Hokkaido Mathematical Jornal Vol. 10 (1981) p. 57-74

Hilbertian support of probability measures on locally convex spaces and their applications

Dedicated to Prof. Kentaro Murata
on his sixtieth birthday

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(Received June 7, 1979)

§ 1. Introduction

In [4] J. Kuelbs has shown that every tight probability measure on a Fréchet space has a separable Banach support and then that every tight probability measure on a countable strict inductive limit of Fréchet spaces which satisfies a 0-1 law (*) has a separable Banach support.

On the other hand, in [9] H. Sato has shown that every Borel probability measure on a Fréchet space has a Banach support and then that every convex-tight Radon probability measure on a locally convex Hausdorff space which satisfies a 0-1 law (**) has a Banach support.

In this paper, we shall give a refinement of their results for some suitable class of locally convex spaces.

It is well known (cf. [2]) that every Fréchet space E is topologically isomorphic to a projective limit of a sequence of Banach spaces E_n. In particular, if for each n a Banach space E_n is topologically isomorphic to a subspace of an $\mathscr{L}_p$-space, then we shall say that a Fréchet space E is a complete countably $\mathscr{L}_p$-space $(1 \leqq p < \infty)$.

In Section 4, it can be shown that every Borel probability measure on a complete countably $\mathscr{L}_p$-space E has a Banach support F which is linearly isometric to a subspace of $L^p(\nu)$, for some measure ν. Furthermore if E is separable then the Banach space F is linearly isometric to a subspace of $L^p(0, 1)$. As a corollary, we have that every Borel probability measure on a nuclear Fréchet space has a separable Hilbertian support.

Now, it is easily seen that for every (LF)-space E which is not isomorphic to a Fréchet space there exists a convex-tight Radon probability measure μ on E which has no Banach support. However, if we assume the following condition :

$$(**)\qquad \begin{cases} \text{For every sequence } x_1^*,\ x_2^*,\ x_3^*, \cdots, \text{ in } E^* \text{ we have} \\ \mu(X\in E;\ \sup|\langle x_n^*, x\rangle|<\infty)=0 \text{ or } 1\,. \end{cases}$$

Then μ has a Banch support (cf. [9]).

It can be shown that every Borel probability measure on an inductive limit of a properly increasing sequence of complete separable countably $\mathscr{L}_p$-spaces which satisfies a 0–1 law (**) has a separable Banach support which is linearly isometric to a subspace of $L^p(0, 1)$. As a corollary, we have that every Borel probability measure on an inductive limit of a properly increasing sequence of nuclear Fréchet spaces which satisfies a 0–1 law (**) has a separable Hilbertian support.

For $2<p<\infty$, it is known that there exists a mean zero Gaussian measure on $L^p(0, 1)$ which has no Hilbertian support. However, if $1\leqq p\leqq 2$, then it can be shown that every mean zero Gaussian measure on an inductive limit of a properly increasing sequence of complete separable countably $\mathscr{L}_p$-spaces has a separable Hilbertian suport.

In Section 5, as their applications we shall discuss the partially admissible shifts of Borel probability measures on locally convex spaces. These generalize the results of D. Xia [17], A. V. Skorohod [11] and the author [12], [14].

Throughout the paper we assume that all linear spaces are with real coefficients.

§ 2. Definitions and notations

Let E be a linear topological space, and let μ be a Borel probability measure on E.

DEFINITION 2.1. Let F be a Borel measurable linear subspace of E equipped with a linear topology τ for which the natural injection from (F, τ) into E is continuous. Then we shall say that μ is supported by F if $\mu(F)=1$. In particular, if (F, τ) is a Hilbert space, then we shall say that μ has a Hilbertian support.

Similarly, Banach support and normed support are defined.

A Borel probability measure μ on E is tight if for each $\varepsilon>0$ there exists a compact subset K of E such that $\mu(K)>1-\varepsilon$. μ is convex-tight if there exists an increasing sequence of convex balanced compact subsets $\{K_n\}$ of E such that $\mu(\cup K_n)=1$. μ is Radon if for each $\varepsilon>0$ and each Borel set A there exists a compact subset $K\subset A$ such that $\mu(A\cap K^c)<\varepsilon$.

Obviously, every convex-tight measure is tight, and every Radon measure is tight. If E is quasi-complete, then every tight measure is convex-tight.

A Borel probability measure is a mean zero Gaussian measure if each continuous linear functional on E has a Gaussian distribution with mean zero.

§ 3. Lemmas

In this section, we shall devote several lemmas which are very useful in the ensuing discussions.

Let E be a linear space and $|\cdot|$ be a seminorm on E. We denote by $\hat{E}_{|\cdot|}$ a Banach space associated with the seminorm $|\cdot|$.

First, we shall study a complete countably $\mathscr{L}_p$-space. For $1 \leqq p < \infty$, a Fréchet space E is said to be a complete countably $\mathscr{L}_p$-space if it can be represented as a topological projective limit of a sequence of Banach spaces E_n such that for each n E_n is topologically isomorphic to a subspace of an $\mathscr{L}_p$-space.

As usual, by $L^p(\nu) = L^p(\Omega, \Sigma, \nu)$; $1 \leqq p \leqq \infty$, we denote the Banach space of equivalence classes of measurable functions on (Ω, Σ, ν) whose p'th power is integrable (respectively, are essentially bounded if $p = \infty$). If (Ω, Σ, ν) is the usual Lebesgue measure space on $[0, 1]$, we denote $L^p(\nu)$ by $L^p(0, 1)$.

LEMMA 3.1. *Let $1 \leqq p < \infty$. In order that a Fréchet space E is a complete countably $\mathscr{L}_p$-space, it is necessary and sufficient that there exists a countable basis of continuous seminorms $|\cdot|_n$ on E such that for each n a Banach space $\hat{E}_{|\cdot|_n}$ is linearly isometric to a subspace of $L^p(\nu_n)$, for some measure ν_n.*

PROOF. First we prove the necessity of the condition. Suppose that E is a complete countably $\mathscr{L}_p$-space. Since every $\mathscr{L}_p$-space is isomorphic to a subspace of $L^p(\nu)$ for some measure ν (cf. [6]), we may assume that E can be represented as a topological projective limit of a sequence of Banach spaces E_n with norm $||\cdot||_n$ such that for each n E_n is linearly isometric to a subspace of $L^p(\nu_n)$, for some measure ν_n. Let T_n be the natural map from E in to E_n.

Define

$$|x|_n = ||T_n x||_n \quad \text{for} \quad x \in E.$$

Since a Frecht space E is a projective limit of a sequence of Banach spaces E_n, $\{|\cdot|_n\}$ is a countable basis of continuous seminorms on E. It is obvious that for each n a Banach space $\hat{E}_{|\cdot|_n}$ is linearly isometric to a subspace of $L^p(\nu_n)$.

Next we prove the sufficiency of the condition. Suppose that there exists a countable basis of continuous seminorms $|\cdot|_n$ on E such that for each n a Banach space $\hat{E}_{|\cdot|_n}$ is linearly isometric to a subspace of $L^p(\nu_n)$, for some measure ν_n. Since a Fréchet space E can be represented as a topological projective limit of a sequence of Banach spaces $\hat{E}_{|\cdot|_n}$, it follows that E is a complete countably $\mathscr{L}_p$-space. This completes the proof.

REMARK 3.1. For $1\leqq p<\infty$, since every $\mathscr{L}_2$-space is isomorphic to a subspace of an $\mathscr{L}_p$-space (cf. [6]), it follows that every complete countably $\mathscr{L}_2$-space is also a complete countably $\mathscr{L}_p$-space. For $1\leqq q\leqq p\leqq 2$, since every $\mathscr{L}_p$-space is isomorphic to a subspace of $L^q(\nu)$ for some measure ν (cf. [1], [6]), it follows that every complete countably $\mathscr{L}_p$-space is also a complete countably $\mathscr{L}_q$-space.

Now, we shall give some examples of complete countably $\mathscr{L}_p$-spaces.

(1) Let Ω be a nonempty open subset of R^n, and let $L^p(\Omega)$ be a usual Banach space $(1\leqq p\leqq\infty)$. We denote by $L^p_c(\Omega)$ the totality of all functions belonging to $L^p(\Omega)$ which have a compact support.

Then, it is well known (cf. [15], p. 132) that $L^p_c(\Omega)$ is a strict inductive limit of an increasing sequence of L^p-spaces. Thus, $(L^p_c(\Omega))^*_b$ (strong dual of $L^p_c(\Omega)$) is a projective limit of a sequence of L^q-spaces, and so it is a complete countably $\mathscr{L}_q$-space $(1/p+1/q=1)$.

(2) A sequence space of Köthe $L=\bigcap_n l^p(\lambda_{m,n})$, $0<\lambda_{m,n}<\lambda_{m,n+1}$ (for all m, $n=1,2,\cdots$), is a complete separable countably $\mathscr{L}_p$-space $(1\leqq p<\infty)$.

(3) A projective limit of a sequence of Hilbert spaces is a complete countably $\mathscr{L}_2$-space. In particular, every nuclear Fréchet space is a complete separable countably $\mathscr{L}_2$-space.

LEMMA 3.2. *Let E be a linear topological space, and let μ be a Borel probability measure on E. If E is semimetrizable, then there exists a bounded balanced closed subset B of E such that $\mu(\bigcup nB)=1$.*

PROOF. Since E is semimetrizable, by the same way as in Theorem 1 of [9], we have that there exists a sequence of bounded balanced closed subsets B_n of E such that $\mu(\bigcup B_n)=1$.

Let $\{V_n\}$ be a countable basis of neighbourhoods of zero in E such that $V_1\supset V_2\supset V_3\supset\cdots$. For each n, since B_n is bounded, there exists an $\varepsilon_n>0$ such that $\varepsilon_n B_n\subset V_n$. We denote by B the balanced closed hull of a set $\bigcup\varepsilon_n B_n$. Then the set B is a desired one. This completes the proof.

REMARK 3.2. In Lemma 3.2., if E is locally convex, then we may assume that B is convex, and hence μ has a seminormed support. Furthermore if E is separated (*i.e.* Hausdorff), then μ has a normed support, and also if E is complete, then μ has a Banach support.

These results were obtained by H. Sato [9].

Let E be a Hausdorff linear topological space. Then, E is said to be quasi-complete if every bounded closed subset of E is complete, and E is said to be a Lusin space if there exists a Polish space F and a continuous bijective mapping from F onto E.

LEMMA 3.3. *Let E be a locally convex Hausdorff space, and let μ be a Borel probability measure on E. If the measure μ is supported by a locally convex quasi-complete Lusin space F, then the restriction on F of μ, we denote it by μ_F, is also a Borel probability measure on F. Furthermore if μ is a mean zero Gaussian measure, then μ_F is also a mean zero Gaussian measure on F.*

PROOF. The first part of a statement is an immediate consequence of Corollary 3 of [10].

Next we prove the second part. Let μ be a mean zero Gaussian measure on E. Then, by the first part, μ_F is a Borel probability measure on F. Since F is a quasi-complete Lusin space, it follows from [10] that μ_F is a convex-thight measure on F. Thus, there exists an increasing sequence of convex balanced compact subsets $\{K_n\}$ of F such that $\mu(\cup K_n)=1$.

Let d be a topology on F^* (the topological dual of F) induced by the uniform convergence on all K_n. Then, d is locally convex semimetrizable, and coarser than the Mackey topology $\tau(F^*, F)$.

Now, let j be a natural injection from F into E, and let j^* be the adjoint of j. Since $j: F \rightarrow E$ is injective, $j^*(E^*)$ is dense in F^* for the Mackey topology $\tau(F^*, F)$. Since the topology d is semimetrizable, and coarser than the Mackey topology $\tau(F^*, F)$, it follows that $j^*(E^*)$ is sequentially dense in F^* for the topology d.

Thus, for each x^* in F^* there exists a sequence $x_1^*, x_2^*, \cdots$, in E^* such that $j^*(x_n^*) \rightarrow x^*$ in F^* for the topology d, and this implies that $j^*(x_n^*) \rightarrow x^*$ in F^* a.s. μ_F. Since for each n $j^*(x_n^*)$ is a Gaussian random variable, it follows that x^* is also a Gaussian one. This completes the proof.

LEMMA 3.4. *Let E be a separable Banach space which is isomorphic to a subspace of $L^p(0,1)$ (for $1 \leqq p \leqq 2$), and let H be a linear subspace of E, and suppose that H itself is a separable Hilbert space. Also, suppose that the inclusion map i from H into E is continuous. Let μ be a continuous cylinder measure on H which satisfies the following two conditions;*

(1) *$\widetilde{M}_\mu$ is second category in H, where $\widetilde{M}_\mu$ denotes the set of partially admissible shifts of the cylinder measure μ. (For the definition of partially admissible shifts; see [14])*

(2) *μ can be extended to a σ-additive measure $\tilde{\mu}$ on E.*
Then, the measure $\tilde{\mu}$ has a separable Hilbertian support.

PROOF. It follows from Corollary 4.3 of [14] that the map $i: H\to E$ can be decomposed through a Hilbert-Schmidt operator. Hence, using the theorem of Minlos [7], we have that $\tilde{\mu}$ has a separable Hilbertian support.

COROLLARY 3.1. *Let E be a separable Banach space which is isomorphic to a subspace of $L^p(0,1)$ (for $1\leqq p\leqq 2$), and let μ be a mean zero Gaussian measure on E. Then μ has a separable Hilbertian support.*

PROOF. It follows from the theorem of Sato [8] that there exists a separable Hilbert space H such that (i, H, E) is an abstract Wiener space and μ is a σ-additive extension of the canonical Gaussian cylinder measure μ_H on H, where i is the inclusion map from H into E.

Since μ_H certainly satisfies the conditions (1) and (2) of Lemma 3.4, it follows that μ has a separable Hilbertian support.

REMARK 3.3. In the proof of Corollary 3.1, a Hilbert space H is not necessarily dense in E. It is well known that (i, H, E) is an abstract Wiener space in a sense of Gross iff for each nonempty open subset U of E $\mu(U)>0$ holds.

§ 4. Hilbertian support of probability measures

In this section, we shall study the Hilbertian support of Borel probability measures on some locally convex spaces.

First, we prove the following fundamental lemma.

LEMMA 4.1. *Let E be the topological inductive limit of a properly increasing sequence $E_1\subset E_2\subset\cdots$ of complete countably $\mathscr{L}_p$-spaces (for $1\leqq p<\infty$). Let F be a Banach space and T be a continuous linear map from F into E. Then, the map T can be decomposed as follows;*

$$F\xrightarrow[J]{} G\xrightarrow[K]{} E$$

$T=K\circ J$ where G is a Banach space which is linearly isometric to a subspace of $L^p(\nu)$ for some measure ν, J is a continuous linear map and K is a one-to-one continuous linear map, respectively. Furthermore if F is separable then the Banach space G can be taken as a subspace of $L^p(0,1)$.

PROOF. It follows from the theorem of Grothendieck (cf. [2], p. 225) that there exists n such $T(F)\subset E_n$ and T is a continuous linear map from F into E_n. Hence, without loss of generality, we may assume that E is a complete countably $\mathscr{L}_p$-space. It follows from Lemma 3.1 that there exists a countable basis of continuous seminorms $|\cdot|_n$ on E such that for each n a Banach space $\hat{E}_{|\cdot|_n}$ is linearly isometric to a subspace of $L^p(\nu_n)$, for some measure space $(\Omega_n, \Sigma_n, \nu_n)$. Let S be a closed unit ball of the Banach space

F. Then $T(S)$ is a bounded subset of E, so that there exists $a_n>0$ such that

$$a_n|x|_n\leqq 1 \quad \text{for all } x\in T(S), \text{ and } n=1,2,\cdots,$$

holds.

Define

$$||x||=\left(\sum_{n=1}^{\infty}2^{-n}a_n|x|_n^p\right)^{1/p}, \qquad \text{for} \quad x\in E,$$

and put $G=\{x\in E;\ ||x||<\infty\}$. Then, obviously G is a linear subspace of E, and $||\cdot||$ is a norm on G since E is Hausdorff.

Since it is easily seen that the set $\{x\in E;\ ||x||\leqq 1\}$ is bounded convex balanced closed in E, it follows from [15] that G is a Banach space with norm $||\cdot||$ and the inclusion map K from G into E is continuous.

Now, it is obvious that $||\cdot||$ is finite on $T(S)$ and, hence, on $T(F)$. From this it follows that T is a linear map from F into G, and the graph of T is closed in $F\times G$ since the inclusion map $K\colon G\to E$ is continuous. Thus, by the closed graph theorem (cf. [16]), the map T is continuous from F into G.

In order to prove the first assertion, it suffices to show that the Banach space G is linearly isometric to a subspace of $L^p(\nu)$ for some measure ν.

Let $\Omega=\bigcup_n \Omega_n$ and $\Sigma=\{A\subset\Omega;\ A\cap\Omega_n\in\Sigma_n$ for all $n\}$, where we may assume that sets $\{\Omega_n\}$ are pairwise disjoint. Then it is obvious that Σ is a σ-algebra consisting of subsets of Ω.

Define a measure ν on (Ω, Σ) and a map f from G into $L^p(\nu)$ as follows;

$$\nu(A)=\sum_{n=1}^{\infty}2^{-n}a_n\nu_n(A\cap\Omega_n), \qquad \text{for} \quad A\in\Sigma,$$

and for each $x\in G$

$$\big(f(x)\big)(\omega)=\big(g_n\circ f_n(x)\big)(\omega), \qquad \text{if} \quad \omega\in\Omega_n,$$

where we denote by f_n the quotient map from E into $\hat{E}_{|\cdot|_n}$ and by g_n the linear isometry from $\hat{E}_{|\cdot|_n}$ into $L^p(\nu_n)$, respectively.

Then, obviously the map f is linear, and furthermore we have

$$\begin{aligned}\|f(x)\|_{L^p(\nu)}^p&=\int_\Omega\left|\big(f(x)\big)(\omega)\right|^p d\nu(\omega)\\&=\sum_{n=1}^{\infty}2^{-n}a_n\int_{\Omega_n}\left|\big(g_n\circ f_n(x)\big)(\omega)\right|^p d\nu_n(\omega)\end{aligned}$$

$$= \sum_{n=1}^{\infty} 2^{-n} a_n \left\| g_n \circ f_n(x) \right\|_{L^p(\nu_n)}^p$$

$$= \sum_{n=1}^{\infty} 2^{-n} a_n |x|_n^p = ||x||^p, \qquad \text{for} \quad x \in G.$$

This shows that the map f from G into $L^p(\nu)$ is linear isometry. Thus, we proved the first assertion.

Next, we prove the second assertion. Suppose that the Banach space F is separable. Then, $T(F)$ is a separable linear subspace of G and, hence, of $L^p(\nu)$. On the other hand, it is well known (cf. [6]) that for every separable subspace X of $L^p(\nu)$, the closed sublattice generated by X is a separable $L^p(\mu)$-space for some measure μ, and every separable $L^p(\mu)$-space is linearly isometric to a subspace of $L^p(0, 1)$.

From this, if we denote by G_0 the closure of $T(F)$ in G, then G_0 is a Banach space which is linearly isometric to a subsapce of $L^p(0, 1)$. This completes the proof.

REMARK 4.1. In Lemma 4.1, as is shown in the proof, the Banach space G is a Borel subset of E_n for some n, but is not necessarily a Borel subset of E. However, if F is separable then G is also separable and, hence, it is a Borel subset of E (cf. [10]).

As an immediate consequence of this lemma, we have

THEOREM 4.1. *Let E be a complete countably $\mathscr{L}_p$-space* $(1 \leqq p < \infty)$, *and let μ be a Borel probability measure on E. Then, μ has a Banach support F which is linearly isometric to a subspace of $L^p(\nu)$, for some measure ν. Furthermore if E is separable, then the Banach space F is linearly isometric to a subspace of $L^p(0, 1)$.*

PROOF. It follows from Remark 3.2 that μ has a Banach support. Furthermore if E is separable, then μ is a Radon measure and, hence, it also follows from the theorem of Kuelbs [4] that μ has a separable Banach support. Thus, using Lemma 4.1 and Remark 4.1, the assertion can be easily proved.

COROLLARY 4.1. *Every Borel probability measure on a nuclear Fréchet space has a separable Hilbertian support.*

Now, it is easily seen that every convex-tight measure on a locally convex Hausdorff space is supported by an inductive limit of an increasing sequence of Banach spaces. On the other hand, it is known (cf. [9]) that every convex-tight Radon measure on a locally convex Hausdorff space is supported by a dual of a locally convex metrizable space.

Since an inductive limit of an increasing sequence of Banach spaces is a barrelled (DF)-space, and also since a strong dual of a locally convex metrizable space is a complete (DF)-space (cf. [2]), it will be very natural to consider the existence of a Banach support of a convex-tight Radon measure on a (DF)-space. Concerning with this, we have

PROPOSITION 4.1. *Let E be a (DF)-space. Then the following conditions are equivalent.*

(1) *Every convex-tight Radon measure on E has a normed support.*

(2) *E is relatively bounded.*

PROOF. First, we prove the part of (1)⇒(2). Since E is a (DF)-space, it has a fundamental sequence of bounded sets $B_1 \subset B_2 \subset \cdots$. We may assume that for each n the set B_n is convex balanced closed since E is a locally convex space. We denote by E_n the linear hull of B_n. Then, obviously we have that $E_n = \bigcup_k kB_n$ for all n, and $E_1 \subset E_2 \subset \cdots$.

Now, we assume the condition (1). To prove the condition (2), it suffices to show that there exists n such that $E_n = E$. For otherwise, without loss of generality, we may assume that $E_1 \subsetneqq E_2 \subsetneqq \cdots \subsetneqq E_n \subsetneqq \cdots$. Thus, there exists a sequence $\{x_n\}$ in E such that $x_1 \in E_1$, $x_{n+1} \in E_{n+1} \cap E_n^c$ for $n=1, 2, \cdots,$. Let $\mu\{x_n\} = 2^{-n}$ for $n=1, 2, \cdots,$. Then, μ is a convex-tight Radon measure on E, so that by the assumption (1) μ has a normed support. Denote by B the closed unit ball of the normed space F. Then, for each n, x_n must be absorbed by B since x_n is contained in F. Since B is bounded, and also since $\{B_n\}$ is a fundamental sequence of bounded sets, it follows that there exists N such that $B \subset B_n$ for all $n \geqq N$, so that $\{x_k\} \subset E_n$ for all $n \geqq N$. As a consequence, we have $x_{N+1} \in E_N$, and this is a contradiction.

The part of (2)⇒(1) is obvious. This completes the proof.

REMARK 4.2. A locally convex Hausdorff space E is called to be relatively bounded if there exists a barrel which is bounded. Then it is obvious that E is relatively bounded iff there exists a norm $\|\cdot\|$ on E such that the original topology on E is coarser than the $\|\cdot\|$ topology.

As an immediate consequence of Proposition 4.1, we have

COROLLARY 4.2. *Let E be a (DF)-space which is barrelled or complete. Then the following conditions are equivalent.*

(1) *Every convex-tight Radon measure on E has a normed support.*

(2) *E is normable.*

PROPOSITION 4.2. *Let E be an inductive limit of a properly increasing sequence $E_1 \subset E_2 \subset \cdots$ of Fréchet spaces. Then there exists a convex-tight Radon measure on E which has no Banach support.*

PROOF. Since the sequence of Fréchet spaces $\{E_n\}$ is properly increasing, there exists a sequence $\{x_n\}$ in E such that $x_1 \in E_1$, $x_{n+1} \in E_{n+1} \cap E_n^c$ for $n=1, 2, \cdots,$. Let $\mu\{x_n\}=2^{-n}$ for $n=1, 2, \cdots,$. Then μ is a convex-tight Radon measure on E. When this, we may show that μ has no Banach support. For otherwise, there exists a Banach subspace F of E such that $\mu(F)=1$. Hence, it follows from the theorem of Grothendieck (cf. [2], p. 225) that $F \subset E_N$ for some N, so that $\{x_n\} \subset E_N$. This is a contradiction, and we complete the proof.

Next, we shall study a Banach support of a Borel probability measure on an (LF)-space. To establish this, it follows from Proposition 4. 2 that we must need some conditions for a probability measure.

Let E be a locally convex Hausdorff space, and let μ be a Borel probability measure on E. Then we introduce the following two conditions:

(*) For any x^* in E^*, we have $\mu(x \in E;\ \langle x^*, x\rangle=0)=0$ or 1.

(**) For any sequence $\{x_n^*\}$ in E^*, we have

$$\mu(x \in E;\ \sup_n |\langle x_n^*, x\rangle| < \infty)=0 \text{ or } 1\,.$$

LEMMA 4. 2. (cf. [4], [9])

Let μ be a Radon measure on a locally convex Hausdorff space E. Then we have

(1) *In order that μ satisfies the condition (*), it is necessary and sufficient that for any closed subspace F of E, $\mu(F)=0$ or 1.*

(2) *In order that μ satisfies the condition (**), it is necessary and sufficient that for any convex balanced closed subset B of E, $\mu(\cup nB)=0$ or 1.*

REMARK 4. 3. In (1), the sufficiency of the condition is obvious, and the necessity of the condition was established by J. Kuelbs in [4]. It is obvious that if μ is not a Radon measure but there exists a topological support of μ, then the same result as (1) holds.

In (2), the sufficiency of the condition can be easily proved, and the necessity of the condition was established by H. Sato in [9].

It follows from Theorem 4. 1 and Lemma 4. 2 that the following result holds.

THEOREM 4. 2. *Let E be a strict inductive limit of an increasing sequence of complete separable countably $\mathscr{L}_p$-spaces ($1 \leqq p < \infty$), and let μ be a Borel probability measure on E which satisfies the condition (*). Then, μ has a separable Banach support F which is linearly isometric to a subspace of $L^p(0, 1)$*

COROLLARY 4. 3. *Let E be a strict inductive limit of an increasing sequence of nuclear Fréchet spaces, and let μ be a Borel probability measure on E which satisfies the condition (*). Then, μ has a separable Hilbertian* *support.*

Here, if we assume that μ satisfies the condition (**), then Theorem 4.2 and Corollary 4.3 hold for an (LF)-space (not necessarily a strict (LF)-space). Those are the followings.

THEOREM 4.3. *Let E be an inductive limit of an increasing sequence $E_1 \subset E_2 \subset \cdots$ of complete separable countably $\mathscr{L}_p$-spaces* $(1 \leqq p < \infty)$, *and let μ be a Borel probability measure on E which satisfies the condition* (**). *Then, μ has a separable Banach support F which is linearly isometric to a subspace of* $L^p(0, 1)$.

PROOF. Since E is an inductive limit of an increasing sequence of Polish spaces, it is easily seen (cf. [10]) that μ is a convex-tight Radon measure on E. Since the measure μ satisfies the condition (**), it follows from Lemma 4.2 that μ has a Banach support G, so that there exists N such that $\mu(E_N)=1$, by the theorem of Grothendieck (cf. [2], p. 225). Hence, using Theorem 4.2, we complete the proof.

COROLLARY 4.4. *Let E be an inductive limit of an increasing sequence of nuclear Fréchet spaces, and let μ be a Borel probability measure on E which satisfies the condition* (**). *Then, μ has a separable Hilbertian support.*

THEOREM 4.4. *Let E be an inductive limit of an increasing sequence of complete separable countably $\mathscr{L}_p$-spaces* $(1 \leqq p \leqq 2)$. *Then, every mean zero Gaussian measure on E has a separable Hilbertian support.*

PROOF. Let μ be a mean zero Gaussian measure on E. Since every Gaussian measure satisfies the condition (**) (cf. [5]), it follows from Theorem 4.3 that μ has a separable Banach support F which is linearly isometric to a subspace of $L^p(0, 1)$. Hence, it follows from Lemma 3.3 that μ can be considered a mean zero Gaussian measure on F. Thus, using Lemma 3.4, we have the assertion.

Finally, we shall study a Banach support of a Borel probability measure on a dual space of a locally convex Hausdorff space.

Let E be a locally convex Hausdorff space, and let $\{\|\cdot\|_\alpha\}_{\alpha \in A}$ be a basis of continuous seminorms on E. We denote by $\hat{E}_\alpha$ the Banach space associated with a seminorm $\|\cdot\|_\alpha$. Then, E is called a Schwarz space if for each continuous seminorm $\|\cdot\|_\alpha$ there exists a continuous seminorm $\|\cdot\|_\beta \geqq \|\cdot\|_\alpha$ such that the natural map from $\hat{E}_\beta$ into $\hat{E}_\alpha$ is compact. It is obvious that every nuclear space is a Schwarz space. E is called a quasi-barrelled space if every barrel in E which absorbs all the bounded sets of E is a neighbourhood of zero. It is obvious that all barrelled spaces and all bornological spaces are quasi-barrelled, and it is known (cf. [2]) that all locally convex metrizable spaces are quasi-barrelled.

THEOREM 4.5. *Let E be a quasi-barrelled Schwarz space, and let μ be a Radon measure on E_b^* (the strong dual E) which satisfies the condition* (**). *Suppose that there exists a basis of continuous seminorms $\{\|\cdot\|_\alpha\}_{\alpha\in\Lambda}$ such that for each $\|\cdot\|_\alpha$ the associated Banach space $\hat{E}_\alpha$ is isomorphic to a quotient space of an $\mathscr{L}_p$-space. Then, μ has a separable Banach support F which is linearly isometric to a subspace of $L^q(0,1)$* $(1<p\leqq\infty,\ 1/p+1/q=1)$.

PROOF. Since μ is Radon, there exists a compact subset K of E_b^* such that $\mu(K)>0$. We denote by B the convex balanced closed hull of K in E_b^* and by B^0 the polar set of B in E, respectively. Then, B is obviously bounded in E_b^* and, hence, B^0 is a barrel in E which absorbs all the bounded sets of E. From this and the assumption of E, it follows that the set B^0 must contain some neighbourhood of zero in E, so that there exists a continuous seminorm $\|\cdot\|_\alpha$ such that $B\subset E_\alpha^*$, where the strong dual of $\hat{E}_\alpha$ be denoted by E_α^*.

On the other hand, since E is a Schwarz space, there exists a continuous seminorm $\|\cdot\|_\beta\geqq\|\cdot\|_\alpha$ such that the natural map from $\hat{E}_\beta$ into $\hat{E}_\alpha$ is compact and, equivalently, the inclusion map from E_α^* into E_β^* is compact. From this, E_α^* can be considered a separable linear subspace of the Banach space E_β^* and, hence, if we denote by F the closure of E_α^* in the Banach space E_β^* then F is a separable Banach space and the inclusion map from F into E_b^* is continuous.

Since the measure μ satisfies the condition (**), it follows from Lemma 4.2 that $\mu(B)>0$ implies $\mu(\cup nB)=1$ and, hence, $\mu(F)=1$. This shows that μ has a separable Banach support F.

Now, to complete the proof, it sufficies to show that the Banach space F is linearly isometric to a subspace of $L^q(0,1)$. For by the assumption the Banach space $\hat{E}_\beta$ is isomorphic to a quotient space of an $\mathscr{L}_p$-space, so that the Banach space E_β^* is isomorphic to a subspace of an $\mathscr{L}_q$-space (cf. [6]) and, hence, F is also so. Thus, the assertion can be proved in a quite similar way as in the proof of Lemma 4.1. This completes the proof.

COROLLARY 4.5. *Let E be a quasi-barrelled nuclear space, and let μ be a Radon measure on E_b^* which satisfies the condition* (**). *Then μ has a separable Hilbertian support. In particular, every mean zero Gaussian Radon measure on E_b^* has a separable Hilbertian support.*

THEOREM 4.6. *Let E be a quasi-barrelled Schwarz space, and let μ be a mean zero Gaussian Radon measure on E_b^*. Suppose that there exists* *a basis of continuous seminorms* $\{\|\cdot\|_\alpha\}_{\alpha\in\Lambda}$ *such that for each* $\|\cdot\|_\alpha$ *the associated Banach space* $\hat{E}_\alpha$ *is isomorphic to a quotient space of an* $\mathscr{L}_p$*-space* $(2\leqq p\leqq\infty)$. *Then* μ *has a separable Hilbertian support.*

Using Theorem 4. 5, the proof can be done in a quite similar way as in the proof of Theorem 4. 4, and so we omit it.

§ 5. Partially admissible shifts of probability measures

In this section, as an application to the previous section we shall study the partially admissible shifts of probability measures. Throughout this section, we assume that all linear spaces are infinite dimensional.

Let E be a linear topological space, and let μ be a Borel probability measure on E. Let μ_x (for $x\in E$) denote the Borel probability measure on E defined by

$$\mu_x(A)=\mu(A-x) \quad \text{for any Borel set } A \text{ of } E.$$

DEFINITION 5. 1. An element x of E is called an admissible shift for the measure μ if μ_x is absolutely continuous with respect to μ. The set of admissible shifts of the measure μ will be denoted by M_μ.

An element x of E is called a partially admissible shift for the measure μ if μ_x contains a component absolutely continuous with respect to μ. The set of partially admissible shifts of the measure μ will be denoted by $\tilde{M}_\mu$.

It is easily seen that $M_\mu\subset\tilde{M}_\mu$, but in general M_μ does not coincide with $\tilde{M}_\mu$. In the case E is a Banach space, we proved in [14] that the set $\tilde{M}_\mu$ is first category in E.

First we shall show that this fact is also true for every Fréchet space.

THEOREM 5. 1. *Let* μ *be a Borel probability measure on a Fréchet space* E. *Then the set* $\tilde{M}_\mu$ *is first category in* E. *In particular,* $\tilde{M}_\mu$ *does not coincide with* E.

PROOF. Since a Fréchet space E is second category, it sufficies to prove that the first statement holds. By Remark 3. 2 there exists a bounded convex balanced closed subset B of E such that $\mu(\cup nB)=1$, so that using Proposition 3. 2 of [14], we have $\tilde{M}_\mu\subset\cup nB$.

Now, assume the contrary. Then the the set $\cup nB$ is second category in E. Since every set nB is closed, at least one of them must have a nonempty interior and, hence, B itself must have at least one interior point. Since B is convex balanced, the origin must be an interior point of B. Thus, B is a neighbourhood of zero in E which is bounded, so that E is normable (cf. [2]). Since E is complete, E is isomorphic to a Banach space, and this is a contradiction to Corollary 4.4 of [14]. This completes the proof.

Next, we shall give a finer result than Theorem 5.1 for some Fréchet spaces.

THEOREM 5.2. *Let E be a Fréchet space, and let μ be a Borel probability measure on E. Suppose that for each neighbourhood U of zero in E the polar set of U in E^* is sequentially compact for the topology $\sigma(E^*, E)$. Then, there exists a compact convex balanced subset K of E such that $\tilde{M}_\mu \subset \bigcup nK$ holds.*

PROOF. First we show that there exists a Borel probaility measure ν on E such that the measure ν is equivalent to μ, and satisfies the following condition (a);

(a) For each continuous seminorm $p(x)$ on E, we have $\int_E p(x)\, d\nu(x) < \infty$.

For since E is a Fréchet space, it follows from Remark 3.2 that there exists a bounded convex balanced closed subset B of E such that $\mu(\bigcup nB)=1$. Let $\{||\cdot||_n\}$ be a countable basis of continuous seminorms on E. Then we may assume that $\sup\{||x||_n;\ x\in B\}\leqq 1$ for all n since B is bounded.
Define

$$||x|| = \sum_{n=1}^{\infty} 2^{-n}||x||_n \quad \text{for} \quad x\in E,$$

and put $F=\{x\in E;\ ||x||<\infty\}$. Then, obviously we have that the set F is a Borel measurable linear subspace of E, and $\bigcup nB\subset F$ implies $\mu(F)=1$.
Define

$$f(x) = \begin{cases} ||x|| & \text{if} \quad x\in F. \\ 0 & \text{if} \ x\in E\cap F^c. \end{cases}$$

Then the function $f(x)$ on E is obviously Borel measurable.
Define

$$\nu(A) = C\int_A e^{-f(x)}\, d\mu(x) \quad \text{for all Borel sets } A \text{ of } E,$$

where C is a normalized constant.

Then, obviously ν is a Borel probability measure on E which is equivalent to the measure μ, and so it sufficies only to show that ν satisfies the condition (a). Let $p(x)$ be a continuous seminorm on E. Then there exist a positive constant C_1 and a positive integer N such that the inequality $p(x)\leqq C_1||x||_N$, $x\in E$, holds.

Hence, we have

$$\int_E p(x)\,d\nu(x) = C\int_E p(x)e^{-f(x)}\,d\mu(x)$$
$$= C\int_F p(x)\,e^{-||x||}\,d\mu(x)$$
$$\leqq C\cdot C_1\cdot 2^N\int_F ||x||e^{-||x||}\,d\mu(x) < \infty\,;$$

Thus, the measure ν certainly satisfies the condition (a).

Now, define the set S as follows:

$$S = \left\{x \in E^*\,; \quad \int_E \left|\langle x^*, x\rangle\right| d\nu(x) \leqq 1\right\},$$

and denote by K the polar set of S in E. Then obviously the set K is convex balanced closed, and by the condition (a) K is bounded in E, and it follows from Lemma 3. 1 of [14] that $\widetilde{M}_\mu = \widetilde{M}_\nu \subset \bigcup nK$.

Consequently, to prove this theorem, it sufficies to show that for each continuous seminorm $p(x)$ on E the set K is totally bounded with respect to the seminorm $p(x)$ since K is complete. Denote by $\hat{E}_p$ the Banach space associated with a seminorm $p(x)$, and also denote by E_K the linear subspace of E generated by K. Then it follows from [2] that E_K is a Banach space with the norm $||\cdot||_K$ defined by

$$||x||_K = \inf\{\lambda > 0\,;\ x \in \lambda K\} \text{ for } x \in E_K,$$

and the natural map J from E_K into $\hat{E}_p$ is continuous, where the map J is the composition of the inclusion map: $E_K \to E$ and the quotient map: $E \to \hat{E}_p$.

We shall finish the proof by showing that the map J is compact. To do this, it sufficies to show that map J^* (the adjoint of J) from $(\hat{E}_p)^*$ into $(E_K)^*$ is compact. Let $\{x_i^*\}$ be a bounded sequence of the Banach space $(\hat{E}_p)^*$. Then, by the assumption of E, there exists a subsequence $\{x_{m_i}^*\}$ of $\{x_i^*\}$ and $x^* \in E^*$ such that $x_{m_i}^* \to x^*$ for the topology $\sigma(E^*, E)$.

On the other hand, we have

$$\left\|J^*(x_{m_i}^*) - J^*(x^*)\right\|_{(E_K)^*} = \sup_{x\in K}\left|\langle x_{m_i}^* - x^*, x\rangle\right|$$
$$\leqq \int_E \left|\langle x_{m_i}^* - x^*, x\rangle\right| d\nu(x)\,.$$

Since it can be shown that the sequence $\{x_{m_i}^* - x^*\}$ is contained in $(\hat{E}_p)^*$ and bounded in it, by the condition (a), we can apply the Lebesgue's dominated convergence theorem for this case as follows;

$$\overline{\lim_{i\to\infty}}\left\|J^*(x^*_{m_i})-J^*(x^*)\right\|_{(E_K)^*}\leqq\overline{\lim_{i\to\infty}}\int_E\left|\langle x^*_{m_i}-x^*,\ x\rangle\right|d\nu(x)=0\,.$$

This shows that the map J^* from $(\hat{E}_p)^*$ into $(E_K)^*$ is compact, and we complete the proof.

REMARK 5.1. In Theorem 5.2, if E is a reflexive Banach space, then by the theorem of Eberlein every bounded set in a Banach space E^* is sequentially relatively compact for the topology $\sigma(E^*, E)$, so that the assumption of E can be satisfied. This shows that Theorem 5.2 generalizes the author's result (cf. [14]). On the other hand, in the case E is a separable Fréchet space, every Borel probability measure on E is Radon, so that Theorem 5.2 is obviously valid.

Using the closed graph theorem [16], we have

COROLLARY 5.1. *Let E be a Fréchet space as in Theorem 5.2, F be a barrelled space and T be a continuous linear map from F into E. Let μ be a Borel probability measure on E, and suppose that $T(F)\subset\widetilde{M}_\mu$. Then the map T from F into E is compact.*

Finally, we shall prove that in Corollary 5.1 the map T is of Hilbert-Schmidt type when E and F belong to some suitable class of Fréchet spaces.

THEOREM 5.3. *Let E be a complete separable countably $\mathscr{L}_2$-space, F be a barrelled space and T be a continuous linear map from F into E. Let μ be a Borel probability measure on E, and suppose that $T(F)\subset\widetilde{M}_\mu$. Then the map T from F into E can be decomposed through a Hilbert-Schmidt one.*

PROOF. If follows from Theorem 4.1 that μ has a separable Hilbertian support H. Hence, by the assumption of μ and Proposition 3.2 of [14], we have that $T(F)\subset H$. Since the inclusion map from H into E is continuous, it follows from the closed graph theorem [16] that T is a continuous linear map from F into H. We denote by μ_H the restriction of μ on H. They by Lemma 3.3 μ_H is a Borel probability measure on H, and obviously $\widetilde{M}_\mu=\widetilde{M}_{\mu_H}$ holds. Thus, by Corollary 4.2 of [14] we complete the proof.

THEOREM 5.4. *Let E be a complete separable countably $\mathscr{L}_1$-space, F be a barrelled space and T be a continuous linear map from F into E. Let μ be a Borel probability measure on E. Suppose that the topology of F is defined by a family of Hilbertian seminorms, and also suppose that $T(F)\subset\widetilde{M}_\mu$. Then the map T from F into E can be decomposed through a Hilbert-Schmidt one.*

PROOF. It follows from Theorem 4.1 that μ has a separable Banach support G which is linearly isometric to a subspace of $L^1(0, 1)$. Hence, using Corollary 4. 3 of [14], the proof can be done by the same way as in the proof of Theorem 5. 3. This completes the proof.

Acknowledgement

The author wishes to thank Professor D. H. Fremlin for his useful comments.

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