Hokkaido Mathematical Journal Vol. 9 (1980) p. 184-189

On Sasakian manifolds with vanishing contact Bochner curvature tensor

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(Received November 21, 1979; Revised January 23, 1980)

§ 1. Introduction.

Recently, S. I. Goldberg and M. Okumura [3] proved

THEOREM A. *Let M be an n-dimensional compact conformally flat Riemannian manifold with constant scalar curvature R. If the length of the Ricci tensor is less than $R/\sqrt{n-1}$, $n\geqq 3$, then M is a space of constant curvature.*

For a Kaehlerian manifold, Y. Kubo [7] proved

THEOREM B. *Let M be a real n-dimensional Kaehlerian manifold with constant scalar curvature R whose Bochner curvature tensor vanishes. If the length of the Ricci tensor is not greater than $R/\sqrt{n-2}$, $n\geqq 4$, then M is a space of constant holomorphic sectional curvature.*

Note that the square of the length of the Ricci tensor is greater than or equal to R^2/n, so the Ricci tensor has been "pinched".

We have the following remarks [5] on Theorem B.

REMARK 1. The condition with respect to the length of the Ricci tensor can be replaced by

$$(*)\qquad R_{ab}R^{ab}\leqq\frac{R^2}{n-2}\,.$$

REMARK 2. Moreover the condition (*) can be replaced by the best condition

$$R_{ab}R^{ab}<\frac{n^3-2n^2+32}{(n+2)^2(n-4)^2}R^2\qquad\text{for}\quad n>4\,.$$

REMARK 3. In paticular, when M is of dimension 4, if the scalar curvature does not vanish, then M is of constant holomorphic sectional curvature.

The purpose of this paper is to obtain the theorems, analogous to the above theorems, for a Sasakian manifold with vanishing contact Bochner curvature tensor.

THEOREM 1. *Let M be a $(2n+1)$-dimensional Sasakian manifold with* *constant scalar curvature R whose contact Bochner curvature tensor vanishes. If the square of the length of the η-Einstein tensor is less than*

$$\frac{(n-1)(n+2)^2(R+2n)^2}{2n(n+1)^2(n-2)^2}, \quad n \geqq 3,$$

then M is a space of constant ϕ-holomorphic sectional curvature.

THEOREM 2. *Let M be a 5-dimensional Sasakian manifold with constant scalar curvature whose contact Bochner curvature tensor vanishes. If the scalar curvature is not -4, then M is a space of constant ϕ-holomorphic sectional curvature.*

§ 2. Preliminaries.

Let M be a $(2n+1)$-dimensional Sasakian manifold with the Riemannian metric g_{ij} and an almost contact structure (ϕ, ξ, η) satisfying

$$\phi_a{}^i \phi_j{}^a = -\delta^i{}_j + \eta_j \xi^i, \quad \phi_a{}^i \xi^a = 0, \quad \eta_a \phi_i{}^a = 0, \quad \eta_a \xi^a = 1,$$

$$N_{ij}{}^k + (\partial_i \eta_j - \partial_j \eta_i)\, \xi^k = 0, \quad \phi_{ij} = \frac{1}{2}(\partial_i \eta_j - \partial_j \eta_i),$$

$$g_{ab} \phi_i{}^a \phi_j{}^b = g_{ij} - \eta_i \eta_j, \quad \xi^i = g^{ia} \eta_a,$$

where $N_{ij}{}^k$ is the Nijenhuis tensor with respect to ϕ_{ij} and $\partial_i = \partial / \partial x^i$ is the partial differential operator with respect to the local coorsinate (x^i). In view of the last equation we shall write η^i instead of ξ^i in the sequel.

In the following, let R_{hijk}, R_{ij}, R and ∇_i denote the Riemannian curvature tensor, the Ricci tensor, the scalar curvature and the operator of covariant differentiation with respect to g_{ij} respectively. It is well known that in a $(2n+1)$-dimensional Sasakian manifold we have the following identities:

(2.1) $$\nabla_i \eta^j = \phi_i{}^j,$$

(2.2) $$\nabla_i \phi_j{}^k = -g_{ij} \eta^k + \delta_i{}^k \eta_j,$$

(2.3) $$\eta^a R_{aijk} = \eta_k g_{ij} - \eta_j g_{ik},$$

(2.4) $$\eta^a R_{ai} = 2n \eta_i,$$

(2.5) $$R_{hij}{}^a \phi_{ak} - R_{hik}{}^a \phi_{aj} = \phi_{ij} g_{hk} - \phi_{ik} g_{hj} + g_{ij} \phi_{hk} - g_{ik} \phi_{hj},$$

(2.6) $$\phi^{ab} R_{aijb} = R_i{}^a \phi_{aj} - (2n-1)\, \phi_{ij},$$

(2.7) $$R_i{}^a \phi_{aj} + R_j{}^a \phi_{ai} = 0,$$

(2.8) $$\phi^{ab} R_{abij} = -2 R_i{}^a \phi_{aj} + 2(2n-1)\, \phi_{ij}.$$

DEFINITION 1. If a tensor $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ of a Sasakian manifold M satisfies

$$\phi_{i_1}{}^{a_1}\cdots\phi_{i_p}{}^{a_p}\phi_{b_1}{}^{j_1}\cdots\phi_{b_q}{}^{j_q}\phi_k{}^c\nabla_c T_{a_1\cdots a_p}{}^{b_1\cdots b_q}=0\,, \tag{2.9}$$

then we say that the tensor $T_{i_1\cdots i_p}{}^{j_1\cdots j_q}$ of M is η-parallel. If the Riemannian curvature tensor of a Sasakian manifold M is η-parallel, then we say that M is locally ϕ-symmetric.

DEFINITION 2. If the Ricci tensor R_{ij} of a Sasakian manifold M satisfies

$$R_{ij}=\left(\frac{R}{2n}-1\right)g_{ij}-\left(\frac{R}{2n}-2n-1\right)\eta_i\eta_j\,, \tag{2.10}$$

then we say that M is an η-Einstein mainfold.

We define the η-Einstein tensor S_{ij} by

$$S_{ij}=R_{ij}-\left(\frac{R}{2n}-1\right)g_{ij}+\left(\frac{R}{2n}-2n-1\right)\eta_i\eta_j\,. \tag{2.11}$$

If the η-Einstein tensor S_{ij} of a Sasakian manifold M vanishes, then M is an η-Einstein mainfold.

(2.1), (2.3), (2.5) and the second Bianchi identity

$$\nabla_l R_{hijk}+\nabla_h R_{iljk}+\nabla_i R_{lhjk}=0 \tag{2.12}$$

give

$$\eta^a\nabla_a R_{hijk}=0\,. \tag{2.13}$$

From this fact and brief computations, we have

LEMMA 1 [4]. *A Sasakian manifold M is locally ϕ-symmetric if and only if the Riemannian curvature tensor of M satisfies*

$$\begin{aligned}\nabla_l R_{hijk}=&-\phi_l{}^a(\eta_h R_{aijk}+\eta_i R_{hajk}+\eta_j R_{hiak}+\eta_k R_{hija})\\&+\eta_h(\phi_{lk}g_{ij}-\phi_{lj}g_{ik})+\eta_i(\phi_{lj}g_{hk}-\phi_{lk}g_{hj})\\&+\eta_j(\phi_{li}g_{hk}-\phi_{lh}g_{ik})+\eta_k(\phi_{lh}g_{ij}-\phi_{li}g_{hj})\,.\end{aligned} \tag{2.14}$$

LEMMA 2. *Let M be a $(2n+1)$-dimensional Sasakian manifold. The Ricci tensor of M is η-parallel if and only if the Ricci tensor satisfies the following identity*:

$$\nabla_k R_{ij}=-\phi_k{}^a(R_{aj}\eta_i+R_{ai}\eta_j)+2n(\phi_{kj}\eta_i+\phi_{ki}\eta_j)\,. \tag{2.15}$$

From brief computations, we see that the Ricci tensor is η-parallel if a Sasakian manifold M is locally ϕ-symmetric.

For a $(2n+1)$-dimensional Sasakian manifold, we define the contact Bochner curvature tensor B_{hijk} of M by

$$
\begin{aligned}
B_{hijk} = R_{hijk} - \frac{1}{2(n+2)}(R_{ij}g_{hk} - R_{ik}g_{hj} + g_{ij}R_{hk} - g_{ik}R_{hj} \\
-R_{ij}\eta_h\eta_k + R_{ik}\eta_h\eta_j - \eta_i\eta_j R_{hk} + \eta_i\eta_k R_{hj} + H_{ij}\phi_{hk} \\
-H_{ik}\phi_{hj} + \phi_{ij}H_{hk} - \phi_{ik}H_{hj} - 2H_{hi}\phi_{jk} - 2\phi_{hi}H_{jk}) \\
+\frac{R-6n-8}{4(n+1)(n+2)}(g_{ij}g_{hk} - g_{ik}g_{hj}) \\
+\frac{R+4n^2+6n}{4(n+1)(n+2)}(\phi_{ij}\phi_{hk} - \phi_{ik}\phi_{hj} - 2\phi_{hi}\phi_{jk}) \\
-\frac{R+2n}{4(n+1)(n+2)}(g_{ij}\eta_h\eta_k - g_{ik}\eta_h\eta_j + \eta_i\eta_j g_{hk} - \eta_i\eta_k g_{hj}),
\end{aligned}
\tag{2.16}
$$

where $H_{ij} = R_i^a\phi_{aj}$.

Taking the covariant differentiation of (2.16) and contraction, we have the following

LEMMA 3. *Let M be a Sasakian manifold with vanishing contact Bochner curvature tensor. If the scalar curvature is constant, then the Ricci tensor of M is η-parallel.*

Also, we know the following

LEMMA 4. *Let M be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M is an η-Einstein manifold, then M is a space of constant ϕ-holomorphic sectional curvature.*

M. Okumura [8] proved

LEMMA 5. *Let c_i, $i=1, 2, \cdots, m$, be real numbers satisfying*

$$
\sum_{i=1}^{m} c_i = 0 \quad \textit{and} \quad \sum_{i=1}^{m} c_i^2 = k^2 \qquad (k = \textit{constant} \geqq 0).
$$

Then we have

$$
-\frac{m-2}{\sqrt{m(m-1)}}k^3 \leqq \sum_{i=1}^{m} c_i^3 \leqq \frac{m-2}{\sqrt{m(m-1)}}k^3 .
$$

§ 3. Proofs of theorems.

PROOF of THEOREM 1.

Let M be a $(2n+1)$-dimensional Sasakian manifold. Assume that the contact Bochner curvature tensor vanishes, then we have

$$
\begin{aligned}
R_{abcd}R^{ad}R^{bc} = \frac{2}{n+2}R_a{}^b R_b{}^c R_c{}^a + \frac{(2n+1)R - 2(5n^2+8n+2)}{2(n+1)(n+2)}R_{ab}R^{ab} \\
+\frac{-R^3 + 2(5n+4)R^2 + 12n^2R - 8n^3(2n+1)}{4(n+1)(n+2)} .
\end{aligned}
\tag{3.1}
$$

Since the contact Bochner curvature tensor vanishes and the scalar curvature is constant, from Lemma 3, the Ricci tensor is η-parallel. Therefore we have (2. 15) by virtue of Lemma 2. Then, from this fact and the Ricci identity, we obtain

$$\begin{aligned} R_{abcd}R^{ad}R^{bc}-R_a{}^bR_b{}^cR_c{}^a &= R^{bc}g^{ad}(\nabla_b\nabla_aR_{cd}-\nabla_a\nabla_bR_{cd}) \\ &= -R_{ab}R^{ab}+4nR-4n^2(2n+1)\,. \end{aligned} \tag{3. 2}$$

Substituting (3. 1) into (3. 2), we have

$$\begin{aligned} &4n(n+1)\,R_a{}^bR_b{}^cR_c{}^a+\left[4n(4n+5)-2(2n+1)\,R\right]R_{ab}R^{ab}+R^3 \\ &\quad -2(5n+4)\,R^2+4n(4n^2+9n+8)\,R-8n^2(2n+1)\,(2n^2+5n+4)=0\,. \end{aligned} \tag{3. 3}$$

From the definition of the η-Einstein tensor S_{ij}, we have $S_i{}^a\phi_a{}^j=\phi_i{}^aS_a{}^j$. Moreover we see that

$$trace\ S=S_a{}^a=0\,, \tag{3. 4}$$

$$trace\ S^2=S_{ab}S^{ab}=R_{ab}R^{ab}-\frac{1}{2n}R^2+2R-4n^2-2n\geqq 0\,, \tag{3. 5}$$

$$\begin{aligned} trace\ S^3 &= S_a{}^bS_b{}^cS_c{}^a=R_a{}^bR_b{}^cR_c{}^a-3\left(\frac{R}{2n}-1\right)R_{ab}R^{ab}+\frac{1}{2n^2}R^3 \\ &\quad -\frac{3}{n}R^2+6(n+1)\,R-4n(n+1)\,(2n+1) \\ &= R_a{}^bR_b{}^cR_c{}^a-3\left(\frac{R}{2n}-1\right)S_{ab}S^{ab}-\frac{1}{4n^2}R^3+\frac{3}{2n}R^2-3R \\ &\quad -2n(2n+1)\,(2n-1)\,. \end{aligned} \tag{3. 6}$$

M is an η-Einstein mainfold if and only if *trace* S^2 vainshes.

Substituting (3. 5) and (3. 6) in (3. 3), we have

$$2n(n+1)\,trace\ S^3+(n+2)\,(R+2n)\,trace\ S^2=0\,. \tag{3. 7}$$

We put $f^2=trace\ S^2$ $(f\geqq 0)$. From $S_i{}^a\eta_a=0$ and the comutativity of $S_i{}^j$ and $\phi_i{}^j$, we see that the characteristic roots of $S_i{}^j$ are $c_1, \cdots, c_n, c_1, \cdots, c_n$ and 0. Combining this fact with Lemma 5, we have

$$-\frac{n-2}{\sqrt{2n(n-1)}}f^3\leqq trace\ S^3\leqq \frac{n-2}{\sqrt{2n(n-1)}}f^3\,. \tag{3. 8}$$

Applying the above inequality, (3. 7) yields the following inequality

$$\begin{aligned} &f^2\left[(n+2)\,(R+2n)-\frac{2n(n+1)\,(n-2)}{\sqrt{2n(n-1)}}f\right]\leqq 0 \\ &\leqq f^2\left[(n+2)\,(R+2n)+\frac{2n(n+1)\,(n-2)}{\sqrt{2n(n-1)}}f\right] \end{aligned}$$

By the assumption of Theorem 1, we see that $f^2=0$, that is M is an η-Einstein mainfold. From Lemma 4, Theorem 1 has been proved.

REMARK. In Theorem 1, the condition with respect to the square of the length of the η-Einstein tensor is equivalent to

$$R_{ab}R^{ab}<\frac{n^3-n^2+4}{2(n+1)^2(n-2)^2}R^2-\frac{2(n^4-3n^3-6n^2+4n+8)}{(n+1)^2(n-2)^2}R+\frac{2n^2(2n^4-3n^3-7n^2+8n+12)}{(n+1)^2(n-2)^2}.$$

PROOF of THEOREM 2.

Let M be a 5-dimensional Sasakian manifold with constant scalar curvature whose contact Bochner curvature tensor vanishes. We see that *trace* $S^3=0$ since the characteristic roots of $S_i{}^j$ are c, $-c$, c, $-c$ and 0. From (3.7) we have $4(R+4)$ *trace* $S^2=0$. Therefore we have Theorem 2 by virtue of Lemma 4.

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