. These results follow from the theorem.

EXAMPLE 5.1. We consider the dual group function ab'_p as an example. Let $a := el'_p$, that is,

$$a(H) = \left\{\alpha \in ab'(H) \middle| p\alpha = 0\right\} = Hom(H, \boldsymbol{Z}/p\boldsymbol{Z}).$$

(See Example 2.7) Let (F, N) be a pair which satisfies the condition of Theorem 5.3. By the speciality of the G-functors el'_p and ab'_p, we can show that (F, N) satisfies further conditions. First, the assumption of (d) does not hold, that is, F is a Sylow p-subgroup of $O_{p'p}(L)$. Next, we are not included in the situation of (e), that is, N and L are p-constrained. Furthermore, if $p=2$ and T/F is neither cyclic nor generalized quaternion, then $k[N/F]$-module $a(F)$ (and also $F/\Phi(F)$) contain the Steinberg modules of the Bender group $N/O_{22'}(N)$. This fact follows from (f) and [20], L. 4.8. As a conclusion, Theorem 5.2 and special properties of the dual group functors as above mentioned yield [20], Th. 4.9. However, these facts which hold in the dual group functor (and the abelian factor functor) are not expected in general cases. We can find counter examples by using, for example, the centralizer functor.

6. Examples

In this section, we give some examples about cohomological G-functors. These examples are well known in finite group theory, but we have not considered some of them as transfer theorems.

EXAMPLE 6.1 ([8], Th. 5.2.3). Let V be a kG-module. Assume that $|G|^{-1} \in k$. Then $V = C_V(G) \oplus [V, G]$, where $[V, G]$ is a k-submodule generated by $vx - v$ for $v \in V$, $x \in G$.

PROOF. We shall consider the centralizer functor c_V (Example 2.5). By Lemma 3.1 (c), $V = c_V(1) = Im\, \rho_1^G \oplus Ker\, \tau_1^G$. Since ρ_1^G is the inclusion map $c_V(G) \to V$, we have that $Im\, \rho_1^G = c_V(G)$. Next it follows from the generalized focal subgroup theorem (Theorem 3.2 (b)) that $Ker\, \tau_1^G = [V, G]$. Hence $V = C_V(G) \oplus [V, G]$, as required.

EXAMPLE 6.2 (Gaschütz, e.g., [11], Th. 15, 8.6). Let V be an abelian p-group on which G acts. Let $\tilde{G}$ be an extension of G with kernel V. Assume that a Sylow p-subgroup of $\tilde{G}$ is split. Then $\tilde{G}$ also is split.

PROOF. Consider the G-functor h_V^2 (Example 2. 4). Then the statement follows from the injectivity of ρ (Lemma 3. 1 (a)).

EXAMPLE 6. 3 (Maschke, D. Higman, e. g., [16], § 5. 1, Example 2). Let W be a kG-module and P a Sylow p-subgroup of G. Assume that $|G: P|$ has an inverse in k. Then W is P-projective.

PROOF. Take any P-split exact sequence of kG-modules

$$(1)\quad 0\longrightarrow U\longrightarrow V\overset{\psi}{\longrightarrow}W\longrightarrow 0$$

Set $D=Hom_k(V, W)$ and $E=End_k(W)$. Then D and E are kG-modules by $(v)\varphi^x:=(vx^{-1})\varphi x$ for $v\in V$, $x\in G$, $\varphi\in D$, etc. Consider the G-functors c_D and c_E. The composition

$$D\times E\longrightarrow D:\ (\varphi, \theta)\longmapsto\varphi\theta$$

induces a pairing (cup product) $c_D\times c_E\rightarrow c_D$. Furthermore, for each $H\leq G$, $c_D(H)$ and $c_E(H)$ are the modules of kH-homomorphisms of D and E, respectively. Now, by Lemma 3. 1 (b), $\tau_P^G: c_E(P)\rightarrow c_E(G)$ is an epimorphism, and so there is $\theta\in c_E(P)$ such that $\theta^G=1_W$. Since (1) is P-split, there is a kP-homomorphism ψ' : $W\rightarrow V$ such that $\psi'\psi=1_W$. By Frobenius axioms,

$$\begin{aligned}(\theta\psi')^G\psi &= (\theta\psi'\psi_P)^G\\ &= (\theta 1_W)^G=\theta^G\\ &= 1_W .\end{aligned}$$

Since $(\theta\psi')^G$ is a kG-homomorphism of W to V, this means that (1) is a split exact sequence of kG-modules. Hence W is P-projective.

REMARK 1. The above proof is no more than rewriting the standard proof ([16], Th. 5. 1). By the similar way, we can prove the defect group contains the vertex ([16], Th. 5. 5).

EXAMPLE 6. 4. Let's examine transfer theorems of three kinds to the dual group functor ab_p'. Let P be a Sylow p-subgroup of G. Then the following holds:

(a) If P is elementary abelian, then $P\cap G'=P\cap N_G(P')$. (Compare with Johnson [13], Th. 2.)

(b) $P\cap G'$ is generated by elements $x^{-1}x^g$, where $g\in G$, $x\in P\cap gPg^{-1}$.

(c) If P has no epimorphism onto $\boldsymbol{Z}_p\wr\boldsymbol{Z}_p$, then $P\cap G'=P\cap N_G(P)'$.

PROOF. Set $a=ab_p'$, $N=N_G(P)$. For any finite group X, the dual group $Hom(X, \boldsymbol{C}^*)$ is denoted by $X^\wedge$ or $\hat{X}$. A unique Sylow p-subgroup of $\hat{X}$ is denoted by $\hat{X}_p$. Thus $a(H)=\hat{H}_p$ for any $H\leq G$.

(a) We shall apply Green's theorem (Theorem 3. 5). Set

$$\mathfrak{X}=\left\{P\cap P^g \middle| g\in G-N\right\}.$$

Then we have that

$$a(P)^G/a(\mathfrak{X})^G\cong a(P)^N/a(\mathfrak{X})^N.$$

Let X be any element of $\mathfrak{X}$. Since X is a proper subgroup of P and P is elementary, we have that $\tau_X^P=0$ by an easy calculation. Thus $a(\mathfrak{X})^P=0$, and so $a(P)^G\cong a(P)^N$. By Lemma 3.1 (b), τ_P^G and τ_P^N are both epimorphisms. Thus $a(G)=(G/G'(p))^\wedge$ is isomorphic to $a(N)=(N/N'(p))^\wedge$, and so $G/G'(p)\cong N/N'(p)$. The statement (a) follows easily from this.

(b) We proved this in Example 3.1. But we again prove it by using $a=ab_p'$. Since $G/G'(p)\cong P/P\cap G'$, we have that $Im\ \rho_P^G=(P/P\cap G')^\wedge$. By the generalized focal subgroup theorem (Theorem 3.2 (a)),

$$Im\ \rho_P^G=\left\{\alpha\in a(P)\middle|\alpha^g{}_{P\cap P}=\alpha_{P\cap P^g}\right\}.$$

It follows from the duality theorem of abelian groups that the right side equals to P/K, where

$$K=\left\langle x^{-1}x^g\middle| g\in G,\ x\in P\cap gPg^{-1}\right\rangle.$$

Thus (b) is proved.

(c) Since ρ_P^G and ρ_P^N are monomorphisms, it will suffice to show that $Im\ \rho_P^G=Im\ \rho_P^N$. Suppose false. Then by Lemma 4.1 (a), there is a nonzero element β of $Im\ \rho_P^N\cap Ker\ \tau_P^G$. We may assume that $p\beta=0$. (See also Lemma 3.3.) By Lemma 4.1 (b), P has a singularity (S,α,X), where $S=P\cap P^g$, $\alpha=\beta^g{}_S-\beta_S$ for an element g of G. Since $\beta\in Im\ \rho_P^N$ and $\alpha\neq 0$, we see that $g\notin N$, and so S is a proper subgroup of P. Thus P has a proper singularity. However, since P has no epimorphism onto $\boldsymbol{Z}_p\wr\boldsymbol{Z}_p$, this is a contradiction by [20], Lemma 3.7, proving (c).

REMARK 2. (a) seems to be weakest of three transfer theorems. However, we can apply this method even if P is not Sylow p-subgroups of G. For example, if W is a weakly closed subgroup in a Sylow p-subgroup S of G, then

$$\Omega_1\big(C_S(W)\big)\cap G'=\Omega_1\big(C_S(W)\big)\cap N_G(W)'.$$

This is Zappa's theorem. The key point of transfer theorems of this type is to characterize p-groups P such that $\tau_M^P=0$ for every maximal subgroup M of P. See [20], Cor. 4.4.1. Green's theorem seems to have further possibilities, because the present theorem does not yields directly only Zappa's theorem. Next (b) extends to the problems about conjugation families and conjugation functors. Some parts of results on them are generalized to cohomological G-functors. Finally, (c) is the strongest of three. The theorems of this type (for the G-functors ab_p, ab'_p) are a little useful to finite group theory. One of the reasons is that we can know many facts about singularities in p-groups. It is usually difficult to study singularities for G-functors, and we cannot often understand what they mean even if all singularities for a given G-functor are known.

EXAMPLE 6.5. Let M be a strongly p-embedded subgroup of G with Sylow p-subgroup P (Definition 5.3). Let a be a G-functor over k. Then the following hold:

(a) Let Q be a p-subgroup of M. Then $a(Q)^G/a(1)^G$ is isomorphic to $a(Q)^M/a(1)^M$.

(b) Assume that a is cohomological and k has an inverse of $|G:P|$. Then $a(G)/a(1)^G$ is isomorphic to $a(M)/a(1)^M$. Furthermore, if $\alpha^g=\alpha$ for all $\alpha\in a(1)$, $g\in G$, then $a(G)$ is isomorphic to $a(M)$.

(c) Assume that a is cohomological and k is a field of characteristic p. If the kP-module $a(1)$ contains no regular kP-submodule, then $a(G)$ is isomorphic to $a(M)$.

PROOF. It is easily proved that the following hold:

(1) Set $b(H)=a(H)/a(1)^H$ for each $H\leq G$. Then b is a quotient functor of a. (Using the notation as in Example 2.2, $b=a/a^1$).

(a) This follows directly from Green's transfer theorem ([8], Th. 2). But we give the proof here. By (1), we may assume that $a(1)=0$. Then τ_M^G induces

$$T:\ a(Q)^M\longrightarrow a(Q)^G:\ \alpha^M\longmapsto\alpha^G\,.$$

Furthermore, let α be an element of $a(Q)$. Then since $\alpha_1=0$ and $Q^g\cap M=1$ for all $g\in G-M$, we have

$$\begin{aligned}\alpha^G{}_M&=\sum_{g\in Q\backslash G/M}\alpha^g{}_{M\cap Q^g}{}^M\\&=\alpha^M\,.\end{aligned}$$

Thus ρ_M^G induces

$$R:\ a(Q)^G\longrightarrow a(Q)^M:\ \alpha^G\longmapsto\alpha^M\,.$$

Since R and T are inverses of each other, we conclude that $a(Q)^G$ is isomorphic to $a(Q)^M$, as required.

(b) Assume first that $\alpha^g=\alpha$ for each $\beta\in a(1)$, $g\in G$. For each $g\in G$, set

$$I(g)=\left\{\alpha\in a(P)\,\middle|\,\alpha^g{}_{P\cap P^g}=\alpha_{P\cap P^g}\right\}.$$

By the focal subgroup theorem (Theorem 3. 2 (a)),

$$Im\ \rho_P^G = \bigcap_{g \in G} I(g)\,, \quad \text{and}$$

(2) $Im\ \rho_P^M = \bigcap_{m \in M} I(m)\,.$

Let $g \in G-M$. Since M is strongly p-embedded, $P \cap P^g = 1$. Thus by the assumption,

$$I(g) = \left\{\alpha \in a(P) \middle| \alpha_1{}^g = \alpha_1\right\}$$
$$= a(P)\,.$$

By (2), $Im\ \rho_P^G = Im\ \rho_P^M$, and so $a(G)$ is isomorphic to $a(M)$ by the injectivity of ρ (Lemma 3. 1 (a)), as required. Next, for a general cohomological G-functor a, define the quotient G-functor b of a by (1), so $b(H) = a(H)/a(1)^H$ and $b(1)=0$. Thus $b(G)$ is isomorphic to $b(M)$ by the isomorphism which is already proved. Hence (b) is proved.

(c) Assume that the kP-module $a(1)$ contains no regular kp-submodules and suppose $a(G)$ is not isomorphic to $a(M)$. Then we have that $Im\ \rho_P^M \neq Im\ \rho_P^G$ by Lemma 3. 1 (a). By Lemma 4. 1, we have that there are $0 \neq \beta \in Ker\ \tau_P^G \cap Im\ \rho_P^M$, and $g \in G$, and $X \leq P$ such that (S, α, X) is a singularity for $a_{|P}$, where $S = P \cap P^g$ and $\alpha = \beta^g{}_S - \beta_S$. If $g \in M$, then $\alpha = 0$ by Theorem 3. 1 (a) or an easy calculation. But this is a contradiction by the definition of singularities (S. 1). Thus $g \notin M$, and so $S = P \cap P^g = 1$. By Lemma 4. 5, $a(1)$ has a regular kP-submodule, a contradiction. Hence $a(G)$ is isomorphic to $a(M)$, proving (c).

EXAMPLE 6. 6 (Hall-Higman, e. g., [8], Th. 11. 1. 1). Let G be a p-solvable group of linear transformation in which $O_p(G)=1$ acting on a finite dimensional vector space V over a field k of characteristic p. Let x be an element of G of order p^n of which minimal polynomial on V is of degree r. Then one of the following holds:

(a) $r = p^n$, or

(b) p is a Fermat prime, and G has a nonabelian Sylow 2-subgroup, and $p^n - p^{n-1} \leq r \leq p^n$, or

(c) $p = 2$, and G has a nonabelian Sylow q-subgroup, and q is a Mersenne prime $< 2^n$, and $2^n q/(q+1) \leq r \leq 2^n$.

PROOF. Suppose false and let (G, V) be a counterexample in which $|G| + dim_k V$ is minimal. Then $P = \langle x \rangle$ is a Sylow p-subgroup of G, and G has a normal p-complement Q, and Q is an elementary or extraspecial q-group for a prime q such that P acts irreducibly on Q/Q' and trivially on Q'. These facts follow easily from the induction argument ([8], Th.

11. 1. 4). First assume that Q is elementary. Then $P\cap P^g=1$ for each $g\in G-P$, and so $0=\hat{c}_v(G)=\hat{c}_V(P)$ by Example 6. 5 (b) or an easy calculation. Thus V is kP-free by Corollary 4. 6, and so $r=p^n$. Assume next that Q is extra-special. We may assume that k is algebraic closed. Set $E=End_k(V)$, so that E is a kG-module by $(v)\varphi^g=(vg^{-1})\varphi g$ for $v\in V$, $\varphi\in E$, $g\in G$. By the irreducibility of V, we have that V_Q is a direct sum of isomorphic irreducible kQ-modules. (Note that G acts on $\{v\in V|vz=\omega v\}$, where z is a generator of Q' and $\omega\in k$.) The fact that V_Q is an irreducible kQ-module is proved as follows (see also the Proof of [8], Th. 11. 1. 4 (ii)). Assume that V_Q is the direct sum of t isomorphic irreducible kQ-submodules $Y_1, \cdots, Y_t$. Then we have that $M:=c_E(Q)=End_{kQ}(V)\cong M(t, k)$, the algebra of all $t\times t$ matrices over k (e. g., see [8], Th. 3. 5. 4). Since P normalizes Q and $c_E(G)=k$, we have that the element x of P acts on M as an algebra-automorphis, and $c_M(x)=k$. By a calculation (e. g., Jordan's canonical form), we have that the automorphisms of $M=M(t, k)$ are all inner. Since $c_M(x)=k$, we have that $t=1$. Thus Q acts irreducibly on V, and so $c_E(Q)\cong M(1, k)=k$. Since a generator of Q' induces a scalar transformation on V, we have that Q' acts trivially on E. Thus E is regarded as a G/Q'-module. Set $\bar{G}=G/Q'$ and $\bar{Q}=Q/Q'$. By Example 6. 1, $E=c_E(\bar{Q})\oplus[E, \bar{Q}]=k\oplus[E, \bar{Q}]$. By the similar way as the elementary abelian case, Corollary 4. 6 yields that $[E, \bar{Q}]$ is a free kP-module. By these facts and a calculation, the example is proved. See [21] for details.

EXAMPLE 6. 7. Let V be a finite generated kG-module, k a field of characteristic p, P a Sylow p-subgroup of G. Assume that for each maximal subgroup Q of P, the $k[P/Q]$-module $C_V(Q)$ (Lemma 2. 2) has no regular $k[P/Q]$-submodules. Then $C_V(G)=C_V(N_G(P))$.

PROOF. We shall apply Lemma 4. 4 to the G-functor $c_V=(c_V, \tau, \rho, \sigma)$ (Example 2. 5). Suppose P has a proper singular subgroup S. Then by Lemma 4. 2 (d), a maximal subgroup Q of P which contains S is also a singular subgroup of P. Since $Q\trianglelefteq P$, it follows from Lemma 4. 5 that the $k[P/Q]$-module $c_V(Q)$ has a regular $k[P/Q]$-submodule. This contradicts the assumption. Thus P has no proper singularities. By Lemma 4. 4, we have the desired conclusion.

REMARK 3. Set $G=SL(n, q)$, $q=p^r$, and let V be a standard kG-module of G, where $k=GF(q)$. Let P be a Sylow p-subgroup of G. We can observe when the assumption of this example holds. In fact, for any maximal subgroup Q of P, the three cases occur:

(1) If $r\geqq 2$, then $C_V(P)=C_V(Q)$, and so the assumption of this example holds.

(2) If $p \geqq 3$ and $r=1$, then $dim\, C_V(P)=1$ and $dim\, C_V(Q) \leqq 2$. Thus $C_V(Q)$ can not have regular $k[P/Q]$-submodules, and so the assumption of this example holds.

(3) If $p=2$ and $r=1$, then $C_V(Q) \neq C_V(P)$ for some Q. Thus in this case, the assumption of this example doesn't hold.
Furthermore, it is easily checked that the conclusion of this example holds in the cases (1) and (2), and doesn't hold in the case (3).

REMARK 4. Let V be a nontrivial finite generated irreducible kG-module. Applying this example, we have that either

(1) for some maximal subgroup Q of P, the $k[P/Q]$-module $C_V(Q)$ has a regular submodule, or

(2) $$C_V\big(N_G(P)\big) = C_V(G) = 0\,.$$

If $p=2$, then (1) is equivalent to the fact that

(1)′ $C_V(P) \neq C_V(Q)$ for some maximal subgroup Q of P.
If $p=2$ and $N_G(P)=P$, then (2) cannot hold, and so (1)′ holds. We have similar results about $\hat{c}_V$, etc. However, the author doesn't know whether these results are valuable and whether they can be proved by the ordinary modular representation theory.

EXAMPLE 6.8. Assume that G has a Sylow p-subgroup P of order p. Take $x \in P$ such that $P=\langle x \rangle$. Let V be a finite generated kG-module, where k is a field of characteristic p. Set $N=N_G(P)$. Then the following hold:

(a) $\hat{c}_V(G) \cong \hat{c}_V(N)$, $\hat{d}_V(G) \cong \hat{d}_V(N)$,

(b) If the minimal polynomial of x on V is of degree $<p$, then $c_V(G)= c_V(N)$, $d_V(G)=d_V(N)$.

(c) Assume that the minimal polynomial of x on V is of degree $\leq p/2$ and that V is indecomposable as a kG-module. Then V is indecomposable also as a kN-module.

PROOF. (a) Since $\hat{c}_V(1)=\hat{d}_V(1)=0$ by the definition (Example 2.5), P has no proper singularities for these G-functors. Thus (a) follows directly Lemma 4.4. See also Example 6.5 (b), (c).

(b) This is proved by Example 6.7.

(c) Set $E=End_k(V)$. Then E is a kG-module by $(v)\varphi^g := (vg^{-1})\varphi^g$ for $v \in V$, $g \in G$, $\varphi \in E$. By the assumption, the minimal polynomial of x on E is of degree $<p$. By (b), $c_E(G)=c_E(N)$. Thus $End_{kG}(V)=End_{kN}(V)$. Since V is kG-indecomposable, $End_{kG}(V)$ and also $End_{kN}(V)$ are local rings, and so V is indecomposable as a kN-module, as required.

7. Concluding remark

I note down here my personal thought on G-functors. So much remains to be studied on this subject and there appears to be five courses that we have to follow. Intuitionally saying, the philosophy of the theory of G-functors is to handle representations not only of G but also of all subgroups of G. First, the concept of G-functors is an excellent language to describe representation theory, cohomology theory, and transfer theory of finite groups (see the cohomology exact sequence in Example 2. 4). By rewriting these theories in the language of G-functors, we can separate parts about transfer from them. Secondly, it is also interesting to apply the results (transfer theorems, etc.) on G-functors to modular representation theory (Examples 6. 7, 6. 8). We get perhaps some new results.

Next, let's change our subject to studying G-functors themselves. The category of G-functors is very like the category of G-modules. There are rings and modules in the category (Definition 2. 4), and we can furthermore define tensor products, induced G-functors, relative projective G-functors, etc. The category of G-functors contains all necessities to develop "an abelian group theory." Following modular representation theory and homology algebra of modules, we can develop the theory of G-functors to some extent. A great difference between G-functors and G-modules is the fact that every G-functor involves a complex internal structure. Transfer theorems which is the subject of this paper tells the information on the inside of a G-functor.

I am much interested in using G-functors in order to join some fields. Using methods in transfer theory [20], we can prove some results about representation and cohomology of groups (Section 6), but these results have no relation to G-functors in appearance and we can prove them even without using G-functors. I want to try using G-functors as mediators on a large scale. Can we state any transfer theorem for Dress' Mackey functors? This is not an easy problem (but see Dress' works, [4], § 5 etc.) It was difficult even to define "cohomological" Mackey functors. But once they are done, we find ourselves being in a peculiar world which is not related to finite group theory in any way but in which we can develop transfer theory as before. This theory will be presented in future papers.

Acknowledgment

I am indebted to H. Sasaki in Hokkaido University. He gave independently proofs of a part of this paper (Lemma 4. 2, 4. 4, etc.) in his paper [17]. I state that I referred to it in the writing.

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