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Finitely generated projective modules over hereditary noetherian prime rings

Dedicated to Professor Goro AZUMAYA
on his 60th birthday

By Kenji NISHIDA

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Introduction. In this paper we study finitely generated projective modules over a hereditary noetherian prime ring (henceforth, we denote an HNP ring, for abbreviation). We mainly concern with the genus of finitely generated projective modules. When one deals with an order over a Dedekind domain, the genus can be investigated by localization (cf. [7, § 27, 35]). In our (noncommutative) case, the localization at a maximal invertible ideal studied in [5, § 3] is very useful. As is stated in [5, § 3], the localization of an HNP ring at a maximal invertible ideal is either a Dedekind prime ring or a semilocal HNP ring defined in § 1. Although finitely generated projective modules over a Dedekind prime ring were perfectly studied in [1], the another case is not treated anywhere. Therefore, we investigate those over a semilocal HNP ring in § 1 and give a necessary and sufficient condition when two finitely generated projective modules are in the same genus and also prove the following.

(1. 15) THEOREM. *Let R be a semilocal HNP ring with its radical I and M, N, K finitely generated projective modules in the same genus such that K/KI contains all $S_\lambda \in \mathfrak{S}_I$. Then there exists a finitely generated projective module L in the genus such that $N \oplus K \cong M \oplus L$.*

In § 2, we treat of an HNP ring R with enough invertible ideals and show that two finitely generated projective modules are in the same genus iff their localization M_I and N_I are in the same genus as R_I-modules for all maximal invertible ideals I, where R_I is the localization of R at I. The generalization of (1. 15) is obtained in (2. 7).

Finally, in § 3, applying the above results we try to define the ideal class group for some HNP ring.

Throughout this paper, R is an HNP ring which is not artinian and Q is the maximal quotient ring of R. We shall shortly mention definitions and notation which will be frequently used in this paper. For more detailed description, the reader is referred to [1, 2, 3]. Let $X^*=\mathrm{Hom}_R(X, R)$ for a right R-module X. Let $P_0, \cdots, P_{t-1}$ be a nontrivial *cycle* of idempotent maximal ideals of R with $t>1$ [2]. We consider an invertible maximal ideal as a trivial cycle. However, in this paper, a cycle is always nontrivial unless otherwise stated. When a maximal invertible ideal I is an intersection of a cycle $P_0, \cdots, P_{t-1}$, we say that "$I=P_0\cap \cdots \cap P_{t-1}$" or "$I=P_0\cap \cdots \cap P_{t-1}$ is an intersection of a cycle" or "I is an intersection of a cycle". We denote the Goldie dimension of a module M by rank M and the length of composition series of M (if it exists) by $l(M)$. A *right bounded* ring is a ring such that every nonzero right ideal contains a nonzero ideal. A right and a left bounded ring is called *bounded*. When every nonzero ideal of R contains an invertible ideal, we say that R has *enough invertible ideals.*

1. Semilocal HNP rings. Let R be an HNP ring and M, N finitely generated projective R-modules. We say that M and N are in the same *genus*, denoted by $M\vee N$, if rank $M=$rank N and $M/MP\cong N/NP$ for all maximal ideals P of R [3, § 4]. We shall investigate the behavior of the genus in this sense.

(1. 1) LEMMA. *Let R be an HNP ring and M, N finitely generated projective modules with $N\subset M$. Then N is an essential submodule of M iff M/N is artinian.*

PROOF. Let X be an essential submodule of Y and $f(X)=0$ for some $f\in Y^*$. We can find a regular element c of R with $yc\in X$ for any $y\in Y$. Then $0=f(yc)=f(y)c$ implies $f(y)=0$, that is $f=0$. Thus we have $Y^*\subset X^*$. Let N be essential in M and $M=M_0\supset M_1\supset\cdots\supset N$ a descending chain of submodules of M. Then $M_0^*\subset M_1^*\subset\cdots\subset N^*$ is an ascending chain of submodules of N^* which terminates. Thus we have $M_n^*=M_{n+1}^*=\cdots$ for some integer n, and then $M_n=M_{n+1}=\cdots$ by the reflexivity of M_i. The converse is obtained by the similar way to [1, Theorem 1. 3].

(1. 2) LEMMA. *Let R be an HNP ring, M, N finitely generated projective R-modules and $I=P_0\cap\cdots\cap P_{t-1}$ a maximal invertible ideal. Then $M/MI\cong N/NI$ iff $M/MP_i\cong N/NP_i$ for every $i\,(0\le i\le t-1)$.*

PROOF. This follows directly from $M/MI\cong M/MP_0\oplus\cdots\oplus M/MP_{t-1}$.

By (1. 2) if idempotent maximal ideals $P_0, \cdots, P_{t-1}$ form a cycle, then we study the genus at a maximal invertible ideal $I=P_0\cap\cdots\cap P_{t-1}$ instead of all P_i's. We shall state the lemma which plays a critical role in this paper.

(1. 3) LEMMA. *Let R, M, and N be the same as* (1. 2), *$M/N=S$ simple, and I a maximal invertible ideal of R.*

1) *If* $I=P$ *is a maximal ideal, then* $M/MP\cong N/NP$.

2) *Assume that* $I=P_0\cap\cdots\cap P_{t-1}$ *is an intersection of a cycle.*

i) *If* $SI\neq 0$, *then* $M/MI\cong N/NI$.

ii) *If* $SI=0$, *that is,* $SP_i=0$ *for some* $i\,(0\leq i\leq t-1)$, *then* $M/MI\oplus S_{i-1}\cong N/NI\oplus S_i$, where S_j is a simple R-module with $S_jP_j=0\,(0\leq j\leq t-1)$ *and* $S_{-1}=S_{t-1}$.

PROOF. 1) Let $SP=0$, that is, $MP\subset N$. Then the exact sequences

$$0\to N/MP\to M/MP\to M/N\to 0\,,\qquad 0\to MP/NP\to N/NP\to N/MP\to 0$$

are those of R/P-modules. Thus they split and we have;

$$M/MP\cong N/MP\oplus M/N\,,\qquad N/NP\cong N/MP\oplus MP/NP\,.$$

On the other hand, it holds by [3, Theorem 33] that;

(1. 4) rank $M=$rank $N\Rightarrow l(M/MI')=l(N/NI')$ for every maximal invertible ideal I'.

Therefore, $l(MP/NP)=l(M/N)=1$, that is, MP/NP is a simple R/P-module. Hence $MP/NP\cong M/N$, and then $M/MP\cong N/NP$.

Let $SP\neq 0$, that is, $MP+N=M$. Then an exact sequence

$$0\to(N\cap MP)/NP\to N/NP\to M/MP\to 0$$

is one of R/P-modules. Thus it splits and we have;

$$N/NP\cong M/MP\oplus(N\cap MP)/NP\,.$$

Again, by (1. 4) $l(N/NP)=l(M/MP)$, and then $(N\cap MP)/NP=0$, that is, $N/NP\cong M/MP$.

2) i) This is proved by the same way as the case $SP\neq 0$ of 1). ii) Since $MI\subset N$, we have;

$$M/MI\cong N/MI\oplus M/N\,,\qquad N/NI\cong N/MI\oplus MI/NI$$

by the similar way to 1). Again, by (1. 4) MI/NI is simple. Since $P_i=IP_{i-1}I^{-1}$ by [3, Theorem 14], $MIP_{i-1}I^{-1}\subset N$, that is, $MIP_{i-1}\subset NI$. Thus $MI/NI\cong S_{i-1}$ and $M/MI\oplus S_{i-1}\cong N/NI\oplus S_i$.

DEFINITION. A ring R is called a *semilocal HNP ring* if R satisfies the following;

1) R is an HNP ring.

2) The Jacobson radical of R is $I=P_0\cap\cdots\cap P_{t-1}$, where $\{P_0,\cdots,P_{t-1}\}$ is a trivial or nontrivial cycle of idempotent maximal ideals and they are the only maximal ideals of R.

3) I is a maximal invertible ideal of R and all invertible ideals of R are powers of I.

Remark. If R is a semilocal HNP ring, then R is bounded by [2, Theorem 4. 13]. A localization R_I of an HNP ring R at a maximal invertible ideal I is a semilocal HNP ring by [5, Theorem 3. 6].

Let R be an HNP ring and $I=P_0\cap\cdots\cap P_{t-1}$ a maximal invertible ideal of R. Put $\mathfrak{S}_I=\{S_0, \cdots, S_{t-1}\}$ be a set of the representatives of the noniso-morphic simple R/I-modules, that is, S_i's are the simple R/I-modules with $S_iP_i=0$. We shall regard an index j of P_j or S_j as an element of $\boldsymbol{Z}/t\boldsymbol{Z}$, where $\boldsymbol{Z}$ is the ring of rational integers. Thus $j\equiv j'$ mod. t means $S_j=S_{j'}$ and $P_j=P_{j'}$. When we fix a maximal invertible ideal I, we sometimes write $\mathfrak{S}$ instead of $\mathfrak{S}_I$.

(1. 5) Theorem. *Let R be a semilocal HNP ring with its radical $I=P_0\cap\cdots\cap P_{t-1}$, N, M finitely generated projective R-modules such that $N\subset M$ and* rank $N=$rank M. *Then $M\vee N$ iff every simple module of $\mathfrak{S}_I$ appears in the composition factors of M/N for the same times, that is, for some integer $k\geq 0$, the set of the composition factors of M/N coincides with $\mathfrak{S}\cup\cdots\cup\mathfrak{S}$ (k-times).*

Proof. By (1. 1) we let $l(M/N)=s$ and $N=N_0\subset N_1\subset\cdots\subset N_s=M$ a composition series of M/N. We put $N_i/N_{i-1}=S_{\lambda_i}$, $S_{\lambda_i}\in\mathfrak{S}$ $(1\leq i\leq s)$ by hypothesis. Since $S_{\lambda_i}P_{\lambda_i}=0$, it holds that $N_i/N_iI\oplus S_{\lambda_i-1}\cong N_{i-1}/N_{i-1}I\oplus S_{\lambda_i}$ $(i=1, \cdots, s)$ by (1. 3). Therefore, we have

(1. 6) $M/MI\oplus S_{\lambda_1-1}\oplus\cdots\oplus S_{\lambda_s-1}\cong N/NI\oplus S_{\lambda_1}\oplus\cdots\oplus S_{\lambda_s}$.

If $M\vee N$, then by (1. 6), we have

(1. 7) $S_{\lambda_1-1}\oplus\cdots\oplus S_{\lambda_s-1}\cong S_{\lambda_1}\oplus\cdots\oplus S_{\lambda_s}$.

Consider the two sets $A=\{\lambda_1-1, \cdots, \lambda_s-1\}$ and $B=\{\lambda_1, \cdots, \lambda_s\}$ of indices of S_μ's in (1. 7). Let n_i be the number of times of $i\in\boldsymbol{Z}/t\boldsymbol{Z}$ in A. Then $i+1$ appears n_i times in B. Thus $n_0=n_1=\cdots=n_{t-1}=k$, since $A=B$ by (1. 7). Hence the set of composition factors of M/N equals $\mathfrak{S}\cup\cdots\cup\mathfrak{S}$ (k-times). The converse follows from (1. 6) at once.

We have now learned that, under the situation of (1. 5), the set of composition factors of M/N is $\mathfrak{S}\cup\cdots\cup\mathfrak{S}$ provided $M\vee N$. We shall further study this situation and show that M/N has a 'good' composition series. To do so, we prepare definitions and lemmas.

Definition. 1) Let $\alpha\in\boldsymbol{Z}/t\boldsymbol{Z}$ and β a positive integer. A *normal chain* Γ_α^β of mod. t is an ordered set $\{\alpha, \alpha+1, \cdots, \alpha+\beta-1\}$ where $\alpha+j\in\boldsymbol{Z}/t\boldsymbol{Z}$ $(0\leq j\leq\beta-1)$. We write $\Gamma_\alpha^\beta=\langle\alpha, \alpha+1, \cdots, \alpha+\beta-1\rangle$ in order to distinguish from a mere set $\{\alpha, \alpha+1, \cdots, \alpha+\beta-1\}$. β is called the length of a normal chain Γ_α^β.

2) For the case $\Gamma_\alpha^\beta=\langle\alpha, \cdots, \gamma, \cdots, \alpha+\beta-1\rangle$, we say that γ *appears* in Γ_α^β.

3) We call $\Gamma_\alpha^{\beta+\delta}$ a *composition* of Γ_α^β and Γ_γ^δ provided $\alpha+\beta=\gamma$.

4) We call $\Gamma_\alpha^{\beta-t}$ a *contraction* of Γ_α^β provided $\beta>t$.

5) We let $[\Gamma_{\alpha_1}^{\beta_1}, \Gamma_{\alpha_2}^{\beta_2}, \cdots]=[\alpha_1, \cdots, \alpha_1+\beta_1-1, \alpha_2, \cdots, \alpha_2+\beta_2-1, \cdots]$ for normal chains $\Gamma_{\alpha_i}^{\beta_i}$ of mod. t and call it a *chain* of mod. t. Note that $[\alpha_1, \alpha_2, \cdots]=[\Gamma_{\alpha_1}^1, \Gamma_{\alpha_2}^1, \cdots]$ for $\alpha_i \in \boldsymbol{Z}/t\boldsymbol{Z}$.

When there is no confusion, we simply call normal chain or chain.

Let $\mathscr{P}$ be a nonempty set of normal chains of mod. t. Consider the following condition;

(1.8) let $c(\mathscr{P})=\cup_{\Gamma_\alpha^\beta \in \mathscr{P}}\{\alpha, \alpha+1, \cdots, \alpha+\beta-1\}$, then $c(\mathscr{P})=\boldsymbol{Z}/t\boldsymbol{Z} \cup \cdots \cup \boldsymbol{Z}/t\boldsymbol{Z}$ (k-times) for some positive integer k.

(1.9) Proposition. *Let $\mathscr{P}$ be a nonempty set of normal chains of mod. t which satisfies* (1.8). *Then we have a set $\mathscr{P}_0$, consisting of the compositions of normal chains in $\mathscr{P}$, all of whose normal chains have the length divided by t.*

Proof. We shall prove by induction on k in (1.8). For $k=1$, the number of elements of $c(\mathscr{P})$ is t. If $\Gamma_\alpha^t \in \mathscr{P}$, then $\mathscr{P}=\{\Gamma_\alpha^t\}$ which satisfies the assertion. Thus we assume that every normal chain of $\mathscr{P}$ has the length $\beta<t$. Put $\Gamma_\alpha^\beta \in \mathscr{P}$ with $\beta<t$. If $\Gamma_{\alpha+\beta}^\gamma \notin \mathscr{P}$ for all γ, then there is $\Gamma_\varepsilon^\eta \in \mathscr{P}$ such that $\varepsilon \neq \alpha+\beta$ and $\alpha+\beta$ appears in Γ_ε^η. Since $\alpha+\beta-1$ also appears in Γ_ε^η, $c(\mathscr{P})$ contains $\alpha+\beta-1$ twice which is a contradiction. Thus we have $\Gamma_{\alpha+\beta}^\gamma \in \mathscr{P}$, and get $\Gamma_\alpha^{\beta+\gamma}$ from Γ_α^β and $\Gamma_{\alpha+\beta}^\gamma$. By iteration we reach Γ_α^t at last.

Let $k>1$ and assume the proposition for $k-1$. If $\Gamma_\alpha^t \in \mathscr{P}$, then we have the assertion for $\mathscr{P}'=\mathscr{P}-\{\Gamma_\alpha^t\}$. Thus the proposition holds. If $\Gamma_\alpha^\beta \in \mathscr{P}$ with $\beta>t$, then we have the assertion for $\mathscr{P}'=(\mathscr{P}-\{\Gamma_\alpha^\beta\}) \cup \{\Gamma_\alpha^{\beta-t}\}$. Let Γ_γ^δ be a normal chain with $t|\delta$ which is composed of the subset A of $\mathscr{P}'$ including $\Gamma_\alpha^{\beta-t}$. Then we can get the composition $\Gamma_\gamma^{\delta+t}$ from the set $(A-\{\Gamma_\alpha^{\beta-t}\}) \cup \{\Gamma_\alpha^\beta\}$ such that $t|(\delta+t)$. Thus the proposition holds.

Finally, we assume that every normal chain Γ_α^β of $\mathscr{P}$ satisfies $\beta<t$. By the similar way to the case of $k=1$, we can find a positive integer β_2 and obtain the composition $\Gamma_\alpha^{\beta_1+\beta_2}$ of $\Gamma_\alpha^{\beta_1}$ and $\Gamma_{\alpha+\beta_1}^{\beta_2} \in \mathscr{P}$. By iteration we reach the composition $\Gamma_\alpha^{\beta_1+\cdots+\beta_m}$ of normal chains in $\mathscr{P}$ such that $\beta_1+\cdots+\beta_m \geq t$. Let m be the smallest such integer and $\mathscr{P}'=\mathscr{P}-\{\Gamma_\alpha^{\beta_1}, \Gamma_{\alpha+\beta_1}^{\beta_2}, \cdots, \Gamma_{\alpha+\beta_1+\cdots+\beta_{m-1}}^{\beta_m}\}$ $(\mathscr{P}'=(\mathscr{P}-\{\Gamma_\alpha^{\beta_1}, \Gamma_{\alpha+\beta_1}^{\beta_2}, \cdots, \Gamma_{\alpha+\beta_1+\cdots+\beta_{m-1}}^{\beta_m}\}) \cup \{\Gamma_\alpha^{\beta_1+\cdots+\beta_m-t}\})$ for the case of $\beta_1+\cdots+\beta_m=t$ $(\beta_1+\cdots+\beta_m>t)$. Then by the similar way to the preceding paragraph we can show the proposition in this case. This completes the proof.

Now, let $I=P_0 \cap \cdots \cap P_{t-1}$ be a maximal invertible ideal of an HNP ring R and M, N finitely generated projective R-modules such that $N \subset M$ and $l(M/N)=kt$. We let

(1.10) $$N=N_0\subset N_1\subset\cdots\subset N_{kt}=M$$
be a composition series of M/N with $N_i/N_{i-1}\cong S_{\lambda_i}\in\mathfrak{S}_I$ $(i=1,\cdots,kt)$. Consider the chain $[\lambda_1,\cdots,\lambda_{kt}]$ of indices of S_{λ_i}. Then we say that the composition series (1.10) is good with respect to I, if the following holds;

(1.11) $[\lambda_{it+1},\cdots,\lambda_{(i+1)t}]=[\Gamma^t_{\alpha_i}]$ for some $\alpha_i\in\boldsymbol{Z}/t\boldsymbol{Z}$ $(0\leq i\leq k-1)$.
If (1.11) holds, then (1.10) has the property that $N_{it+j}/N_{it+j-1}\cong S_{\alpha_i+j-1}$ where $S_{\alpha_i+j-1}\in\mathfrak{S}$ $(0\leq i\leq k-1;\ 1\leq j\leq t)$. The importance of the existence of such a series will become clear in (1.14).

Next, consider the nonzero uniserial modules C, C' of finite length such that every composition factor of the composition series of them is in $\mathfrak{S}_I$. If $CI^i\neq 0$, then CI^i/CI^{i+1} is simple, for R/I is semisimple and C is uniserial. Thus $C\supset CI\supset\cdots\supset CI^\beta=0$ is a composition series of C for some β. Let $CI^{\beta-1}\cong S_\alpha\in\mathfrak{S}$. Since $CI^{\beta-1}P_\alpha=0$ and $P_{\alpha+i}=I^iP_\alpha I^{-i}$ by [3, Theorem 14], we have $CI^{\beta-i-1}P_{\alpha+i}=CI^{\beta-1}P_\alpha I^{-i}=0$, that is, $CI^{\beta-i-1}/CI^{\beta-i}\cong S_{\alpha+i}$, $S_{\alpha+i}\in\mathfrak{S}$ $(1\leq i\leq\beta-1)$. Thus we get the ordered set $\{S_\alpha, S_{\alpha+1},\cdots,S_{\alpha+\beta-1}\}$ of composition factors of C whose indices yields the normal chain Γ^β_α of mod. t and we call this normal chain Γ^β_α the normal chain of (the composition series of) C. Let Γ^β_α, $\Gamma^\gamma_{\alpha+\beta}$ be normal chains of C, C', respectively. Then a composition series $0\subset CI^{\beta-1}\subset\cdots\subset C\subset C\oplus C'I^{\gamma-1}\subset\cdots\subset C\oplus C'$ of $C\oplus C'$ corresponds with the composition $\Gamma^{\beta+\gamma}_\alpha$. This composition series has the ordered set $\{S_\alpha, S_{\alpha+1},\cdots,S_{\alpha+\beta-1}, S_{\alpha+\beta},\cdots,S_{\alpha+\beta+\gamma-1}\}$ of composition factors. Summarizing the above results we get the following.

(1.12) LEMMA. *The notation and the assumption are the same as above. A nonzero uniserial module C with $l(C)=\beta$ and $CI^{\beta-1}\cong S_\alpha$, $S_\alpha\in\mathfrak{S}$, gives a normal chain Γ^β_α of C according to the above way. Further, if another nonzero uniserial module C' gives a normal chain $\Gamma^\gamma_{\alpha+\beta}$, then the composition series $0\subset CI^{\beta-1}\subset\cdots\subset C\subset C\oplus C'I^{\gamma-1}\subset\cdots\subset C\oplus C'$ corresponds with $\Gamma^{\beta+\gamma}_\alpha$.*

(1.13) THEOREM. *Let R be a semilocal HNP ring with its radical $I=P_0\cap\cdots\cap P_{t-1}$ and M, N finitely generated projective modules such that $N\subset M$ and $M\vee N$. Then M/N has a composition series* (1.10) *which satisfies* (1.11).

PROOF. Since M/N is an R/I^{kt}-module and R/I^{kt} is a serial ring, M/N is a direct sum of nonzero uniserial modules, say $M/N=C_1\oplus\cdots\oplus C_a$. As in (1.12), every C_j gives a normal chain $\Gamma^{\beta_j}_{\alpha_j}$ and the set $\mathscr{P}=\{\Gamma^{\beta_j}_{\alpha_j};\ 1\leq j\leq a\}$ satisfies the condition (1.8) by (1.5). Thus, by (1.9) and (1.12), we have $M/N\cong D_1\oplus\cdots\oplus D_b(b\leq a)$ such that $l(D_i)=tn_i$ and each D_i has the composition series $0=E_0\subset E_1\subset\cdots\subset E_{tn_i}=D_i$ and $E_j/E_{j-1}\cong S_{\alpha_i+j}$, $S_{\alpha_i+j}\in\mathfrak{S}$ for $\alpha_i\in\boldsymbol{Z}/t\boldsymbol{Z}$ and positive integers n_i. Hence we can construct the composition series of M/N which satisfies (1.11) from those of D_i's.

(1.14) LEMMA. *Let R be a semilocal HNP ring with its radical $I=P_0\cap\cdots\cap P_{t-1}$ and M, N, K finitely generated projective R-modules in the same genus such that $N\subset M$, $l(M/N)=t$, and K/KI contains all $S_\lambda\in\mathfrak{S}_I$. Then there exists a finitely generated projective R-module L in the genus with $K\oplus N\cong M\oplus L$.*

PROOF. By (1.13) there exist $\lambda\in \boldsymbol{Z}/t\boldsymbol{Z}$ and a composition series $N=N_0\subset N_1\subset\cdots\subset N_t=M$ of M/N such that $N_i/N_{i-1}\cong S_{\lambda+i-1}$, $S_{\lambda+i-1}\in\mathfrak{S}$ $(1\leq i\leq t-1)$. Consider the exact sequences $0\to N\to N_1\to S_\lambda\to 0$ and $0\to L_1\to K\to S_\lambda\to 0$. Then we have $N\oplus K\cong N_1\oplus L_1$ by Schanuel's Lemma, and $K/KI\oplus S_{\lambda-1}\cong L_1/L_1I\oplus S_\lambda$ by (1.3). Thus we can get the exact sequence $0\to L_2\to L_1\to S_{\lambda+1}\to 0$ and go up the next step. By iteration we get finitely generated projective modules L_i $(1\leq i\leq t)$ such that $N_i\oplus L_i\cong N_{i+1}\oplus L_{i+1}$, $L_i/L_iI\oplus S_{\lambda+i-1}\cong L_{i+1}/L_{i+1}I\oplus S_{\lambda+i}$ $(1\leq i\leq t-1)$. Therefore, putting $L=L_t$ we have $N\oplus K\cong M\oplus L$ and $K/KI\oplus S_{\lambda-1}\cong L/LI\oplus S_{\lambda+t-1}\cong L/LI\oplus S_{\lambda-1}$, that is, $L\vee K$.

(1.15) THEOREM. *Let R be a semilocal HNP ring with its radical I and M, N, K finitely generated projective modules in the same genus such that K/KI contains all $S_\lambda\in\mathfrak{S}_I$. Then there exists a finitely generated projective module L in the genus such that $N\oplus K\cong M\oplus L$.*

PROOF. This follows from [1, Lemma 1.4], (1.5), (1.13), (1.14).

(1.16) COROLLARY. *If M, N are right R-ideals of Q in the same genus as R, then $M\oplus N\cong L\oplus R$ for a right R-ideal L of Q in the genus.*

2. HNP rings with enough invertible ideals. We shall apply the results of § 1 to the case that R is an HNP ring with enough invertible ideals and study the genus of finitely generated projective right R-modules. However, we let R be an HNP ring with not necessarily enough invertible ideals for a while. Let I be a maximal invertible ideal of R and R_I a localization of R at I [5, § 3]. Then R_I is a bounded HNP ring with enough invertible ideals, moreover, R_I is either a Dedekind prime ring provided I is a maximal ideal or a semilocal HNP ring provided $I=P_0\cap\cdots\cap P_{t-1}$ is an intersection of a cycle [5, Theorem 3.6]. Let $A=\cup\{B^{-1};\ B$ is an invertible ideal of $R\}$ be a quotient ring of R by [5, Proposition 2.3].

(2.1) LEMMA. *Let R be an HNP ring and S a simple right R-module. Then the following hold.*

1) *If $SI=0$ for some maximal invertible ideal I of R, then $S\otimes R_I\cong S$ and $S\otimes R_{I'}=0$ for any maximal invertible ideal $I'\neq I$ of R.*

2) *If S is faithful or $SP=0$ for some idempotent maximal ideal P of R which doesn't belong to a cycle, then $S\otimes R_I=0$ for any maximal invertible ideal I of R.*

PROOF. 1) If I' is a maximal invertible ideal with $I' \neq I$, then $I'+I=R$ which implies $IR_{I'}=R_{I'}$. Thus $S\otimes R_{I'}=SI\otimes R_{I'}=0$. Since $IA=A$, $S\otimes A=SI\otimes A=0$, and then $S\otimes R_I \neq 0$ by [5, Theorem 3. 12]. By $S\otimes R_I I=0$, $S\otimes R_I$ is a semisimple R/I-module which yields $S\otimes R_I \cong S$ by [8, Corollary 1. 5, Proposition 1. 7].

2) If $S\otimes R_I \neq 0$ for a faithful simple module S where I is a maximal invertible ideal, then $S\otimes R_I I=SI\otimes R_I=0$ by [8, Corollary 1. 5]. However, we have $SI=S$, and then $S\otimes R_I=0$ which is a contradiction. Hence $S\otimes R_I=0$. The latter case is proved by the similar way to 1).

In the rest of this section, we always assume that R is an HNP ring with enough invertible ideals. Therefore, every simple R-module S is either faithful or $SI=0$ for a maximal invertible ideal I, moreover, since every maximal ideal of R either is invertible or belongs to a cycle, we have that, for finitely generated projective R-modules M, N, $M\vee N$ iff $\operatorname{rank} M=\operatorname{rank} N$ and $M/MI\cong N/NI$ for all maximal invertible ideals I. We write $M_I=M\otimes R_I$ for a finitely generated projective R-module M.

(2. 2) LEMMA. *Let $I=P_0\cap\cdots\cap P_{t-1}$ be a maximal invertible ideal and M, N finitely generated projective R-modules such that $M\supset N$ and $M_I\vee N_I$. Then every simple module in $\mathfrak{S}_I$ appears k-times in the composition factors of M/N for an integer $k\geq 0$.*

PROOF. This follows directly from (1. 5) and (2. 1).

(2. 3) THEOREM. *Let M, N be finitely generated projective R-modules such that* $\operatorname{rank} M=\operatorname{rank} N$ *and $N\subset M$. Then $M\vee N$ iff $M_I\vee N_I$ for all maximal invertible ideals I.*

PROOF. If $M\vee N$, then $M_I\vee N_I$ by $M_I/M_I I\cong M/MI\otimes R_I$. Conversely, by (1. 3) we have $M/MI\cong N/NI$ for a maximal invertible ideal I which is either maximal or an intersection of a cycle such that I doesn't annihilate any composition factor of M/N. Let $I_1, \cdots, I_m$ be all maximal invertible ideals which are intersections of a cycle such that every I_j annihilates some composition factors of M/N. Then by (2. 2) the composition factors of M/N includes each simple module of $\mathfrak{S}_{I_j}$ by k_j-times for a positive integer $k_j (1\leq j\leq m)$. Therefore, by (1. 3) the similar method to (1. 6) yields $M/MI_j\oplus X_j^{(k_j)}\cong N/NI_j\oplus X_j^{(k_j)}$ where $X_j=\oplus_{S_\lambda\in\mathfrak{S}_{I_j}} S_\lambda (1\leq j\leq m)$. Hence $M/MI_j\cong N/NI_j$ $(1\leq j\leq m)$. This completes the proof.

(2. 4) LEMMA. *Let X, Y, Z be finitely generated projective R-modules such that $X\subset Y\subset Z$ and $Y/X\cong S$, $Z/Y\cong T$, where S and T are simple modules. If one of the following holds, then there exists a module Y' such that $X\subset Y'\subset Z$ and $Y'/X\cong T$, $Z/Y'\cong S$.*

1) *T is faithful and S is unfaithful.*

2) *T is annihilated by an invertible maximal ideal and $S \not\cong T$.*

3) *T is annihilated by an idempotent maximal ideal P such that $P = P_0, \cdots, P_{t-1}$ form a cycle and S is annihilated by a maximal ideal $P' \neq P_{t-1}$.*

PROOF. Consider an exact sequence $0 \to S \to Z/X \to T \to 0$. If $\mathrm{ext}_R^1(T, S) = 0$, then this sequence splits and we have a desired module Y'. However, it holds by [3, Propositions 2, 4 and Corollary 9] that $\mathrm{ext}_R^1(T, S) = 0$ under the cases 1), 2), and 3).

(2. 5) LEMMA. *Let C be a nonzero uniserial R-module of finite length. Then all composition factors of C are either faithful simple modules, or annihilated by an invertible maximal ideal, or elements of $\mathfrak{S}_I$ for a maximal invertible ideal I which is an intersection of a cycle. Further, in the third case, there exists a normal chain of C.*

PROOF. Let $0 = C_0 \subset C_1 \subset \cdots \subset C_n = C$ be the composition series of C. If there exists C_i' for some i $(1 \leq i \leq n-1)$ such that $C_{i-1} \subset C_i' \subset C_{i+1}$ and $C_i'/C_{i-1} \cong C_{i+1}/C_i$, $C_{i+1}/C_i' \cong C_i/C_{i-1}$, then $C_i = C_i'$ by assumption. Thus we have $C_{i+1}/C_i \cong C_i/C_{i-1}$, whence (2. 5) follows from (2. 4) and (1. 12).

(2. 6) THEOREM. *Let M, N be finitely generated projective R-modules with $N \subset M$. Then $M \vee N$ iff there exists a composition series $N = N_0 \subset N_1 \subset \cdots \subset N_n = M$ of M/N which satisfies the following; there exist integers $k, k_0, \cdots, k_m = n$ with $0 \leq k \leq k_0 \leq \cdots \leq k_m$* and maximal invertible ideals $I_1, \cdots, I_m$, *where each I_j is an intersection of t_j idempotent maximal ideals, such that;*

1) *all the composition factors of N_k/N_0 are faithful simple modules,*

2) *all the composition factors of N_{k_0}/N_k are annihilated by invertible maximal ideals,*

3) *all the composition factors of $N_{k_j}/N_{k_{j-1}}$ are annihilated by I_j, moreover, $l(N_{k_j}/N_{k_{j-1}}) = t_j s_j$ for some positive integer s_j and the composition series $N_{k_{j-1}} \subset N_{k_{j-1}+1} \subset \cdots \subset N_{k_j}$* of $N_{k_j}/N_{k_{j-1}}$ *satisfies* (1. 11) *with respect to I_j $(1 \leq j \leq m)$.*

PROOF. By the iterative use of (2. 4) 1) we can find an integer k such that 1) holds and all the composition factors of N_n/N_k are unfaithful. Let $J_1, \cdots, J_s, I_1, \cdots, I_m$ are all maximal invertible ideals which annihilate some composition factor of N_n/N_k such that J_i's are maximal ideals and I_i's are intersections of a cycle. Put $I = J_1 \cap \cdots \cap J_s \cap I_1 \cap \cdots \cap I_m$. Then I is an invertible ideal by [2, Propositions 2. 5, 2. 8] and N_n/N_k is annihilated by some power I^b of I. Since R/I^b is a serial ring, N_n/N_k is a direct sum of nonzero uniserial modules, say $N_n/N_k = C_1 \oplus \cdots \oplus C_a$. Thus we get the integers $k_0, \cdots, k_m = n$ which satisfy the theorem by (2. 2), (2. 5), and the proof of (1. 13). The converse is obtained from (1. 3), (1. 5), and (2. 3).

Applying the above theorem we obtain the generalization of (1. 15).

(2. 7) THEOREM. *Let M, N, K be finitely generated projective R-modules in the same genus such that K/KI contains each $S\in\mathfrak{S}_I$ for every maximal invertible ideal I which is an intersection of a cycle. Then there exists a finitely generated projective R-module L in the genus such that $N\oplus K\cong M\oplus L$.*

PROOF. By (1. 3) 1), (1. 14), and (2. 6) we only need to prove the theorem when M/N is a faithful simple module. Let $S\cong M/N$ be a faithful simple module and $S\cong R/X$ for a maximal right ideal X of R. Then it is well-known that there exists an epimorphism $K\to S$ (cf. [3, § 4]). However, we state the proof for the completeness. Let T be a trace ideal of K. Then T is a nonzero ideal of R and is not contained in X, since R/X is faithful. Thus there exist $f\in K^*$ and $w\in K$ with $f(w)\notin X$ which yields an epimorphism $K\to S$ at once. Let $L=\mathrm{Ker}(K\to S)$. Then we have $N\oplus K\cong M\oplus L$ by Schanuel's Lemma.

(2. 8) COROLLARY. *Let M, N be right R-ideals of Q which are in the same genus as R. Then there exists a right R-ideal L of Q in the genus such that $M\oplus N\cong R\oplus L$.*

(2. 9) EXAMPLE. We investigate an example of an HNP ring with enough invertible ideals. The example is one in [2, § 5]. Let D be a non-commutative Dedekind domain which is a primitive principal ideal domain with a unique maximal ideal $xD=Dx$ such that D/xD is a field. We shall study the HNP ring

$$R=\begin{pmatrix} D & D \\ xD & D \end{pmatrix}.$$

Let $P_0=\begin{pmatrix} xD & D \\ xD & D \end{pmatrix}$, $P_1=\begin{pmatrix} D & D \\ xD & xD \end{pmatrix}$, and $I=P_0\cap P_1=\begin{pmatrix} xD & D \\ xD & xD \end{pmatrix}$. Then P_0 and P_1 are idempotent maximal ideals which are the only maximal ideals of R and form a cycle by [2, § 5], whence I is an invertible maximal ideal and R has enough invertible ideals by [2, Corollary 4. 7]. Consider the following;

$$R\supset P_1\supset I\supset P_1P_0=P_1I\supset I^2\supset P_1IP_1=P_1P_0I.$$

Let $S_0\cong R/P_0$, $S_1\cong R/P_1$ be the unfaithful simple modules. Then by a routine computation we have $P_1/I\cong S_0$, $I/P_1P_0\cong S_0$, $P_1P_0/I^2\cong S_1$, $I^2/P_1IP_1\cong S_1$. Thus $R\vee I\vee I^2$, $P_1\vee P_1IP_1$ by (2. 3), in fact, $R/I\cong I/I^2\cong I^2/I^3\cong S_0\oplus S_1$ and $P_1/P_1I\cong P_1IP_1/P_1IP_1I\cong S_0\oplus S_0$. Since $P_1\vee P_1IP_1$, we must have $P_1\supset N_1\supset N_2\supset N_3\supset P_1IP_1$ such that $N_3/P_1IP_1\cong S_\lambda$, $N_2/N_3\cong S_{\lambda+1}$, $N_1/N_2\cong S_\lambda$, $P_1/N_1\cong S_{\lambda+1}$,

$\lambda \in \boldsymbol{Z}/2\boldsymbol{Z}$, by (2. 6). Indeed, the chain $P_1 \supset I \supset P_0P_1 \supset I^2 \supset P_1IP_1$ satisfies $I/P_0P_1 \cong S_1$, $P_0P_1/I^2 \cong S_0$.

3. In this section, we make a brief attempt to define an ideal class group for some HNP ring R. Let R be an HNP ring with enough invertible ideals and $\mathrm{Cl}(R)$ the set of stable isomorphism classes of right R-ideals of Q which are in the same genus as R, where, as usual, right R-modules M and N are stably isomorphic if there exists a non-negative integer r such that $M \oplus R^{(r)} \cong N \oplus R^{(r)}$. We denote a stable isomorphism class of a right R-ideal M in $\mathrm{Cl}(R)$ by $[M]$. Define an additive structure on $\mathrm{Cl}(R)$ by $[M]+[N]=[K]$ for $[M]$, $[N]$, $[K] \in \mathrm{Cl}(R)$, provided $M \oplus N \cong K \oplus R$ (cf. (2. 8)). Keeping the above notation and hypotheses, we have

(3. 1) Theorem. *If R has a unique maximal invertible ideal I which is an intersection of a cycle and* $\mathrm{rank}\, R = l(R/I)$, *then* $\mathrm{Cl}(R)$ *is an abelian group with* $[R]$ *its identity.*

Proof. Let $[M]=[M']$, $[N]=[N']$ in $\mathrm{Cl}(R)$ and $M \oplus N \cong R \oplus K$, $M' \oplus N' \cong R \oplus K'$ by (2. 8). Then we have $M \oplus R^{(s)} \cong M \oplus R^{(s)}$ and $N \oplus R^{(s)} \cong N' \oplus R^{(s)}$ for some s. Thus $K \oplus R^{(2s+1)} \cong M \oplus N \oplus R^{(2s)} \cong M \oplus N' \oplus R^{(2s)} \cong K' \oplus R^{(2s+1)}$, whence $[K]=[K']$. Since rank $R=l(R/I)$, we have rank $M=l(M/MI)$ for each finitely generated projective module M by [3, Theorem 33]. Let M be a right R-ideal with $M \vee R$ and $M \oplus N \cong R^{(s)}$ for a right R-module N and a positive integer s. Then $M/MI \oplus N/NI \cong (R/I)^{(s)}$ implies $N/NI \cong (R/I)^{(s-1)}$. Assume $R/I \cong S_1^{(n_1)} \oplus \cdots \oplus S_k^{(n_k)}$, where S_i's are simple R/I-modules. Then $N/NI \cong S_1^{(n_1(s-1))} \oplus \cdots \oplus S_k^{(n_k(s-1))}$ concludes rank $N=(s-1)(n_1+\cdots+n_k)=m$. We let $N \cong K_1 \oplus \cdots \oplus K_m$ where each K_i is uniform. Then $K_1/K_1I \oplus \cdots \oplus K_m/K_mI \cong S_1^{(n_1(s-1))} \oplus \cdots \oplus S_k^{(n_k(s-1))}$ and each K_i/K_iI is simple by hypothesis, whence $N \cong L_1 \oplus \cdots \oplus L_{s-1}$ with $L_i \vee R$ by (1. 3) 1) $(1 \leq i \leq s-1)$ which yields $N \cong R^{(s-2)} \oplus L$ for a right R-ideal L of Q with $L \vee R$ by (2. 8). Again, by (2. 8) we have $M \oplus L \cong R \oplus K$ for a right R-ideal K in the genus and $K \oplus R^{(s-1)} \cong M \oplus L \oplus R^{(s-2)} \cong M \oplus N \cong R^{(s)}$, whence $[K]=[R]$ and $[M]+[L]=[R]$ in $\mathrm{Cl}(R)$. Therefore, $[L]$ is the inverse of $[M]$ in $\mathrm{Cl}(R)$. This completes the proof.

Remark. By [2, § 5] the ring R in (2. 9) satisfies the assumption of (3. 1). It also holds that the completion $\hat{R}$ of an HNP ring R with respect to a maximal invertible ideal satisfies this assumption by [4, Theorem 2. 3 or 6, Theorem 1. 1].

We note that a Dedekind prime ring doesn't necessarily satisfy the assumption of (3. 1) by [4, Note 3. 9]. However, this assumption is used only to prove the existence of the inverse, so that we can prove the following by virtue of [1, Theorems 2. 2, 2. 4].

(3.2) THEOREM. *Let R be a Dedekind prime ring and* $\mathrm{Cl}(R)$ *the set of stable isomorphism classes of right R-ideals of Q (cf.* (1.3) 1)). *Then* $\mathrm{Cl}(R)$ *is an abelian group with* $[R]$ *its identity.*

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Kitami Institute of Technology