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Some studies on group algebras

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In this paper we study a ring theoretical approach to the theory of modular representations of finite groups which is studied by several authors ([5], [6], [8], [9], [10], e. t. c.). Most of results in this note is not new but is proved by a character-free method.

Let F be a fixed algebraically closed field of characteristic p, a rational prime. If G is a finite group, we let FG denote the group algebra of G over F. If X is a subset of G, we let $\hat{X}$ be the sum of elements of X in FG. Other notations are standard and we refer to [2] and [5].

In section 1 we shall give a proof of the result of Brauer which appears in [1] without proof. In section 2, using results in section 1 we investigate the center of a group algebra and an alternating proof of the result of Osima [7] is given.

1. In this section we give a characterization of elements of the radical of a group algebra which appears in [1] without proof. Related results also appear in [12]. Let G be a finite group of order $p^{a}k$, $(p, k)=1$. Choose an integer b so that $p^{b}\equiv 1 \pmod{k}$ and $b\geqq a$. Let U be the F-subspace of FG generated by all commutators in FG. Then $U=\left\{\sum_{g\in G}a_{g}g\,;\ \sum_{g\in C}a_{g}=0\right.$ for every conjugacy class C of $\left.G\right\}$ and it holds that $(\alpha+\beta)^{p}\equiv\alpha^{p}+\beta^{p} \pmod{U}$ for α and β in FG. For these results see [2].

LEMMA (1. A). *Let $e=\sum_{g\in G}a_{g}g$ be an idempotent of FG. then we have $\sum_{g\in C}a_{g}=0$ for every p-singular conjugacy class C of G.*

PROOF. As $e^{p^{b}}=e$, we have $\sum_{g\in G}a_{g}g\equiv\sum_{g\in G}a_{g}^{p^{b}}g^{p^{b}} \pmod{U}$. Since coefficients of p-singular elements in the right-hand side of the above equation are all 0, we have the lemma.

LEMMA (1. B). *Let $e=\sum_{g\in G}a_{g}g$ be a primitive idempotent of FG. Then there exists a p'conjugacy class C of G such that $\sum_{g\in C}a_{g}\neq 0$.*

PROOF. Since e is primitive, $e\notin U$. Thus the lemma follows from (1. B).

Using the above lemmas, we can prove the assertion of Brauer stated in the begining of this section. Let $S_{1}, S_{2}, \cdots$ be p'-sections of G. If X and

Y are subsets of FG, then we set $Ann_Y X=\{\alpha\in Y;\ \alpha X=0\}$. We denote the radical of FG by $J(FG)$.

THEOREM (1. C) (Brauer [1]). $J(FG)=\bigcap_i Ann_{FG}\hat{S}_i$.

PROOF. First we shall prove the following.

(1. D). $J(FG)\supseteq\bigcap_i Ann_{FG}\hat{S}_i$.

PROOF of (1. D). Since $\bigcap_i Ann_{FG}\hat{S}_i$ is an ideal of FG, if it contains an idempotent, then also contains a primitive idempotent e. Considering the coefficient of 1 in $e\hat{S}_i$ this contradicts to (1. A) and (1. B). Thus $\bigcap_i Ann_{FG}\hat{S}_i$ contains no idempotent of FG and therefore (1 .D) follows.

Next we prove;

(1. E). $J(FG)\subseteq Ann_{FG}\hat{S}_i$ *for each* i.

PROOF of (1. E). Let $\alpha=\sum_{g\in G}a_g g\in FG$ and assume $\alpha^{p^b}\in Ann_{FG}\hat{S}_i$. Then we have $\sum_{g^{p^b}\in S_i^{-1}} a_g{}^{p^b}=0$ since $\alpha^{p^b}=\sum_{g\in G}a_g{}^{p^b}g^{p^b}$ (mod U), where $S_i^{-1}=\{s^{-1};\ s\in S_i\}$. So $\sum_{g\in S_i^{-1}}a_g{}^{p^b}=(\sum_{g\in S_i^{-1}}a_g)^{p^b}=0$ and $\sum_{g\in S_i^{-1}}a_g=0$. This implies that the coefficient of 1 in $\alpha\hat{S}_i$ is 0. Thus $FG/Ann_{FG}\hat{S}_i$ has no nilpotent ideal and therefore $J(FG)\subseteq Ann_{FG}\hat{S}_i$.

COROLLARY (1. F). *$\sum_i\hat{S}_iFG$ is the socle of FG. In particular, for a primitive idempotent e of FG there exists i such that $e\hat{S}_iFG$ is an irreducible FG-module.*

PROOF. This follows from the fact that FG is a symmetric algebra and (1. C).

COROLLARY (1. G). *Let e be a primitive idempotent of FG and α an element of the form $\sum_i a_i\hat{S}_i$. Then $e\alpha=0$ if and only if the coefficient of 1 in $e\alpha$ is 0.*

PROOF. It is sufficient to show that if the coefficient of 1 in $e\alpha$ is 0 then $e\alpha=0$. Let t be the F-homomorphism from FG to F defined by the rule; $FG\ni\sum_{g\in G}a_g g\to a_1\in F$. Then the kernel of t has no non-zero right ideal of FG. Since $e\beta=e\beta e+(ee\beta-e\beta e)$ for $\beta\in FG$, we have $e\alpha FG\subseteq e\alpha FGe+U$. $eFGe=Fe+eJ(FG)e$ as $eFGe/eJ(FG)e\cong F$. Thus by (1. C) $e\alpha FG\subseteq Fe\alpha+U$. $U\subseteq \mathrm{Ker}\,t$ and $Fe\alpha\subseteq\mathrm{Ker}\,t$ by our assumption. Therefore $e\alpha FG\subseteq\mathrm{Ker}\,t$ which implies that $e\alpha=0$.

PROPOSITION (1. H). *Let $\alpha=\sum_{g\in G}a_g g$ be an element of the center of FG*

with $a_g \neq 0$ *for some* p'*-element* g. *Then there is a primitive idempotent* e *of* FG *such that the coefficient of* 1 *in* $e\alpha$ *is not* 0.

PROOF. Let $\beta = \sum_{g \in G_0} a_g g$ where G_0 is the set of all p'-elements of G. And write $\beta = \sum_i b_i \hat{C}_i$ where C_i is the p'-conjugacy class of G contained in S_i and set $\gamma = \sum_i b_i \hat{S}_i$. By (1. A) for an idmpotent f of FG the coefficient of 1 in $f\alpha$ is equal to that in $f\gamma$. Since $\gamma \neq 0$, the result follows from (1. G).

2. Let $Z(FG) = Z$ denote the center of FG. For $\alpha = \sum_{g \in G} a_g g$ FG we set $\sup \alpha = \{g \in G ; a_g \neq 0\}$. The result of Osima [7] shows that for a central idempotent e of FG $\sup e$ does not contain any p-singular element. Ring-theoretical proofs of this fact appear in [5] and [8]. Furthermore we have the following.

THEOREM (2. A) (Osima [7]). *Let* α *be in* Z *and* T *a* p*-section of* G. *Then* $\sup \alpha \cap T = \phi$ *if and only if* $\sup e\alpha \cap T = \phi$ *for every idempotent* e *of* Z.

PROOF. If $\sup e\alpha \cap T = \phi$ for every idempotent e of Z, then it is clear that $\sup \alpha \cap T = \phi$. Conversely assume that $\sup \alpha \cap T = \phi$. Let x be a p-element in T and C the conjugacy class of G containing x. Considering the Brauer homomorphism from Z to $Z(FC_G(x))$ defined by the rule; $Z \ni \sum_{g \in G} a_g g \rightarrow \sum_{g \in C_G(x)} a_g g \in Z(FC_G(x))$, we may assume that $G = C_G(x)$ and $C = \{x\}$. Then we may also assume that $x = 1$ and T is the set of all p'-elements of G. Suppose that $\sup e\alpha \cap T \neq \phi$. Then by (1. H) there exists a primitive idempotent f of FG such that the coefficient of 1 in $fe\alpha$ is not 0. Since f is primitive, $fe = f$ and then $fe\alpha = f\alpha$. Thus by (1. A) $\sup \alpha \cap T \neq \phi$ which is a contradiction.

The following is the result of Reynolds and is proved in [11]. We shall give here an elementary proof of it.

THEOREM (2. B) (Reynolds [11]). $Z_{p'} = \sum_i F\hat{S}_i$ *is an ideal of* Z.

PROOF. Let S be a p'-section and C a conjugacy class of G. Let M be a p'-conjugacy class and N a p-singular conjugacy class of G such that M and N are contained in the same p'-section of G. Let $\hat{S}\hat{C} = a\hat{M} + b\hat{N} + \cdots$. To prove the theorem it will suffice to show that $a = b$. Let $z \in N$ and $z = xy = yx$ where x is a p-element and y is a p'-element of G. Since $S \cap C_G(x)$ is a union of p'-sections of $C_G(x)$, considering the Brauer homomorphism with respect to $C_G(x)$ we may assume $G = C_G(x)$. Then $\hat{S}x = \hat{S}$ and $\hat{M}x = \hat{N}$. Thus $\hat{S}\hat{C} = a\hat{M} + b\hat{N} + \cdots = a\hat{M}x + b\hat{N}x + \cdots$ and we have $a = b$.

LEMMA (2. C). *Let* e *be an idempotent of* FG *such that* $e + J(FG)$ *is* *central in $FG/J(FG)$. Then $e\hat{S}_i$ is in $Z_{p'}$.*

PROOF. By (1. C) $e\hat{S}_i$ is in Z. Let $e\hat{S}_i=\alpha+\beta$ where α is in $Z_{p'}$ and sup β consists of p-singular elements. Such elements α and β can be chosen. Then for a primitive idempotent f of FG the coefficient of 1 in $f(e\hat{S}_i-\alpha)$ is 0 by (1. A). Since f is primitive, $fe\hat{S}_i=0$ or $=f\hat{S}_i$. Thus by (1. G) we have $f(e\hat{S}_i-\alpha)=0$. Therefore $f\beta=0$ for every primitive idempotent f of FG and then $\beta=0$. So the proof of the lemma is complete.

PROPOSITION (2. D). *Let e be an idempotent of FG such that $e+J(FG)$ is centrally primitive in $FG/J(FG)$. Then* $\dim_F eZ_{p'}=1$.

PROOF. Let $e=e_1+\cdots+e_n$ where e_i's are mutually orthogonal primitive idempotents of FG. Then $e_1FG\cong e_iFG$ for all i (see [2]). It is easily shown that there are elements $\alpha_i\in e_1FG$ and $\beta_i\in e_iFG$ such that $e_1=\alpha_i\beta_i$ and $e_i=\beta_i\alpha_i$. Therefore $e_i-e_1\in U$. By (1. G) $\dim_F e_1Z_{p'}=1$ and again by (1. G) and the fact that $e_i-e_1\in U$ we have $\dim_F eZ_{p'}=1$.

As a consequence of (2. C) and (2. D) we have the following. For this result see [3].

THEOREM (2. E). *Let B be a p-block of G with corresponding centrally primitive idempotent e. Then the number of irreducible FG-modules in B equals to* $\dim_F eZ_{p'}$.

PROOF. Let $e=e_1+\cdots+e_n$ where e_i's are mutually orthogonal idempotent and $e_i+J(FG)$ is centrally primitive. Then n is the number of FG-modules in B. Thus the result follows from (2. C) and (2. D).

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