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On the radical of the center of a group algebra

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1. Let kG denote the group algebra of G over a field k of characteristic $p>0$ and $Z=Z(kG)$, the center of kG. In this short note we shall prove the following ;

THEOREM. *Let e be a block idempotent of kG with defect d. If $J(Z)$ denotes the Jacobson radical of Z, then the following hold* ;

(1) $J(Z)^{p^{d}}e=0$.

(2) *If k is algebraically closed, then $J(Z)^{p^{d}-1}e\neq 0$ if and only if the block of kG corresponding to e is p-nilpotent with a cyclic defect group.*

As a corollary of the theorem we have the following which extends the result of Passman (Theorem, [8]) ;

COROLLARY. *Let $|G|=p^{a}m$ with $(p, m)=1$. Then $J(Z)^{p^{a}}=0$.*

2. Proof of the theorem.

To prove the theorem we may assume that k is algebraically closed (see Corollary 12. 12, [6] and (9. 10) Chapter III, [4]). We shall prove the statement (1) by induction on d. If $d=0$, then $J(Z)e=0$ and the result follows easily. Assume $d>0$ and we shall show ;

(a). *Let x be a p-element of G of order p^{b}, $b>0$ which is contained in a defect group of e and σ the Brauer homomorphism from Z to $Z(kC_{G}(x))$. Then*

$$\sigma\left(J(Z)^{p^{d}-1}e\right)\subseteq\alpha kC_{G}(x)$$

where $\alpha=\sum_{i=0}^{p^{b}-1}x^{i}$.

PROOF of (a). Let f be a block idempotent of $kC_{G}(x)$ with $f\sigma(e)=f$. Then the defect of f is at most d (see § 9, Chapter III, [4]). Consider the homomorphism τ from $kC_{G}(x)$ onto $kC_{G}(x)/\langle x\rangle$ induced by the natural homomorphism of $C_{G}(x)$ to $C_{G}(x)/\langle x\rangle$. The kernel of τ is $(x-1)kC_{G}(x)$ and $\tau(f)$ is a block idempotent of $kC_{G}(x)/\langle x\rangle$ with defect at most $d-b$ (see § 4, Chapter V, [4]). Thus by induction it follows that $J(Z(kC_{G}(x)))^{p^{d-b}}f\subseteq(x-1)kC_{G}(x)$. Since $p^{d}-1\geq p^{d-b}(p^{b}-1)$, $J(Z(C_{G}(x)))^{p^{d}-1}f\subseteq((x-1)kC_{G}(x)^{p^{b}-1}$

$=(x-1)^{p^b-1}kC_G(x)=\alpha kC_G(x)$. Therefore the assertion (a) follows since $\sigma(J(Z)) \subseteq J(Z(kC_G(x)))$.

Next we shall prove;

(b) *If $Z_{p'}$ denotes the k-subspace of Z spanned by all p'-section sums in kG ($Z_{p'}$ is an ideal of Z (see* [9]*)), then*

$$J(Z)^{p^d-1}e \subseteq Z_{p'}.$$

PROOF of (b). Let y be a p'-element of g and x a p-element of G of order p^b such that $xy=yx$. It will suffice to show that for any β in $J(Z)^{p^d-1}e$ the coefficients of y and xy in β are equal. If x is not contained in any defect group of e, then the coefficients of y and xy in β are 0 by definition of defect groups of blocks. Thus we may assume that x is contained in a defect group of e. Let σ be the Brauer homomorphism from Z to $Z(kC_G(x))$. Then by (a) $\sigma(\beta)\in\alpha kC_G(x)$ where $\alpha=\sum_{i=0}^{p^b-1} x^i$. Thus the coefficients of y and xy in $\sigma(\beta)$ are equal and therefore the result follows.

From the result of Brauer [1] $J(Z)\,Z_{p'}=0$ (see also Theorem (1. C), [7]). Hence the statement (1) of the theorem follows.

If the blcok of kG corresponding to e is p-nilpotent with a cyclic defect group, the $J(Z)\,e \simeq J(kP)$ where P is a defect group of e from the result of Broué and Puig [2]. Thus $J(Z)^{p^d-1}e\neq 0$.

Conversely assume that $J(Z)^{p^d-1}e\neq 0$. We claim that a defect group of e is cyclic. If a defect group of e is not cyclic, then any element in the defect group of e is of order p^b with $b<d$. Since $p^d-2\geq p^{d-b}(p^b-1)$ for such b, the proofs of the statements (a) and (b) in the above show that $J(Z)^{p^d-2}e\subseteq Z_{p'}$ and therefore $J(Z)^{p^d-1}e=0$ which is a contradiction. Thus a defect group of e is cyclic and our claim follows. By the result of Dade [3] $\dim_k Ze\leq p^d$. From this we can conclude that $\dim_k Ze=p^d$ and $dim_k\, Z_{p'}e=1$. Since $dim_k\, Z_{p'}e$ is the number of irreducible kG-modules in the block B of kG corresponding to e (see [5]), it follows that B has the inertial index 1 and its is p-nilpotent. Thus the statement (2) is proved and the proof of the theorem is complete.

REMARK. If $J(Z)^{p^a-1}\neq 0$ in Corollary, then the proof of the statement (2) of the theorem shows that G has a cyclic Sylow p-subgroup. But in general it is not true that G is p-nilpotent. For example let H be a cyclic group of odd order $p^a n$, $n\neq 1$ and $(p, n)=1$, $t\in \mathrm{Aut}\,H$ of order 2 such that $h^t=h^{-1}$ for any $h\in H$ and $G=\langle t\rangle H$. Then for any non-principal block idempotent e of kG it holds that $J(Z)^{p^a-1}e\neq 0$.

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