Hokkaido Mathematical Journal Vol. 10 (1981) p. 343-405

On standard involutions of homotopy spheres

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(Received July 18, 1980)

INTRODUCTION

This paper studies fixed point free smooth involutions on homotopy $(2n-1)$-spheres. It has been proved in [10], [24] that every element of bP_{2n} $(n\geqq 3)$ admits free involutions, where bP_{2n} is the group of homotopy $(2n-1)$-spheres which bound parallelizable manifolds. When such actions exist, it is natural to ask how they behave.

We will make an approach to this problem in the study of the following involutions. *Let T be a free involution on a homotopy sphere $\sum^{2n-1}\in bP_{2n}$ $(n\geqq 3)$. If there exists a parallelizable manifold M with boundary Σ such that T extends to an involution with isolated fixed points on M, then we call (T, Σ) a standard involution.*

We shall establish an explicit description of standard involutions on homotopy $(4k-1)$-spheres and then prove that they give a classification of standard involutions.

Contents

Certain examples of satndard involutions are constructed by an equivariant plumbing technique in chapter III. F. Hirzebruch and K. H. Mayer [13], [21] gave examples of free involutions of homotopy 7-spheres using the equivariant plumbing. However, the plumbing we need here is different from it and is based upon the plumbing which is originally motivated by S. Weintraub [34]. In general, if we are given a free involution T on a homotopy sphere Σ and even though T extends to an involution on M which Σ bounds, it may be considered that T does not extend "uniquely". So to construction we need to construct Z_2-actions on M in a "uniform way". Then the invariants for the actions on M —for examples, Atiyah-Singer invariants, Spin invariants, Eells-Kuiper μ-invariants— give informations to the classification of (T, Σ). In particular, we have the following table of standard involutions.

Table 1.

Type	(A, S^{4k-1})	$(T_{2l-1}^{-}, \Sigma_{(2l-1,-)}^{4k-1})$	$(T_{2l-1}^{+}, \Sigma_{(2l-1,+)}^{4k-1})$	$(T_{2l}^{-}, \Sigma_{(2l,-)}^{4k-1})$	$(T_{2l}^{+}, \Sigma_{(2l,+)}^{4k-1})$
Browder-Livesay invariant	0	$2l-1$	$2l-1$	$2l$	$2l$
Normal cobordism class	(P, id) $(P=P^{4k-1})$	$(8(2l-1)-1)(P, id)$	$(8(2l-1)+1)(P, id)$	$(8(2l)+1)(P, id)$	$(8(2l)-1)(P, id)$
Spin invariant mod 2^{2k}	± 1	$\pm(8(2l-1)-1)$	$\pm(8(2l-1)+1)$	$\pm(8(2l)+1)$	$\pm(8(2l)-1)$
Matrix rank	—	H_{2l-1}^{-} $8(2l-1)$	H_{2l-1}^{+} $8(2l-1)$	H_{2l}^{-} $8(2l)$	H_{2l}^{+} $8(2l)$
Differentiable structure	S^{4k-1}	$(2l-1)\,\Sigma_1$	$(2l-1)\,\Sigma_1$	$(2l)\,\Sigma_1$	$(2l)\,\Sigma_1$

Chapter IV serves as the classification of standard involutions. For this, if we are given a standard involution, then we require that a bounded manifold with a Z_2-action is highly-connected. The equivariant surgery enables us to do so within the normal cobordism class. In such a situation, we calculate the normal cobordism class of the standard involution and seek the same class out of the table. And then choosing one with the same Browder-Livesay invariant, we have the main result of chapter IV.

THEOREM 4.4.3. *Let T be a standard involution on a homotopy sphere $\Sigma^{4k-1} \in bP_{4k}$ $(k \geqq 2)$. Then, (T, Σ^{4k-1}) is equivariantly diffeomorphic to the equivariant connected sum of the definite element $\Sigma' \in bP_{4k}$ with the unique representative (T_h, Σ_h) in the table, i.e., the quotient*

$$\Sigma^{4k-1}/T \cong \Sigma_h/T_h \# \Sigma' .$$

For the $(4k+1)$-dimensional case, we have

THEOREM 4.9.1. *Let T be a standard involution on a homotopy sphere $\Sigma^{4k+1} \in bP_{4k+2}$ $(k \geqq 1)$. Then, (T, Σ^{4k+1}) is equivarianly diffeomorphic to the equivariant connected sum of some $\Sigma' \in bP_{4k+2}$ with the unique re-*

presentative (T_d, Σ_d^{4k+1}) *in table* (*cf.* § 3. 8), *i. e., the quotient*

$$\Sigma^{4k+1}/T \cong \Sigma_d^{4k+1}/T_d \# \Sigma' .$$

Here T_d *is the Brieskorn involution on the Brieskorn sphere* Σ_d^{4k+1}.

Chapter V studies the models which are constructed in chapter III. The bundles we used to construct the models are stably trivial. However, if we are able to take bundles with Z_2-actions which are not stably trivial, then by the same method we have spin manifolds (almost closed manifolds in his definition of Wall [31]) with Z_2-actions. We then receive free involutions on the boundaries which are elements of *bspin*. If we call such involutions "spin involutions", then some of these are identified with "curious involutions" due to Hirsch-Milnor [12] (See also [20, p. 63]). In particular, we can re-establish the classification on homotopy projective 7-spaces using the standard involutions and spin involutions.

Certain notational conventions will be used throughout. Z denotes the integers and $Z_2 = Z/2Z$ the quotient group. (Co)-homology coefficients are assumed to be Z. By $D^{n+1}(S^n)$ we mean the $(n+1)\,(n)$-dimensional unit disk (sphere) in the $(n+1)$-dimensional euclidean space R^{n+1}. A denotes the antipodal map on $D^{n+1}(S^n)$,

$$A(x_1, x_2, \cdots, x_{n+1}) = (-x_1, -x_2, \cdots, -x_{n+1}) ,$$

and $P^n = S^n/A$ is the n-dimensional standard projective space. By I we mean the unit interval $[0, 1]$ in R^1.

The author would like to thank Professor H. Suzuki and Professor T. Yoshida for their valuable suggestions.

1. Preliminary results and definitions

1. 1. For details on the surgery on manifolds, we refer to Browder [4], López De Medrano [20] and Wall [33].

Let T be a free involution on a homotopy n-sphere Σ^n. Two (T_i, Σ_i^n) $(i=0, 1)$ are equivalent if there is an equivariant diffeomorphism $g: (T_0, \Sigma_0) \to (T_1, \Sigma_1)$. If n is odd, g is required to be orientation preserving. Denote by Π_n the set of equivalence classes of free involutions on homotopy n-spheres. Denote by $\sigma(T, \Sigma)$ the Browder-Livesay invariant which is the obstruction whether (T, Σ) has a codimension 1 invariant sphere, i. e., desuspends or not. It lies in the following groups (see [20])

$$\sigma(T,\Sigma^n)=\begin{cases}0 & n \text{ even}\\ Z & n\equiv -1(4)\\ Z_2 & n\equiv 1(4).\end{cases}$$

THEOREM (Browder-Livesay [7]). *If $n\geqq 6$, (T,Σ^n) desuspends if and only if $\sigma(T,\Sigma^n)=0$.*

To distinguish free involutions, we may put them into the frame work of the surgery theory. Taking quotient spaces and choosing homotopy equivalences (of degree 1 if n is odd), Π_n is in one to one correspondence with $hS(P^n)$, where $hS(P^n)$, is the set of homotopy smoothings of P^n.

THEOREM ([20, p. 47]). *Two free involutions (T_i,Σ_i^n) $(i=0,1)$ are equivalent, modulo the action of bP_{n+1}, if and only if they have the same normal cobordism class and the same Browder-Livesay invariant.*

This theorem follows easily from the surgery exact sequence on p^n,

$$L_{n+1}(Z_2,w)\xrightarrow{\omega}hS(P^n)\xrightarrow{\eta}[P^n,G/0]\xrightarrow{\theta}L_n(Z_2,w).$$

1.2. DEFINITION. Let T be a free involution on a homotopy sphere $\Sigma^n\in bP_{n+1}(n\geqq 5)$. We say that T is *standard* if there exists a parallelizable manifold M^{n+1} which Σ bounds such that T extends to an involution with isolated fixed points on M. If there exist no such paralleliziable manifold, we call T *non-standard.*

Under such a situation, we write $(T,\Sigma)=\partial(T,M)$ (we do not distinguish an extended involution from T). To make clear the notion of non-standard it will be discussed in chapter V.

1.3. NOTATION. Let M^{n+1} be a parallelizable manifold whose boundary is Σ^n. Denote by $\sigma(M)$ the index (resp. Kervaire invariant) of M for $n+1\equiv 0$ (resp. 2) mod 4. Otherwise we set $\sigma(M)=0$.

Note that these invariants are the surgery obstructions of normal maps into the $(n+1)$-disk D^{n+1} which are homotopy equivalences on the boundary.

2. Properties of standard involutions

PROPOSITION 2.1. *Suppose that T is a standard involution on Σ^n so that $(T,\Sigma^n)=\partial(T,M^{n+1})$ and $n\geqq 5$. Then,*

$$\sigma(T,\Sigma)\equiv\sigma(M)\qquad \textit{if } n \textit{ is even and } n\equiv 1\ (4),$$

$$\sigma(T,\Sigma)\equiv 1/8\sigma(M)\ (2)\qquad \textit{if } n\equiv -1\ (4).$$

PROOF. If n is even, it follows from the above definition and theorem that $\sigma(T,\Sigma)=\sigma(M)=0$.

Assume $n\equiv -1\,(4)$. The Atiyah-Singer invariant and the Browder-Livesay invariant agree for free involutions (see [2]). Hence, it follows that $\sigma(T,\Sigma)=1/8\,(\mathrm{Sign}\,(T,M)-L(T,M))$. Since $L(T,M)=L(\mathrm{Fix}\,T\cdot\mathrm{Fix}\,T)=0$ (in this case) and $\mathrm{Sign}\,(T,M)\equiv\sigma(M)$ (2), the result follows.

Assume $n=4k+1$. Let (T,N^{4k}) be a characteristic submanifold of (T,Σ). We quote the result of P. Orlik [25] to look for a specific geometry of (T,N) for $k\geqq 2$. One can perform equivariant surgery on N to yield a submanifold W_c for $c=0,1$. W_c is as follows. Let $\sigma(T,\Sigma)=c$. If $\sigma(\mathrm{T},\Sigma)=0$, then $W_0=S^{4k}$. If $\sigma(T,\Sigma)=1$, then W_1 is the double of plumbing J two copies of the tangent disk bundles of S^{2k}. More precisely, let $\Sigma=\mathrm{A}\cup\mathrm{TA}$, $\mathrm{A}\cap\mathrm{TA}=\mathrm{W}_1$. Then, $0\to H_{2k}(W_1)\to H_{2k}(A)+H_{2k}(TA)\to 0$, and $H_{2k}(W_1)=\mathrm{Ker}\,i_A\oplus\mathrm{Ker}\,i_{TA}$, where $\mathrm{Ker}\,i_A=\mathrm{Ker}\,\{i_{A*}:\ H_{2k}(W_1)\to H_{2k}(A)\}$ and $\mathrm{Ker}\,i_{TA}=T\cdot\mathrm{Ker}\,i_A$. $\mathrm{Ker}\,i_A$ has a sympletic basis $\{e,f\}$ with respect to the skew-symmetric bilinear form B defined by $B(x,y)=x\cdot Ty$ on $\mathrm{Ker}\,i_A$. The quadratic form ψ_0: $\mathrm{Ker}\,i_A\to Z_2$ associated with B satisfies that $\psi_0(e)=\psi_0(f)=1$ (of course, if $\psi_0(e)=\psi_0(f)=0$, W_1 reduces to S^{4k}). If we put $a=e-(f-Tf)$, $b=Tf+(e-Te)$, then by the contribution to the intersection numbers, we see that a neighborhood J of $\{a\cup b\}$ in W_1 is the result of plumbing two copies of the tangent disk bundle of S^{2k} and the same is true for TJ of $\{Ta\cup Tb\}$. Then, it is proved in [25] that $W_1-(J\cup TJ)$ is an h-cobordism and hence W_1 is the double of plumbing J. We note the following facts which are used later.

(2.2) If we put $c=e-Te$, $d=f-Tf$, then $\{a, b, Ta, Tb, c, d\}$ generates $H_{2k}(W_1)$. By the above remark, we have

(2.3) W_0 bounds $V_0=D^{4k+1}$ and W_1 bounds $V_1=J\times I$.

(2.4) $\mathrm{Ker}\left\{i_*:\ H_{2k}(W_1)\longrightarrow H_{2k}(J\times I)\right\}=\{c,d\}$.

Now, $(T,\Sigma^{4k+1})=\partial(T,M^{4k+2})$, $k\geqq 2$. Let F_c^{4k+1} be a parallelizable manifold which separates M equivariantly such that (T,W_c) extends to an involution with isolated fixed points on F_c, i.e.,

$$M=B\cup TB,\ B\cap TB=Fc\quad\text{and}\quad (T,W_c)=\partial(T,F_c).$$

We can do this by using the relative transversality theorem. We will compute the Kervaire invariant $\sigma(M)$ of M^{4k+2}. For this, we show that there is a normal cobordism between F_c and V_c rel. boundary $W_c\times I$.

First, $W_1=\partial(J\times I)$. Divide S^{4k+1} into D_-^{4k+1} and D_+^{4k+1} along S^{4k}. Take a degree 1 map g of J onto the disk D^{4k}. Define a map

(2.5) $f_1:\ J\times I\longrightarrow D^{4k}\times I\cong D_-^{4k+1}$, by setting $f_1=g\times 1$. Put

(2. 6) $\qquad f_0=f_1|\partial(J\times I):\ W_1\longrightarrow S^{4k}.$

Then f_0 extends to a map

(2. 7) $\qquad f_2:\ F_1^{4k+1}\longrightarrow D_+^{4k+1}.$

We choose a framing $b':\nu_F\to R^l$, l sufficiently large. There is a commutative diagram

$$\begin{tikzcd} \nu_{F_1} \arrow[r, "b"] \arrow[d, "\pi"] & \xi \arrow[d] \\ F_1 \arrow[r, "f_2"] & D_+^{4k+1}, \end{tikzcd}$$

where $b(v)=(f_2\pi(v), b'(v))$, $\xi=D_+^{4k+1}\times R^l$.

Hence, f_2 is a normal map of F onto D_+^{4k+1}. Since $J\times 0\subset W_1\subset F_1$, we have $\nu_{J\times I}+\varepsilon^1=((\nu_{F_1}|J)+\varepsilon^1)\times I$. By (2. 5), $f_1=g\times 1$, there is a commutative diagram

$$(2.9)\qquad \begin{tikzcd} \nu_{J\times I}+\varepsilon^1 \arrow[r, "(b|J)\times 1"] \arrow[d] & (\xi|D^{4k}+\varepsilon^1)\times I \arrow[d] \\ J\times I \arrow[r, "f_1"] & D^{4k}\times I=D_-^{4k+1} \end{tikzcd}$$

Hence f_1 is a normal map.

In the diagram (2. 10)

$$\begin{array}{ccc} F_1 & \xrightarrow{\quad f_2\quad} & D_+^{4k+1} \\ \cup & & \cup \\ W_1 & \xrightarrow{\quad f_0\quad} & S^{4k} \\ \cup & & \cup \\ \partial J\times I & \xrightarrow{\quad g\times 1\quad} & S^{4k-1}\times I, \end{array}$$

the normal bundle

$$\begin{aligned} \nu_{S^{4k-1}\times I} &= \varepsilon^1+\xi|S^{4k-1}\times I \\ &= (\varepsilon^1+\xi|S^{4k-1})\times I, \end{aligned}$$

thus we have $\nu_{\partial J\times I}=(g\times 1)^*(\nu_{S^{4k-1}\times I})=(\varepsilon^1+\nu_{F^1}|\partial J)\times I$. Therefore, the restriction of the bundle map

$$b|\partial J\times I:\ \nu_{\partial J\times I}\longrightarrow \nu_{S^{4k-1}\times I} \quad\text{is}\quad b|\partial J\times I=(b|\partial J)\times 1\,.$$

Since $f_1=f_2=f_0$ on $W_1=\partial(J\times I)$, two bundle maps b, $(b|J)\times 1$ are compatible on W_1. Hence f_1 and f_2 define a normal map

$$f\colon F_1^{4k+1}\underset{W_1}{\cup}(J\times I)\longrightarrow S^{4k+1}.$$

Rearrange f to a normal map

$$(2.11)\qquad f'\colon V^{4k+1}\longrightarrow\partial(D^{4k+1}\times I),$$

where

$$(V_+,f'_+)=(F_1^{4k+1},f_2)\longrightarrow(D^{4k+1}\times 0)$$
$$(V_-,f'_-)=(J\times I,f_1)\longrightarrow(D^{4k+1}\times 1)$$

and

$$(\partial V,f'|\partial V)=(W_1\times I,f_0\times 1)\longrightarrow S^{4k}\times I.$$

Removing a small disk D_ε^{4k+1} from inside $J\times I$, f' gives a normal map

$$(2.12)\qquad f''\colon(V-\mathrm{int}\,D_\varepsilon^{4k+1},S_\varepsilon^{4k})\longrightarrow\left(\partial(D^{4k+1}\times I)-\mathrm{int}\,D_\varepsilon^{4k+1},S_\varepsilon^{4k}\right)$$
$$\cong(D^{4k+1},S^{4k})$$

which is a homotopy equivalence on the boundary. Since the surgery obstruction of f'' in $L_{4k+1}(1)$ is zero, f'' is normally cobordant to a homotopy equivalence $h'\colon D'\to D^{4k+1}$ rel. boundary. Let $H\colon X\to D^{4k+1}\times I$ be its cobordism. Then, X is viewed as a normal cobordism between $(X_+,H|X_+)=(F_1^{4k+1},f_2)$ and $(X_-,H|X_-)=((J\times I-\mathrm{int}\,D_\varepsilon^{4k+1})\underset{S_\varepsilon^{4k}}{\cup}D',f_1\cup h')$, and $(\partial X,H|\partial X)=(W_1\times I,f_0\times 1)$. Note that

$$(2.13)\qquad X_-\ \text{ is again }\ (J\times I).$$

For, in the exact sequence

$$\begin{tikzcd} H_*(J\times I-D_\varepsilon)\oplus H_*(D') \arrow[r] \arrow[d, "id+h'_*"] & H_*(X_-) \arrow[r] \arrow[d] & H_*(S^{4k}) \arrow[d, "id=(h'|)_*"] \\ H_*(J\times I-D_\varepsilon)\oplus H_*(D) \arrow[r] & H_*(J\times I) \arrow[r] & H_*(S^{4k}), \end{tikzcd}$$

it follows that $H_*(J)\to H_*(X_-)\cong H_*(J\times I)$ is isomorphic. The boundary of X_- is already $\partial J\times I$, hence by the relative h-cobordism theorem, X_- is $J\times I$. Thus we get

(2.14) There is a normal map $H\colon X\to D^{4k+1}\times I$ between $(X_+,H_+)=(F_1^{4k+1},f_2)$ and $(X_-,H_-)=(J\times I,H_-)$, $(\partial X,H|\partial X)=(W_1\times I,f_0\times 1)$.

Second, $W_0=S^{4k}$. There is a normal map $f_2\colon(F_0^{4k+1},W_0)\to(D^{4k+1},S^{4k})$ which is a homotopy equivalence on the boundary. As above, there is a normal cobordism

(2. 15) $H: X \rightarrow D^{4k+1} \times I$ between $(X_+, H_+) = (F_0^{4k+1}, f_2)$ and $(X_-, H_-) = (D', H_-)$, where $H_- : D' \rightarrow D^{4k+1}$ is a homotopy equivalence, and $(\partial X, H|\partial X) = (W_0 \times I, f_0 \times 1)$.

Next we extend these cobordisms (2. 14), (2. 15) to normal cobordims of M. Since F_c separates M, there is a normal map $h: M \rightarrow D^{4k+2}$ which is a normal map $H: M \rightarrow D^{4k+2}$ which is a homotcpy equivalence on the boundary $\Sigma \rightarrow S^{4k+1}$ and satisfies that $h|F_c = f_2 : F_c^{4k+1} \rightarrow D^{4k+1}$ is the restricted normal map. Applying the normal cobordism extension theorem to 2. 14 (2. 15) (see [20, p. 45]), we have a normal map $h_1 : M_1 \rightarrow D^{4k+2}$ which is normally cobordant to (M, h) rel. boundary, $h_1^{-1}(D^{4k+1}) = X_-$, $h_1|X_- = H_-$. The normal bundle ν of D^{4k+1} in D^{4k+2} is trivial, so is true for the pull back $H^*(\nu)$, i. e., $H^*(\nu) = X \times I$.
Hence M_1 has the following form (see Figure 1),

$$M_1 = B \cup X_0 \cup (X_- \times I) \cup X_1 \cup TB, \ \partial M_1 = \Sigma .$$

Here X_i is a copy of X for $i = 0, 1$.
Since $\sigma(M) = \sigma(M_1)$, we compute the obstruction of M_1. We notice that

(2. 16) $\quad A \underset{W_c}{\cup} (X_- \times 0), \ TA \underset{W_c}{\cup} (X_- \times 1)$ are homotopy spheres.

For, if $W_0 = S^{4k}$, then $A = D^{4k+1}$. $X_- = D'$ is a homotopy disk. And so,

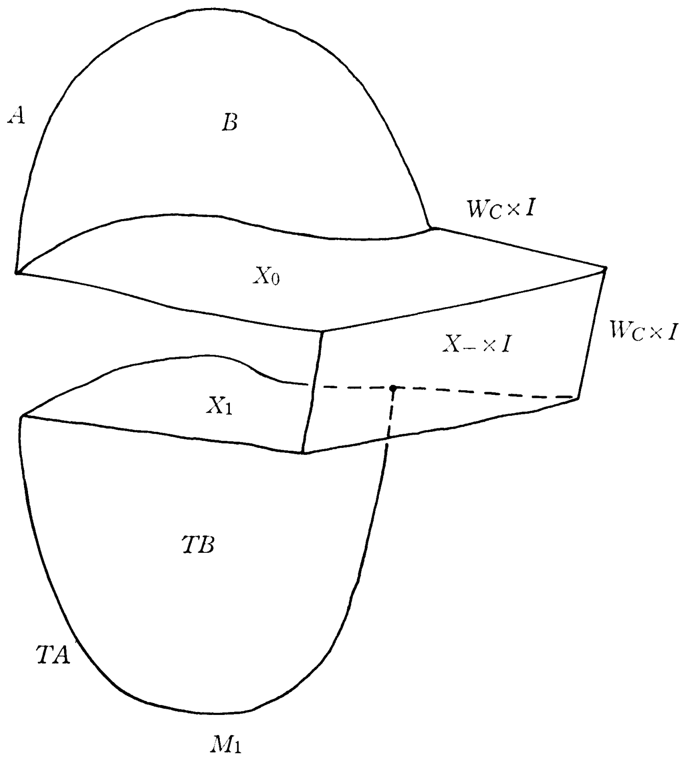


Fig. 1.

$A\underset{W_0}{\cup}(X_-\times 0)$ is a homotopy sphere (the same is true for $TA\underset{W_0}{\cup}(X_-\times 1)$). If $W_1=\partial(J\times I)$, then $X_-=J\times I$. Consider the following exact sequence

$$H_{2k+1}\left(A\underset{W_1}{\cup}(X_-\times 0)\right)\longrightarrow H_{2k}(W_1)\longrightarrow H_{2k}(J\times I)+H_{2k}(A)$$
$$\longrightarrow H_{2k}\left(A\underset{W_1}{\cup}(X_-\times 0)\right).$$

Then by (2.2) and (2.4), the central map is an isomorphism. Hence $A\underset{W_1}{\cup}(X_-\times 0)$ is a homotopy sphere.

Put $B\cup X_0=Y_0$. $A\underset{W_c}{\cup}(X_-\times 0)$ bounds Y_0. Let Y_1 be another copy of Y_0, i.e., $Y_1=TB\cup X_1$ so that $TA\underset{W_c}{\cup}(X_-\times 1)$ bounds Y_1. Take a tube $D^{4k+1}\times I\subset X_-\times I$ and connect $D^{4k+1}\times\{i\}$ to X_i, $i=0, 1$. Set

$$Z^{4k+1}=X_- -\operatorname{int} D^{4k+1}.$$

Z is a cobordism between W_c and S^{4k}.

M_1 splits into $(Z\times I)\cup(Y_0\# Y_1)$. Since $\partial(Y_0\# Y_1)$ is a homotopy sphere by (2.16) and Y_1 is a copy of Y_0, the surgery obstruction of $Y_0\# Y_1$ to making a disk rel. boundary is zero. Hence, $\sigma(M_1)$ is equal to the obstruction of the rest $Z\times I$ to making it homotopy equivalent to $(S^{4k}\times I)\times I$ rel. bondary. Since $k\geqq 2$, $\sigma(M_1)$ is the obstruction whether Z is an h-cobordism between W_c and S^{4k} or not. Therefore, $\sigma(M_1)=c=\sigma(T, \Sigma)$ (if $k=1$, the above argument breaks down because we cannot find an invariant codimension 1 sphere of (T, Σ^5) even though $\sigma(T, \Sigma)=0$).

Suppose $k=1$. Let Σ_d^5, $d>0$, odd, be the Brieskorn sphere which is described by two equations

$$z_0^d+z_1^2+z_2^2+z_3^2=0\,,$$
$$z_0\bar{z}_0+z_1\bar{z}_1+z_2\bar{z}_2+z_3\bar{z}_3=1\,.$$

The involution T_d given by $T_d(z_0, z_1, z_2, z_3)=(z_0, -z_1, -z_2, -z_3)$ on Σ_d is a fixed point free involution. T_d extends to an involution with isolated d-fixed points on the parallelizable mainfold F_d^6 with boundary Σ_d^5, which is the fibre of the fibration $S^7-\Sigma_d^5\to S^1$. The following result is well known (see [11], [20]).

LEMMA 2.17. $hS(P^5)=\{\Sigma_d^5/T_d,\ d=1, 3, 5, 7\}$.

$$\sigma(T_d, \Sigma_d^5)=\sigma(F_d^6)\,.$$

In general, the last fact holds for the $(4k+1)$-dimensional Brieskorn spheres, i.e.,

$$\sigma(T_d, \Sigma_d^{4k+1}) = \sigma(F_d^{4k+2}) = \begin{cases} 0 & d \equiv \pm 1(8) \\ 1 & d \equiv \pm 3(8) . \end{cases}$$

Let $(T, \Sigma^5) = \partial(T, M^6)$. We can classify Σ^5/T in $hS(P^5)$. Assume that $\Sigma^5/T \cong \Sigma_d^5/T_d$ for some d. Let $h : (T, \Sigma^5) \to (T_d, \Sigma_d^5)$ be an equivariant diffeomorphism. Denote a 6-dimensional smooth manifold with an involution by

$$(2.18) \qquad N^6 = M^6 \underset{h}{\cup} F_d^6 .$$

Let τ_N be the tangent bundle of N^6. The obstructions to the triviality of $\tau_N \oplus \varepsilon^1$ lie in the following exact sequence (see [17]).

$$\longrightarrow H^{i-1}\big(\Sigma^5, \pi_{i-1}(SO)\big) \longrightarrow H^i\big(N^6, \pi_{i-1}(SO)\big)$$
$$\longrightarrow H^i\big(M^6, \pi_{i-1}(SO)\big) \oplus H^i\big(F_d^6, \pi_{i-1}(SO)\big) \longrightarrow$$

Write the obstruction $\theta_i(N^6) \in H^i(N^6, \pi_{i-1}(SO))$. Then, for $1 \leqq i \leqq 5$ the above sequence yields that

$$0 \longrightarrow H^i\big(N^6, \pi_{i-1}(SO)\big) \longrightarrow H^i\big(M^6, \pi_{i-1}(SO)\big) \oplus H^i\big(F_d^6, \pi_{i-1}(SO)\big) \longrightarrow 0$$

and since M^6 and F_d^6 are parallelizable, we have $\theta_i(N^6) = 0$. $\theta_6(N^6) = 0$ because of $\pi_5(SO) = 0$. Hence all $\theta_i(N^6)$ vanish. Therefore,

$$(2.19) \qquad \tau_N \oplus \varepsilon^1 \text{ is trivial.}$$

Take a characteristic submanifold $(T, W^4) \subset (T, \Sigma^5)$ (which is also a characteristic submanifold of (T_d, Σ_d^5) under the identification of h). (T, W^4) extends to a (T, F^5) which separates (T, M^6), i. e., $M^6 = A \cup TA$, $A \cap TA = F^5$. Similarly, (T, W^4) extends to a (T, F'^5) which separates (T_d, F_d^6), $F_d^6 = A' \cup TA'$, $A' \cap TA' = F'^5$. Put

$$W^5 = F^5 \underset{W^4}{\cup} F'^5 \quad \text{and} \quad B = A \cup A' .$$

Then, $N^6 = B \cup TB$ and $B \cap TB = W^5$. Take a degree 1 map $f : N^6 \to S^6$ such that $f|W^5 : W^5 \to S^5$ is also a degree 1 map. From the fact (2.19), f is a normal map and hence $f|W^5 : W^5 \to S^5$ is the restricted normal map. Since $f|W^5 : W^5 \to S^5$ is normally cobordant to a homotopy equivalence $g' : K^5 \to S^5$, we can apply the normal cobordism extension theorem to yield a normal map $g' : L^6 \to S^6$ which is normally coordant to f, and $g^{-1}(S^5) = K^5$, $g|K = g'$. The normal bundle of K^5 in L^6 is trivial, so L^6 is written as the following form,

$$L^6 = B' \cup TB',\ B' \cap TB' = K^5 .$$

Then, the surgery obstruction of g' is $\sigma(L^6)=\sigma(B')+\sigma(TB')=0$. Hence we have $\sigma(N^6)=0$. On the other hand, since $N^6=M^6 \cup F_d^6$, $M^6 \cap F_d^6=\Sigma^5$, it follows that $\sigma(M^6)+\sigma(F_d^6)=0$. By (2.17) we have $\sigma(T, \Sigma^5)=\sigma(T, \Sigma_d^5)=\sigma(F_d^6)=\sigma(M^6)$. This completes the proof of $n\equiv 1$ (4).

When we consider $(4k-1)$-dimensional standard involutions, $k\geqq 2$, we have the following result.

LEMMA 2.20. *Let $(T, \Sigma^{4k-1})=\partial(T, M^{4k})$ be a standard involution. There exists a free involution T' on the sphere S^{4k-1} with $\sigma(T', S^{4k-1})=0$ such that Σ/T is normally cobordant to S^{4k-1}/T'.*

PROOF. It follows from [20] that Σ/T is normally cobordant to a homotopy projective space Q^{4k-1} which has a double desuspension. Let W^{4k} be its normal cobordism between Q and Σ/T. Applying the Atiyah-Singer theorem to W^{4k}, we have

$$\sigma(T, \Sigma) = \sigma(T, \Sigma)-\sigma(T, \tilde{Q}) = 1/8\left(2\sigma(W)-\sigma(\widetilde{W})\right). \tag{2.21}$$

It follows by Proposition 2.1 that $\sigma(T, \Sigma)=1/8\,\sigma(M)+2m$ for some $m\in Z$. If we take a parallelizable manifold M' with boundary $\Sigma'\in bP_{4k}$ such that $\sigma(M')=8m-\sigma(W)$, then the connected sum $Q\#\Sigma'$ satisfies the conclusion of lemma. For, put $Q'=Q\#\Sigma'$. Then, clearly Q' has a double suspension, i.e., $\sigma(T', Q')=0$. $(Q\times I)\#M'$ is a normal cobordism between Q and Q'. Hence Q' is normally cobordant to Σ/T. $\tilde{Q}'$ bounds the parallelizable manifold

$$(\tilde{Q}\times I\#2M') \underset{\tilde{Q}}{\cup} \widetilde{W} \underset{\Sigma}{\cup} M,$$

the index of which is zero. Therefore $\tilde{Q}'$ is diffeomorphic to S^{4k-1}.

By Lemma 2.20 and Theorem (Browder-Livesay), we have desuspensions of (T', S^{4k-1}), $(T', S^{4k-2})\supset(T', S^{4k-3})$.

PROPOSITION 2.22. *For a desuspension (T', S^{4k-3}) of (T', S^{4k-1}), it follows that $\sigma(T', S^{4k-3})=0$. Furthermore, there is a sequence of desuspensions of (T', S^{4k-1}). i.e., $(T', S^{4k-2})\supset(T', S^{4k-3})\supset\cdots\supset(T', S^5)$ satisfying $\sigma(T', S^5)=0$.*

PROOF. If W^{4k} is a normal cobordism between Σ/T and S^{4k-1}/T', then by the above argument, (T', S^{4k-1}) extends to an involution with isolated fixed points on the parallelizable manifold $(\widetilde{W} \underset{\Sigma}{\cup} M)$. For $(T', S^{4k-2})\subset(T', S^{4k-1})$, (T', S^{4k-2}) extends to an involution with isolated fixed points on a parallelizable manifold $F^{4k-1}\subset(\widetilde{W}\cup M)$. The same is true for (T', S^{4k-3}), i.e., $(T', S^{4k-3})=\partial(T', F^{4k-2})$, $F^{4k-2}=B\cap T'B$, $B\cup T'B=F^{4k-1}$. And we note that

$$(1) \qquad S^{4k-3}=D_0^{4k-2}\cap T'D_0^{4k-2},\ D_0^{4k-2}\cup T'D_0^{4k-2}=S^{4k-2}.$$

The boundary of B consists of D_0^{4k-2} and F^{4k-2}. Since (T', S^{4k-2}) and (T', S^{4k-3}) are desuspensions, there is a homotopy equivalence $f: S^{4k-1}/T'\to p^{4k-1}$ such that characteristic maps f_1, f_2 are homotopy equivalences in the following diagram.

$$\begin{array}{ccc} S^{4k-1}/T' & \xrightarrow{f} & P^{4k-1} \\ \cup & & \cup \\ S^{4k-2}/T' & \xrightarrow{f_1} & P^{4k-2} \\ \cup & & \cup \\ S^{4k-3}/T' & \xrightarrow{f_2} & P^{4k-3}. \end{array}$$

Since $S^{4k-2}=\partial F^{4k-1}$ and F^{4k-2} is characteristic for F^{4k-1} as above, $\tilde{f}_1: S^{4k-2}\to S^{4k-2}$ extends to a normal map $g_1: F^{4k-1}\to D^{4k-1}$ which is transverse on $D^{4k-2}\subset D^{4k-1}$ and $g_1^{-1}(D^{4k-2})=F^{4k-2}$. g_1 is a normal map since F^{4k-1} is parallelizable. If we put $g_1|g_1^{-1}(D^{4k-2})=g_2: F^{4k-2}\to D^{4k-2}$, we notice that $g_2|\partial F^{4k-2}=\tilde{f}_2: S^{4k-3}\to S^{4k-3}$.

Restricting g_1 to $B\subset F^{4k-1}$, we have a normal map $G: B\to D_+^{4k-1}$. Here $\partial D_+^{4k-1}=D'^{4k-2}\cup D^{4k-2}$, $D'^{4k-2}\cap D^{4k-2}=S^{4k-3}$, the first disk being the half of the boundary S^{4k-2}. Now, $\partial B=D_0^{4k-2}\cup F^{4k-2}$ and $D_0^{4k-2}\cap F^{4k-2}=S^{4k-3}$. It follows by (1) and the above diagram that $G|D_0^{4k-2}=\tilde{f}_1|D_0^{4k-2}: D_0^{4k-2}\to D'^{4k-2}$ is a homotopy equivalence. If we put $f_2=G|D_0^{4k-2}$, then (G, B) is viewed as a normal cobordism between (f_2, D_0^{4k-2}) and (g_2, F^{4k-2}) rel. boundary $(\tilde{f}_2, S^{4k-3})$. Hence, F^{4k-2} is normally cobordant to D_0^{4k-2}, i. e., $\sigma(F^{4k-2})=0$. On the other hand, by proposition 2. 1 and $(T', S^{4k-3})=\partial(T', F^{4k-2})$, it follows that $\sigma(T', S^{4k-3})=\sigma(F^{4k-2})=0$.

For the rest, we have desuspensions $(T', S^{4k-3})\supset(T', S^{4k-4})\supset(T', S^{4k-5})$. Then in particular, we can take (T', S^{4k-5}) with $\sigma(T', S^{4k-5})=0$. This follows from theorem [20, p. 52] that given a desuspension (T_0, Σ_0) of (T', S^{4k-4}), there is another desuspension (T_1, Σ_1) for each $i\in Z$ such that $\sigma(T_1, \Sigma_1)-\sigma(T_0, \Sigma_0)=2i$. In this case, as is $(T', S^{4k-5})=\partial(T', F^{4k-4})$, so $\sigma(T', S^{4k-5})\equiv 1/8\,\sigma(F^{4k-4})\equiv 0\ (2)$ by Proposition 2. 1. Therefore we can apply the above argument to (T', S^{4k-5}) completing the proof.

3. Geometric models

In this chapter, we give examples of standard involutions.

Notation 3. 1. Let $D^n(S^{n-1})$ be the unit disk (sphere) in R^n with the Z_2-action, $t(x_1, \cdots, x_n)=(-x_1, \cdots, -x_n)$. Let S^n be the suspension of S^{n-1}, i. e., the unit sphere in $R^n\times R$ with the Z_2-action, $t(x_1, \cdots, x_n, y)=(-x_1, \cdots, -x_n, y)$.

The following lemma is a special case in [15], [34] when $p=2$ (which is not stated for the $(4k+1)$-dimensional case, but the proof is similar).

LEMMA 3.2. *For any integer $n\geqq 3$, there are D^n-bundles E (specifically, E_+, E_0 and E_-) over S^n with semifree Z_2-actions T satisfying*

(1) *T is a bundle map preserving the 0-section.*

(2) *The action T on the 0-section is S^n with the above action and T has no fixed points outside the 0-section.*

(3) *E has two isolated fixed points each normal representation of which is $D^n\times D^n$ with the above diagonal action.*

(4) *If $n=2k$, the euler class χ of E_+, E_0 and E_- are taken to be 2, 0, and -2 mod any multiple of 4-times respectively (in particular, we distinguish E_+ from E_- for our necessity, see remark of Lemma 3.3).*

(5) *If $n=2k+1$, E_+ and E_- are the tangent disk bundles over S^{2k+1} and E_0 is the trivial disk bundle.*

(6) *These bundles are stably trivial.*

PROOF. Let $d: S^n\rightarrow S^n\times S^n$ be the diagonal embedding which is invariant under the action. Let $H_n(S^n\times S^n)=\langle\alpha\rangle+\langle\beta\rangle$ with the first factor representing α and the second representing β. For any $l\in Z$, take $|l|$-embedded spheres S^n's in the free part of $S^n\times S^n$ each of which represents β. Taking their equivariant connected sum with $d(S^n)$, i. e.,

$$d(S^n)\underset{Z_2}{\#}|l|S^n\subset S^n\times S^n\,,$$

we have a stably trivial normal bundle E_+ over S^n which is invariant under the action. E_+ has the euler class $\chi(E_+)=(\alpha+(2l+1)\beta)(\alpha+(2l+1)\beta)=2+4l$. Clearly E_+ satisfies (1), (2) and (3).

Let g be the equivariant diffeomorphism of S^{n-1} onto S^{n-1} defined by

$$g(x_1, \cdots, x_n)=(-x_1, x_2, \cdots, x_n)\,.$$

Denote the n-dimensional sphere with a Z_2-action obtained by attaching D^n to D^n by means of g by

$$S_1^n=D^n\underset{g}{\cup}D^n$$

which is again S^n.

We then define also equivariant embeddings

$$d' : S_1^n\longrightarrow S_1^n\times S_1^n$$

and

$$\iota : S_1^n\longrightarrow S_1^n\times S^n \quad (\text{similarly, } \iota' : S^n\rightarrow S^n\times S_1^n)$$

by setting

$$d'\left((x_1, \cdots, x_n, y)\right) = \left((x_1, \cdots, x_n, y), (-x_1, x_2, \cdots, x_n, y)\right), \iota(z) = (z, z_1)$$

where $(x_1, \cdots, x_n, y)$, $z \in S_1^n$ and z_1 is a fixed point of S^n. Making use of d', ι, we obtain the desired bundles E_-, E_0 accordingly.

Next we consider normal cobordisms for the resulting manifolds which are obtained by an equivariant plumbing. Let N_1, N_2 be the equivariant neighborhoods of the fixed points in E. We let each of them small so as to be contained in $D^n \times D^n$ of (3) of Lemma 3.2. Put

$$W = E - \text{int}\left\{\bigcup_{i=1}^{2} N_i\right\}/T \quad (\text{if } E = E_+, \text{ we put } W = W_+, \text{ and so on}).$$

LEMMA 3.3. *W defines a "normal cobordism" between $\partial E/T$ and $\left\{\bigcup_{i=1}^{2} \partial N_i/T\right\}$, i.e., there exists a normal map $H: W \to P^{2n-1}$ covered by a bundle map $b: \nu_W \to \nu_P$, where ν_W, ν_P are stable normal bundles of W, P^{2n-1}. Furthermore, if we look at the inclusion maps of the boundary components, then $H|D^n \times D^n - \text{int}\, N_i/T \to P^{2n-1}$ and $H_- = H|\partial N_i/T: \partial N_i/T \to P^{2n-1}$ are as follows.*

Table 0.

W	$(H\|D^n \times D^n - \text{int}\, N_i/T, D^n \times D^n - \text{int}\, N_i/T)$	$(H\|\partial N_i/T, \partial N_i/T)$
W_+	$(Pr(1\times 1), D^n \times D^n - \text{int}\, N_i/T)$ $i=1,2$	$(1\times 1, P^{2n-1})$ $i=1,2.$
W_0	Case ι	
	$(Pr(1\times 1), D^n \times D^n - \text{int}\, N_1/T)$	$(1\times 1, P^{2n-1})$ $i=1$
	$(Pr(c\times 1), D^n \times D^n - \text{int}\, N_2/T)$	$(c\times 1, P^{2n-1})$ $i=2$
	Case ι'	
	$(Pr(1\times 1), D^n \times D^n - \text{int}\, N_1/T)$	$(1\times 1, P^{2n-1})$ $i=1$
	$(Pr(1\times c), D^n \times D^n - \text{int}\, N_2/T)$	$(1\times c, P^{2n-1})$ $i=2$
W_-	$(Pr(1\times c), D^n \times D^n - \text{int}\, N_1/T)$	$(1\times c, P^{2n-1})$ $i=1$
	$(Pr(c\times 1), D^n \times D^n - \text{int}\, N_2/T)$	$(c\times 1, P^{2n-1})$ $i=2$

Here c is the orientation reversing diffeomorphism induced from the map $\tilde{c}: D^n \to D^n$, $\tilde{c}(x_1, \cdots, x_n) = (-x_1, x_2, \cdots, x_n)$.

NOTE. H is not a degree 1 map.

REMARK. The maps raised in the table are natural with respect to the euler classes 2, 0, -2.

So, given E with $\chi(E)\equiv 2\,(4)$, then we are free to take W_+ and the pair $\{1\times 1\}$ or W_- and the pair $\{1\times c,\ c\times 1\}$ for E in view of $\chi(E)=2+4l$ or $\chi(E)=-2+4(l+1)$ respectively.

PROOF OF LEMMA 3. 3. Let D^{n+1} be the unit disk in R^{n+1} with the Z_2-action, $t(x_1, \cdots, x_n, y)=(-x_1, \cdots, -x_n, y)$. The fixed points of S^n are written $z_1=(\bar{0}, 1)$, $z_2=(\bar{0}, -1)$, $\bar{0}=(0, \cdots, 0)\in R^n$. Then, by the construction of E, we have an equivariant embedding

$$E-\mathrm{int}\left\{\bigcup_{i=1}^{2} N_i\right\}\subset D^{n+1}\times D^{n+1}-(\bar{0}\times D^1)\times(\bar{0}\times D^1)$$
$$\cong(D^n\times D^n-\mathrm{int}\,\bar{0}\times\bar{0})\times D^2$$

We notice that the action on the part D^2 is trivial. Hence it induces an embedding of the quotient spaces

(1) $\qquad W^{2n}\subset P^{2n-1}\times I\times D^2$

which has the trivial normal bundle.

The normal bundle ν_W is induced from that of P^{2n-1}. Therefore, W defines a "normal cobordism". That is, there is a normal map $H: W\to P^{2n-1}$ which is covered by a bundle map $b: \nu_W\to\nu_P$. If we look at the inclusion maps of the boundary components carefully and by constructions of E_+, E_0 and E_-, the rest of lemma follows easily.

When the above bundles are plumbed equivariantly, we will show that a normal cobordism of the resulting manifold is obtained from each block of normal cobordisms of Lemma 3. 3.

LEMMA 3. 4. *Suppose that E^i's are plumbed equivariantly one after another at a fixed point on each and denote M' its resulting manifold with a Z_2-action T. Let $N(pts)$ be the equivariant tubular neighborhoods of the fixed points in M' so that they are a union of N_i's for each $i\,(=1, 2)$ as in the Lemma 2. 3. Then, the cobordism $V'=M'-\mathrm{int}\,N(pts)/T$ defines a normal cobordism $G': V'\to P^{2n-1}$ between $\partial M'/T$ and $\bigcup_j\{\partial N_i/T,\ i=1, 2\}$ (j runs over the fixed point set), covered by a bundle map $b': \nu_{V'}\to\nu_P$.*

Under the situation of the above lemma, we prove

LEMMA 3. 5. *If we do further plumbings in the free part of the action in M' equivariantly, and if we denote its manifold with a Z_2-action T by M, then the resulting cobordism $V=M-\mathrm{int}\,N(pts)/T$ defines a normal cobordism $G: V\to P^{2n-1}$ between $\partial M/T$ and $\bigcup_j\{\partial N_i/T,\ i=1, 2\}$. Moreover, G is unaltered on the boundary components, i. e., $G|\bigcup_j\{\partial N_i/T\}=G'|\bigcup_j\{\partial N_i/T\}$.*

PROOF OF LEMMA 3.4.

The normal representations (3) of Lemma 3.2 and the normal maps on them of Lemma 3.3 give us how to plumb equivariantly, i.e., around a fixed point, the two spaces $D^n\times D^n$ are equivariantly diffeomorphic by the map $h: D^n\times D^n\to D^n\times D^n$, $h(x,y)=(y,x)$. When we consider plumbings on the quotient spaces, plumbing E^1 with E^2 together equivariatly at a fixed point (we assume this at $z_2\in N_2\subset E^1$ and $z_1\in N_1'\subset E^2$, for instance) is equivalent to taking $(E^1-\mathrm{int}\{N_1\cup N_2\}/T)\cup(E^2-\mathrm{int}\{N_1'\cup N_2'\}/T)$ and identifying $(D^n\times D^n-\mathrm{int}\,N_2)/T$ with $(D^n\times D^n-\mathrm{int}\,N_1')/T$ by the map h' induced from h. If we put the manifold M' when E^1 and E^2 are plumbed as above, then the resulting cobordism V' is $V'=(M'-\mathrm{int}\{N_1\cup N_2\cup N_2'\})/T$, where $N_1'\subset E^2$ is identified with $N_2\subset E^1$. Now, in view of table 0 of Lemma 3.3, we have the following commutative diagram

$$\begin{array}{ccc} (D^n\times D^n-\mathrm{int}\,N_2)/T & \xrightarrow{H} & P^{2n-1} \\ \downarrow{h'} & & \downarrow{h'} \\ (D^n\times D^n-\mathrm{int}\,N_1')/T & \xrightarrow{H} & P^{2n-1}. \end{array} \tag{1}$$

By the form H of table 0 of Lemma 3.3 and the construction of normal cobordisms, the diagram (1) is compatible with the bundle maps b of stable normal bundles which cover H. Hence, V' defines a normal cobordism between $\partial M'/T$ and $\{\partial N_1/T\cup\partial N_2/T\cup\partial N_2'/T\}$. If E^2 is plumbed further with E^3 equivariantly at the unused fixed point in E^2, then the diagram (1) holds around the point, and hence the resulting cobordism also defines a normal cobordism. Iterating in this way, we obtain the result.

PROOF OF LEMMA 3.5.

In M', we do further plumbings equivariantly in the free part of the action. This can be done by taking two spaces $D_i^n\times D_i^n\subset V'$ and identifying $D_1\times D_1$ with $D_2\times D_2$ by the map $h(x,y)=(y,x)$, $h: D_1\times D_1\to D_2\times D_2$. Lifting gives 2-plumbings in the cover M'. Denote its manifold by M. If $(G',b'):V'\to P^{2n-1}$ is a normal map in Lemma 3.4, we can arrange, using the homotopy extension theorem, that $G'|D_1\times D_1=(G'|D_2\times D_2)\cdot h$ without changing on the boundary components $\bigcup_j\{\partial N_i/T,\ i=1,2\}$. Let V be the resulting cobordism when we identify $D_1\times D_1$ with $D_2\times D_2$ by h. Then, $V=M-\mathrm{int}\,N(pts)/T$. The above compatibility defines a map $G: V\to P^{2n-1}$. By choosing a bundle equivalence of $\nu_{V'}|D_1\times D_1$ with $\nu_{V'}|D_2\times D_2$ covering h, we can arrange, using the bundle covering homotopy theorem, that $b'|(\nu_{V'}|D_1\times D_1)$ and $b'|(\nu_{V'}|D_2\times D_2)$ are compatible to give a bundle map $b:\nu_V\to\nu_P$. Hence, $G: V\to P^{2n-1}$ is a normal cobordism. Repeating further plumbings in the free part of the action, the above argument holds on each step. Therefore we can prove the lemma.

PROPOSITION 3. 6. *Let Σ_1 be the generator of bP_{4k}, $k \geqq 2$. Then, for any $l \geqq 1$, there exist following examples of standard involutions.*

Table 1.

Type	(A, S^{4k-1})	$(T_{2l-1}^{-}, \Sigma_{(2l-1,-)}^{4k-1})$	$(T_{2l-1}^{+}, \Sigma_{(2l-1,+)}^{4k-1})$	$(T_{2l}^{-}, \Sigma_{(2l,-)}^{4k-1})$	$(T_{2l}^{+}, \Sigma_{(2l,+)}^{4k-1})$
Browder-Livesay invariant	0	$2l-1$	$2l-1$	$2l$	$2l$
Normal cobordism class	(P^{4k-1}, id)	$(8(2l-1)-1)(P, id)$ $P=P^{4k-1}$	$(8(2l-1)+1)(P, id)$	$(8(2l)+1)(P, id)$	$(8(2l)-1)(P, id)$
Spin invariant mod 2^{2k}	± 1	$\pm(8(2l-1)-1)$	$\pm(8(2l-1)+1)$	$\pm(8(2l)+1)$	$\pm(8(2l)-1)$
Matrix rank	—	H_{2l-1}^{-} $8(2l-1)$	H_{2l-1}^{+} $8(2l-1)$	H_{2l}^{-} $8(2l)$	H_{2l}^{+} $8(2l)$
Differentiable structure	S^{4k-1}	$(2l-1)\Sigma_1$	$(2l-1)\Sigma_1$	$2l\Sigma_1$	$2l\Sigma_1$

PROOF. Let m be a positive integer. We introduce the unimodular, even, symmetric matrices with the rank $8m$ (the index of which is also $8m$),

$$H_m^{+}=\begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & \ddots & & \mathbf{0} & \\ & \ddots & \ddots & 1 & & \\ \mathbf{0} & & 1 & 2 & 4m+1 \\ & & & 4m+1 & 2m(8m+3) \end{bmatrix}, \quad H_m^{-}=\begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & \ddots & & \mathbf{0} & \\ & \ddots & \ddots & 1 & & \\ \mathbf{0} & & 1 & 2 & 4m-1 \\ & & & 4m-1 & 2(8m^2-5m+1) \end{bmatrix}.$$

We write simply for the above matices as;

$$H=\begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & \ddots & \mathbf{0} & \\ & \ddots & \ddots & 1 & \\ \mathbf{0} & & 1 & 2 & b \\ & & & b & a \end{bmatrix}.$$

Then, $a \equiv \begin{cases} 0\ (4) & \text{when } H = H_{2l}^{+},\ H_{2l-1}^{-} \\ 2\ (4) & \text{when } H = H_{2l-1}^{+},\ H_{2l}^{-} \end{cases}$ and

$b \equiv 1\ (2)$.

We rake bundles E^i, $i=1, \cdots, 8m$ from Lemma 3.2, each euler class of which is 2 for $i=1, \cdots, 8m-1$ and a for $i=8m$. Plumb together the E^i's, plumbing E^i with E^{i+1} at a fixed point of the action on each. Let M' be the resulting mainfold which realizes the plumbing matrix

$$\begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & \ddots & & \mathbf{0} & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & \mathbf{0} & & 1 & 2 & 1 \\ & & & & 1 & a \end{bmatrix}.$$

By Lemma 3.3 and 3.4, the cobordism $V' = (M' - \operatorname{int} N((8m+1)\,pts))/T$ defines a normal cobordism $G' : V' \to P^{4k-1}$ between $\partial M'/T$ and

$$\begin{cases} (8m+1)\,(P^{4k-1}, id) & \text{if } a \equiv 2\,(4) \\ (8m)\,(P^{4k-1}, id) \cup (P^{4k-1}, c \times 1) & \text{if } a \equiv 0\,(4)\,. \end{cases}$$

We do further plumbings equivariantly to realize H. Then, we have a manifold with boundary M which admits a Z_2-action. By Lemma 3.5, there is a normal map

$$G : V = \left(M - \operatorname{int} N\left((8m+1)\,pts\right)\right)/Z_2 \longrightarrow P^{4k-1}$$

between $\partial M/Z_2$ and $(8m+1)\,(P^{4k-1}, id)$ or $((8m)\,(P^{4k-1}, id) \cup (P^{4k-1}, c \times 1))$ if $a \equiv 2\,(4)$ or $0\,(4)$ accordingly. If follows from the plumbing theory that M is connected and $\pi_1(\partial M) \cong \pi_1(M)$ is free, $H_i(\partial M) = H_i(M) = 0$ for $1 < i < 2k-1$ and $H_{2k-1}(M) = 0$ (see Figure 2).

Put $(G_+, \partial_+ V) = (f', \partial M/Z_2)$. Then, clearly $\pi_1(f') = 0$. There is no obstruction to doing a normal surgery on a generator in $\pi_2(f') = \operatorname{Ker}\{f'_* : \pi_1(\partial M/Z_2) \to \pi_1(P^{4k+1})\}$, so there is a trace W and a normal map $F' : W \to P^{4k-1}$ between $\partial M/Z_2$ and $\partial_+ W$ such that f is 2-connected when we set $(F'|\partial_+ W, \partial_+ W) = (f, Q^{4k-1})$. Put

$$V_1 = V \cup W \quad \text{along } \partial M/Z_2 \text{ and}$$
$$M_1 = M \cup \widetilde{W} \quad \left(= \tilde{V}_1 \cup N\left((8m+1)\,pts\right)\right).$$

Then, V_1 is a normal cobordism between

$$(8m+1)\,(P^{4k-1}, id)\,\left((8m)\,(P^{4k-1}, id) \cup (P^{4k-1}, c \times 1)\right)$$

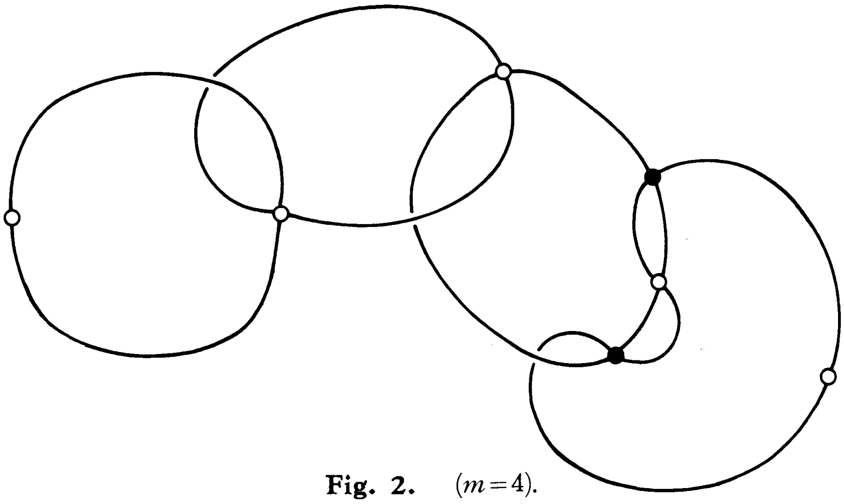


Fig. 2. $(m=4)$.

and Q. The universal cover $\tilde{Q}$ bounds the parallelizable mani- fold M_1. The intersection matrix on $H_{2k}(M_1)$ is the above plumbing matrix H. Since H is unimodular and by the above facts of plumbing, it follows that $H_i(\tilde{Q})=0$, $i\neq 4k-1$. Hence Q is a homotopy projective space (see Remark 1 below). We denote by $T_m^{\pm}$ the actions on M_1 for $H_m^{\pm}$ and put $\tilde{Q}=\Sigma_{(m,\pm)}^{4k-1}$ accordingly. Since the induced action is trivial on homology $H_{2k}(M_1)$, it follows that

$$\begin{aligned}\mathrm{Sign}\,(T_m^{\pm}, M_1) &= \text{Index of the intersection matrix on } H_{2k}(M_1)\\ &= \sigma(H_m) = 8m\,.\end{aligned}$$

M_1 has isolated $(8m+1)$-fixed points, so the local invariants of $(T_m^{\pm}, M_1)$, $L(T_m^{\pm}, M_1)=0$.

Hence, the Browder-Livesay invariant of $(T_m^{\pm}, \Sigma_{(m,\pm)}^{4k-1})$ is $\sigma(T_m^{\pm}, \Sigma_{(m,\pm)})=m$ and $\Sigma_{(m,\pm)}=1/8\sigma(M_1)\,\Sigma_1=m\Sigma_1$. For the rest, $Q=\Sigma_{(m,\pm)}/T_m^{\pm}$ is normally cobordant to

$$\begin{cases}(8m+1)\,(P^{4k-1}, id) & \text{if } a\equiv 2\,(4)\\ (8m)\,(P^{4k-1}, id)\cup(P^{4k-1}, c\times 1) & \text{if } a\equiv 0\,(4)\,.\end{cases}$$

Since $c\times 1$ is the orientation reversing diffeomorphism, $((8m)\,(P^{4k-1}, id)\cup(P^{4k-1}, c\times 1))$ is normally cobordant to $(8m-1)\,(P^{4k-1}, id)$. By the above remark, if $H=H_{2l-1}^{+}$, H_{2l}^{-}, then Q is normally cobordant to $(8m+1)\,(P^{4k-1}, id)$, and if $H=H_{2l}^{+}$, H_{2l-1}^{-}, then Q is normally cobordant to $(8m-1)\,(P^{4k-1}, id)$.

REMARK 1. We have a "normal cobordism" $F: V_1\to P^{4k-1}$ between (Q,f) and $d(P^{4k-1}, id)$, covered by a bundle map $b:\nu_{V_1}\to\nu_P$. Here $d=8m+1$ or $8m-1$. We note that the map of the boundary components into P^{4k-1} has degree d. So, the map f of Q into P^{4k-1} is not a homotopy equivalence even though Q is a homotopy projective space. However, put $d=2s+1$.

Form $P^{4k-1}(d)=d(P^{4k-1})\cup -s(S^{4k-1})$ and define the normal map $d(id)$ of $P^{4k-1}(d)$ into P^{4k-1} to be id on P^{4k-1} and p on S^{4k-1}, where $p: S^{4k-1}\to P^{4k-1}$ is the 2-fold cover. $d(id)$ has degree 1. If we take a connected sum along the boundary Q of V_1 with $(-s)\,S^{4k-1}\times I$, we get a normal cobordism of $Q\#(-s)\,S^{4k-1}=Q$ with $P^{4k-1}(d)$. And hence the induced map $Q\to P^{4k-1}$ is of degree 1, i.e., a homotopy equivalence.

REMARK 2. Let (T_m, Σ_m) and M_m with boundary Σ_m be as in the proof of (3.6). $H_{2k}(M_m)$ consists of a basis of invariant spheres. We notice that $\sigma(T_m, \Sigma_m)=1/8\sigma(M_m)$ and $\Sigma_m=1/8\sigma(M_m)\,\Sigma_1$.

Next, within each normal cobordism class of the above models, we can construct a typical example whose Browder-Livesay invariant takes any value. This can be achieved by López De Medrano's construction [20], however, by the above Remark 2 we must know how the differentiable structure of the homotopy sphere is altered when we obtain an example from each class. An explicit construction has been found in [26]. The following lemma slightly extends the result of [26].

LEMMA 3.7. *Let* (T_m, Σ_m^{4k-1}) *and* M_m^{4k} *be as in Proposition* 3.6. *For each* $i\in Z$ *and* (T_m, Σ_m), *there exists a free involution* (T_i, Σ_i^{4k-1}) *which satisfies*

(1) T_i *extends to an involution with isolated fixed points on a parallelizable manifold* M_i, *i.e.,* T_i *is a standard involution.*

(2) $\sigma(T_i, \Sigma_i)=1/8\sigma(M_i)=i$.

(3) Σ_i/T_i *is normally cobordant to* Σ_m/T_m.

PROOF. We recall the definition of the Browder-Livesay invariant. Let $W^{4k-2}=V\cap TV$ be a characteristic submanifold for (T_m, Σ_m) which we can assume $(2k-2)$-connected. We have the bilinear form

$$B(x,y)=x\cdot T_*y$$

defined on $\mathrm{Ker}\, i_V=\mathrm{Ker}\,\{i_*: H_{2k-1}(W)\to H_{2k-1}(V)\}$. Since $\mathrm{Ker}\, i_{TV}=T\cdot\mathrm{Ker}\, i_V$, it follows that $H_{2k-1}(W)=\mathrm{Ker}\, i_V\oplus T\cdot\mathrm{Ker}\, i_V$. Then, by definition of Browder-Livesay invariant (see [20]), we have

$$\text{(i)}\qquad \sigma(T_m, \Sigma_m)=1/8\,\mathrm{Index}\,\big(B(x,y)\big)=m\,.$$

By adding handles equivariantly to W inside Σ_m, we obtain a characteristic submanifold $W'=W\underset{Z_2}{\#}4(S^{2k-1}\times S^{2k-1})$. Let $W'=V'\cap T'V'$, $V'\cap T'V'=\Sigma_m$.

We have an equivariant map $f:(T', W')\to(T, W)$ such that $\mathrm{Ker} f_*=\mathrm{Ker}\,\{f_*: H_{2k-1}(W')\to H_{2k-1}(W)\}$ is a free group on generators α_i, $T\alpha_i$, $i=$

$1, \cdots, 8$, so that α_i's are contained in $\operatorname{Ker} i_{V'}$ and $T\alpha_i$'s in $\operatorname{Ker} i_{T'V'}$. The matrix of the form B on $\{\alpha_i\}$ consists of 1's on the nonprincipal diagonal and 0's elsewhere. Since W' and W are characteristic submanifolds of Σ_m, there exists a characteristic cobordism Y joining them, i.e.,

$$\Sigma_m \times I = X \cup TX, \quad XT \cap X = Y.$$

Let $F: Y \to W$ be an equivariant normal map between $f: (T', W) \to (T, W)$ and $id: (T, W) \to (T, W)$. Noting that $\partial X = V \cup Y \cup V'$, there is a normal map $G: X \to V$ which extends F.

Choose new generators $\alpha_i^* = p_{ij}\alpha_j + q_{ij} T\alpha_j$, $i = 1, \cdots, 8$. The matrices $P = (p_{ij})$, $Q = (q_{ij})$ are given explicitly in [20, p. 23]. We notice that

(ii) $\quad 1/8 \operatorname{Index} B(\alpha_i^*, \alpha_j^*) = 1$ and $\alpha_i^* \cdot \alpha_j^* = 0$, and $\phi(\alpha_i^*) = 0$.

So, we can perform surgery on the α_i^*'s$\in \operatorname{Ker} f_*$, obtaining a normal cobordism $h: A \to W$ between $f: W' \to W$ and a homotopy equivalence $W' \to W$. Then, we claim that

$(*)$ $\quad V' \bigcup_{W'} A$ is diffeomorphic to V

For, V' is obtained from V by adding handles equivariantly along W, in fact, we can write

$$V' = V \bigcup_{W} \left(W \times I \# 4(S^{2k-1} \times D^{2k}) \# 4(D^{2k} \times S^{2k-1}) \right).$$

Put $V'' = W \times I \# 4(S^{2k-1} \times D^{2k}) \# 4(D^{2k} \times S^{2k-1})$. Then, $\alpha_i \in \operatorname{Ker} \{i_* : H_{2k-1}(W') \to H_{2k-1}(V'')\} \subset \operatorname{Ker} i_{V'}$, $i = 1, \cdots, 8$. Since V'' is a cobordism between W and W', we may show that $V'' \bigcup_{W'} A$ is an h-cobordism between W and W'', so that is diffeomorphic to $W \times I$. W and W' are $(2k-2)$-connected so the non-vanishing homology except 0, $(4k-2)$-dimensions is

$$\begin{array}{ccc}
 & & 0 \\
 & & \uparrow \\
 & & H_{2k-1}(V'' \bigcup_{W'} A) \\
 & & \uparrow \\
H_{2k-1}(W) & \longrightarrow & H_{2k-1}(V'') \oplus H_{2k-1}(A) \\
 & & \uparrow \\
 & & H_{2k-1}(W') \\
 & & \uparrow \\
 & &
\end{array}$$

We notice that

$$H_{2k-1}(W') = H_{2k-1}(W) + \{\alpha_i, T\alpha_i\} \left(= H_{2k-1}(W) + \{\alpha_i^*, T\alpha_i^*\}\right),$$
$$H_{2k-1}(V'') = H_{2k-1}(W) + \{a_i\},$$

where $\{a_i\}$ corresponds to $\{T\alpha_i\}$, $i=1, \cdots, 8$, and $H_{2k-1}(A)$ consists of the generators corresponding to those of $H_{2k-1}(W)$ and the generators $\{b_i\}$ corresponding to $\{T\alpha_i^*\}$. If $x \in H_{2k-1}(W)$, the vertical map is, $x \mapsto (x, -x)$ and for $\{\alpha_i^*, T\alpha_i^*\}$, $\alpha_i^* \mapsto (q_{ij} a_j, 0)$ and $T\alpha_i^* \mapsto (p_{ij} a_j, -b_i)$. We see by [20] that det $Q=1$. Hence, the vertical map maps the elements $\{\alpha_i^*, T\alpha_i^*\}$ onto $\{a_i, b_i\}$ isomorphically. Thus, $H_{2k-1}(V'' \underset{W'}{\cup} A)$ consists of the generators corresponding to $(x, 0)$ for $x \in H_{2k-1}(W)$. Therefore $H_{2k-1}(W) \to H_{2k-1}(V'' \underset{W'}{\cup} A)$ is an isomorphism. Since W'' is obtained from W' by performing surgery on the $\{\alpha_i^*\}$ and $H_{2k-1}(W'')$ consists of generators corresponding to those of $H_{2k-1}(W)$, the similar argument holds for W''. Hence $V'' \underset{W'}{\cup} A$ is an h-cobordism.

Now, by (*) we can attach a copy of V to $V' \underset{W'}{\cup} A$ along W'' so that $(V' \underset{W'}{\cup} A) \underset{W''}{\cup} V$ bounds an h-cobordism B of $V \times I$ (see Figure 3). Put $B' = X \underset{V'}{\cup} B$ along V'. We apply the López's construction. Let B'^* be another copy of B', we can obtain a parallelizable manifold M' with a free involu-

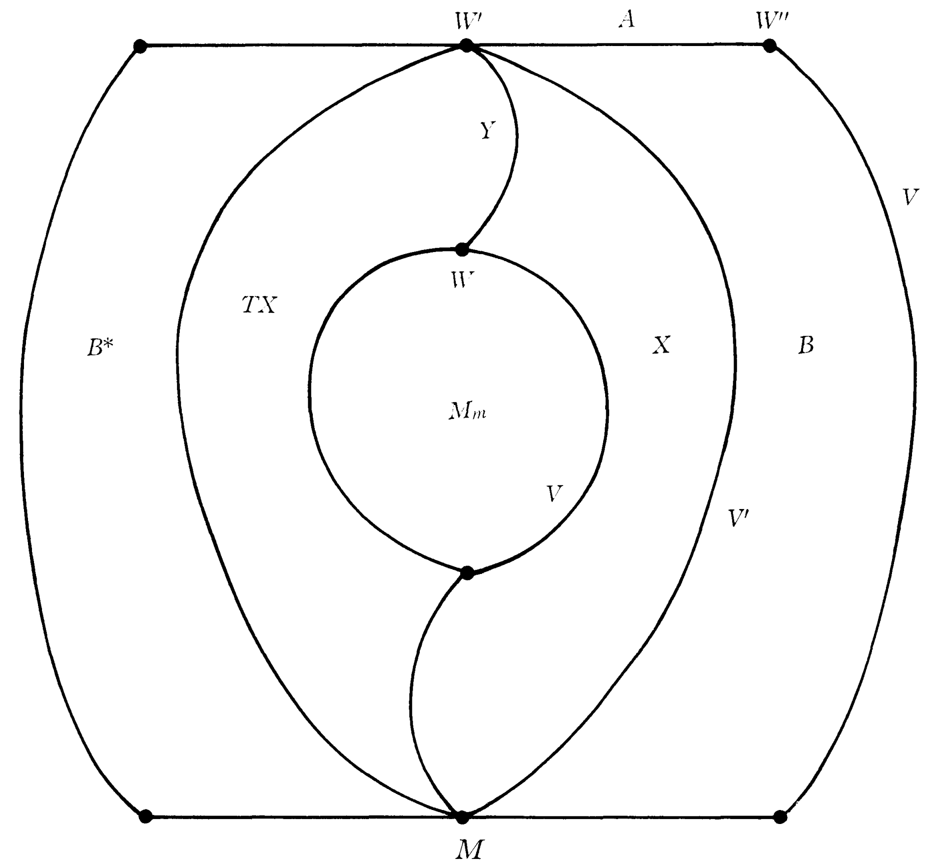


Fig. 3.

tion T, $M'=B'\cup B'^*$, sewed equivariantly on (T, Y). Then, the boundaries of M' are the given (T_m, Σ_m) and a López's involution (T, Σ). M/T is a cobordism between Σ_m/T and Σ/T.

We first show (3). The normal map

$$(G|V')\underset{f}{\cup} h: V'\underset{W'}{\cup} A \longrightarrow V \quad \text{extends to a normal map}$$
$$H': (V'\underset{W'}{\cup} A)\underset{W''}{\cup} V \longrightarrow \partial(V\times I).$$

By the above remark, there is a normal map $H: B\to V\times I$. Combining with G, we have a normal map

$$G': B' = X\underset{V'}{\cup} B \longrightarrow \mathrm{V}.$$

Let E_W be the normal disk bundle of W/T in Σ_m/T_m and E_Y the normal disk bundle of Y/T in M/T. Then, $\partial E_W=W$, $\partial E_Y=Y$ and $\Sigma_m/T_m=E_W\cup V$, $M'/T=E_Y\cup B'$. Since $F/T: Y/T\to W/T$ can be covered by a bundle map between the stable normal bundles, the same is true for $E(F/T): E_Y\to E_W$. Hence, G' and $E(F/T)$ define a normal map of M'/T. This proves (3).

Σ_m bounds M_m, so if we put $M=M'\cup M_m$ along Σ_m, then T and T_m define an involution T on the parallelizable manifold M which satisfies the property of (1). Finally, if we put $A'=A\underset{W''}{\cup} V$, then by the above construction, it follows that $(T,\Sigma)=A'\cup TA'$, $A'\cap TA'=W'$ and $\operatorname{Ker} i_{A'}=\operatorname{Ker} i_V\oplus\{\alpha_i^*\}$. By (i) and (ii), we have

$$\sigma(T,\Sigma)=1/8\big(B(x,y)+B(\alpha_i^*,\alpha_j^*)\big)=m+1.$$

Since $X\cup TX=\Sigma_m\times I$, we notice that

$$\begin{aligned} M &= M'\cup M_m=(B'\cup B'^*)\cup M_m \\ &= B\cup(X\cup TX)\cup M_m\cup B^* \\ &= B\cup M_m\cup B^*. \end{aligned}$$

From the Mayer-Vietoris sequence and noting that $(B\cup B^*)\cap M_m=\Sigma_m$, we have

$$\begin{array}{c} 0 \\ \downarrow \\ 0 \longrightarrow H_{2k}(B\cup B^*)+H_{2k}(M_m) \longrightarrow H_{2k}(M) \longrightarrow 0 \\ \downarrow \\ H_{2k-1}(W') \\ \downarrow \\ H_{2k-1}(B)+H_{2k-1}(B^*) \\ \downarrow \\ 0. \end{array}$$

Since $B \cong V \times I$ and $W' = W \underset{Z_2}{\#} 4(S^{2k-1} \times S^{2k-1})$, $H_{2k}(B \cup B^*)$ consists of the generators $\{\bar{a}_i, \bar{b}^i\}$ corresponding to $\{\alpha_i, T\alpha_i\}$, $i=1, \cdots, 8$. Then, by the result of [26], we can show similarly that the index on $H_{2k}(B \cup B^*)$ is 8. Hence, it follows that

$$\begin{aligned}\sigma(M) &= \text{Index on } H_{2k}(B \cup B^*) + \text{Index on } H_{2k}(M_m) \\ &= 8 + 8m = 8(m+1).\end{aligned}$$

Therefore, $\sigma(T, \Sigma) = 1/8\sigma(M) = m+1$. This proves (2). Repeating the above construction for (T, Σ) and M, then the result follows.

By Lemma 3. 7, we denote the equivalence class of the normal cobordism class by $[T_i, \Sigma_i]$ for each representative (T_i, Σ_i). We have the following table more generally.

Table 2. $k \geqq 2$, $l \geqq 1$ and i any integer

Type	(A, S^{4k-1})	$(T_{2l-1}^{-}, \Sigma_{(2l-1,-)}^{4k-1})$	$(T_{2l-1}^{+}, \Sigma_{(2l-1,+)}^{4k-1})$	$(T_{2l}^{-}, \Sigma_{(2l,-)}^{4k-1})$	$(T_{2l}^{+}, \Sigma_{(2l,+)}^{4k-1})$
general element	$[T_i, \Sigma_i]$	$[T_i, \Sigma_i]$	$[T_i, \Sigma_i]$	$[T_i, \Sigma_i]$	$[T_i, \Sigma_i]$
Browder-Livesay invariant	i	i	i	i	i
Normal cobordism class	(P^{4k-1}, id)	$(8(2l-1)-1)(P, id)$ $P=P^{4k-1}$	$(8(2l-1)+1)(P, id)$	$(8(2l)+1)(P, id)$	$(8(2l)-1)(P, id)$
Spin invariant mod 2^{2k}	± 1	$\pm(8(2l-1)-1)$	$\pm(8(2l-1)+1)$	$\pm(8(2l)+1)$	$\pm(8(2l)-1)$
Differentiable structure	$i\Sigma_1$	$i\Sigma_1$	$i\Sigma_1$	$i\Sigma_1$	$i\Sigma_1$

3. 8. The $(4k+1)$-dimensional case

When we construct $(4k+1)$-dimensional standard involutions by an equivariant plumbing, the resulting involutions are well known "Brieskorn involutions".

Suppose that d is an odd integer. We use the bundles E_+ in Lemma 3. 2 to plumb equivariantly so that the resulting manifold M_d^{4k+2} with a Z_2-action T realizes the plumbing matrix of the rank $(d-1)$,

$$\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}.$$

Then, by the result of Hirzebruch it follows that

$$(T|\partial M_d, \partial M_d) = (T_d, \Sigma_d^{4k+1}).$$

We have the following result.

Table 3.

Type	$[T_d, \Sigma_d^{4k+1}]$
Normal cobordism class	$d(P^{4k+1}, id)$
Browder-Livesay invariant	0 if $d \equiv \pm 1$ (8) 1 if $d \equiv \pm 3$ (8)
Differentiable structure of Σ_d^{4k+1}	S^{4k+1} if $d \equiv \pm 1$ (8) $\Sigma_{\mathcal{K}}^{4k+1}$ is $d \equiv \pm 3$ (8)

Here $\Sigma_{\mathcal{K}}^{4k+1}$ is the Kervaire sphere which is the generator of $bP_{4k+2}(k \geqq 1)$.

4. Classification of standard involutions

4.1. Let (T, Σ^{2n-1}) be a standard involution so that $(T, \Sigma) = \partial(T, M^{2n})$. We require M^{2n} to be $(n-1)$-connected, which will be accomplished in the beginning of this chapter.

LEMMA 4.1.1. *Assume $n \geqq 4$. Let T be a free involution on a homotopy sphere Σ^{2n-1} and W^{2n-2} a characteristic submanifold for (T, Σ). We can perform equivariant surgery to make W $(n-2)$-connected, Then,*

$$\text{if } n = 2k, \qquad (T, W^{4k-2}) = (T, S^{4k-2}) \underset{Z_2}{\#} r(S^{2k-1} \times S^{2k-1})$$

and

$$\text{if } n = 2k+1, \quad (T, W^{4k}) = (T, W_c^{4k}) \underset{Z_2}{\#} r(S^{2k} \times S^{2k}),$$

where W_c is defined in Proposition 2.1 of chapter II.

PROOF. The case for $n = 2k$ has been proved in [20, p. 24], i.e., W^{4k-2} can be thought of as a characteristic submanifold for another involution

(T', Σ') with $\sigma(T', \Sigma')=0$. We can perform equivariant surgery on W until we obtain a sphere S^{4k-2}. The surgery on S^{4k-2} to return to W yields the above form. Suppose $n=2k+1$. Let $W^{4k}=B\cap TB$ and $B\cup TB=\Sigma$. Then, we have

$$H_{2k}(W^{4k}) = \operatorname{Ker} i_B \oplus T\cdot\operatorname{Ker} i_B\,,$$

where

$$\operatorname{Ker} i_B = \operatorname{Ker}\left\{i_* : H_{2k}(W) \longrightarrow H_{2k}(B)\right\}.$$

The bilinear form $B: \operatorname{Ker} i_B \times \operatorname{Ker} i_B \to Z$, defined by $B(x, y) = x\cdot T_* y$, is skew-symmetric and unimodular. Thus, we have a symplectic basis $\{e_1, \cdots, e_r, f_1, \cdots, f_r\}$ on $\operatorname{Ker} i_B$. If S_x^{2k} is an embedded sphere which induces x on $\operatorname{Ker} i_B \subset H_{2k}(W)$, then the quadratic form $\psi_0: \operatorname{Ker} i_B \to Z_2$ is defined to count, mod 2, the number of pairs (p, Tp) of points in $S_x^{2k}\cap TS_x^{2k}$. Then, ψ_0 is associated with B. If $\psi_0(e)=\psi_0(f)=0$ for a symplectic basis $\{e, f\}$, then we can perform surgery on e to obtain W' so that $W=W' \underset{Z_2}{\#} (S^{2k}\times S^{2k})$, i.e., $\{e, Te, f, Tf\}$ are killed in $H_{2k}(W')$. Suppose that $\psi_0(e_1)=\psi_0(f_1)=\psi_0(e_2)=\psi_0(f_2)=1$. Choose a new symplectic basis

$$e_1' = e_1 + e_2,\ f_1' = f_1\,,$$
$$e_2' = f_1 - f_2,\ f_2' = e_2\,.$$

Then, $\psi_0(e_1')=\psi_0(e_2')=0$ and $\psi_0(f_1')=\psi_0(f_2')=1$. So, we put $\lambda_i=e_i'$, $\mu_i=e_i'+f_i'$, $i=1, 2$. $\{\lambda_i, \mu_i, i=1, 2\}$ is again a symplectic basis and $\psi_0(\lambda_i)=\psi_0(\mu_i)=0$. And thus $\operatorname{Ker} i_B$ consists of a symplectic basis $\{\lambda_i, \mu_i, i=1, 2\}\cup\{e_i, f_i, i=3, \cdots, r\}$. As above, we can perform surgery on λ_i, $i=1, 2$. Hence, iterating this process, W^{4k} can be reduced to $W_c \underset{Z_2}{\#} r(S^{2k}\times S^{2k})$. We note that if $c=0$, then $W_0=S^{4k}$, and if $c=1$, W_1 is the double of plumbing two copies of the tangent disk bundles of S^{2k} and $H_{2k}(W)=\{e, f, Te, Tf\}$ with $\psi_0(e)=\psi_0(f)=1$.

Let T be a standard one so that $(T, \Sigma^{2n-1})=\partial(T, M^{2n})$.

LEMMA 4.1.2. *Assume $n\geqq 4$. By performing equivariant surgeries on M^{2n} rel. boundary, we can make M $(n-1)$-connected.*

PROOF. By Lemma 4.1.1, we have an $(n-2)$-connected characteristic submanifold (T, W_1) of (T, Σ). Applying the relative transversality theorem, we have a characteristic submanifold (T, F_1) with boundary W_1 for (T, M). Write $\Sigma=B\cup TB$, $B\cap TB=W_1$, $M=A_1\cup TA_1$, $A_1\cap TA_1=F_1$. Let (T, F_2, W_2) be a characteristic submanifold of (T, F_1, W_1), $F_1=A_2\cup TA_2, A_2\cap TA_2=F_2$. We may assume that F_2 and W_2 are connected. We note the following facts.

(1) M is obtained from the disjoint union $A_1 \cup A_1^*$, A_1^* a copy of A_1, by identifying x^* with Tx for all $x \in F_1$ (we can view A_1^* as TA_1).

(2) The framing $A_1(A_1^*)$ comes from the restriction of a framing M under the above identification. The same is true for (T, F_1) and the framing F_2 comes from A_2.

We can perform framed 1-surgery on A_2 rel. boundary. Let X_2 be a framed cobordism between A_2 and a 1-connected manifold A_2' rel. boundary, i. e., rel. $(F_2 \cup W_2) \times I$. Define an involution T on $(F_2 \cup W_2) \times I$ by $T(x, t) = (Tx, t)$. Let X_2^* be a copy of X_2. Then, we have manifolds with involutions T, $X_1(F_1')$ which are obtained from $X_2 \cup X_2^*$ by identifying x^* with Tx for all $x \in F_2 \times I$ ($A_2' \cup A_2'^*$ by identifying x^* with Tx for all $x \in F_2 \times 1$) accordingly. (T, X_1) is a cobordism between (T, F_1) and (T, F_1'). Since X_2 and X_2^* are framed and the framing on $F_2 \times I$ is unaltered, hence, as the above identification on $F_2 \times 0$ gives the framing F_1, the identification on $F_2 \times I$ gives a framing X_1. Therefore, (T, X_1) is a framed cobordism between (T, F_1) and (T, F_1') rel. boundary $W_1 \times I$ (see Figure 4).

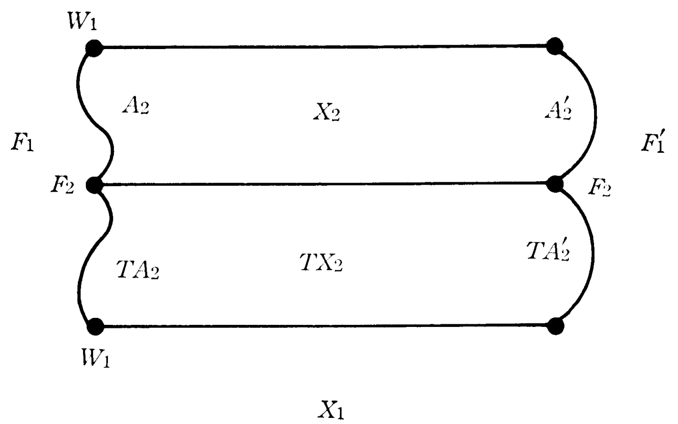


Fig. 4.

By the Van Kampen theorem, it follows that $\pi_1(F_1') = 0$.

Now, $(F_1 \underset{W_1}{\cup} B)$ bounds A_1. We consider the manifold $(A_1 \cup X_1 \underset{F_1}{\cup} B \times I)$. $F_1' \cup B \times 1$ bounds it. Perform framed 1-surgeries on it rel. boundary. Let X be its cobordism. Then, X is viewed as a cobordism between A_1 and a 1-connected manifold A_1' with boundary $F_1' \cup B \times 1$. Let X^* be another copy of X and denote by (T, V) the manifold with an involution, glued on (T, X_1) as above. Then, by the above remark, (T, V) is a framed cobordism between (T, M) and a (T, M') rel. boundary $\Sigma \times I$, where $M = A_1' \underset{F_1'}{\cup} TA_1'$. It follows from the Van Kampen theorem that $\pi_1(M') = 0$.

Let $M = A \cup TA$, $A \cap TA = F$ and $\Sigma = B \cup TB$, $B \cap TB = W$. Assume that W is $(n-2)$-connected and (M, A, F) are $(k-1)$-connected, $k \leqq (n-2)$.

We consider the exact sequence

$$\longrightarrow H_k(F) \xrightarrow{i_*} H_k(A)+H_k(TA) \xrightarrow{j_*} H_k(M) \longrightarrow 0 .$$

Choose a set of generators $\{\alpha_i\}$ in $H_k(M)$. Since j_* is onto, we have elements $\{a_i\}\subset H_k(A)$ and $\{a_i^*\}\subset H_k(TA)$ such that $j_*(a_i, a_i^*)=\alpha_i$. $\{Ta_i^*\}$ (resp. $\{Ta_i\}$) are contained in $H_k(A)$ (resp. $H_k(TA)$). We choose $\{b_j\}$ which generate $\{a_i, Ta_i^*\}$ in $H_k(A)$ so that $\{Tb_j\}$ generate $\{a_i^*, Ta_i\}$ in $H_k(TA)$.

Given $x\in H_k(A)$, by general position, we can find an embedded sphere S^k which induces x on homology. By the dimensional reason, we can assume that $S^k\cap TS^k=\phi$. Hence we can represent b_j by a disjoint embedded sphere S_j^k inside A so that TS_j^k represents Tb_j. Perform framed surgeries on $\{S_j^k\}\subset A$ rel. boundary, we have a framed cobordism Y between A and A' such that $\{b_j\}$ vanish in $H_k(A')$. Noting that Y has $F\times I$ as a boundary part, we define an involution T to be $T(x, t)=(Tx, t)$ on $F\times I$. Let Y^* be a copy of Y and denote by (T, Z) the manifold with an involution, obtained from $Y\cup Y^*$ glued on $(T, F\times I)$. Then, by the previous remark (T, Z) is a framed cobordism between (T, M) and (T, M) rel. boundary $\Sigma\times I$. It is easily checked that $H_k(M')=0$. And thus, the above sequence becomes

$$\longrightarrow H_{k+1}(M') \longrightarrow H_k(F) \longrightarrow H_k(A')+H_k(TA') \longrightarrow 0 .$$

Let $\operatorname{Ker} i_{A'}=\operatorname{Ker}\{i_*: H_k(F)\to H_k(A')\}$. We note that $\operatorname{Ker} i_{TA'}=T\cdot\operatorname{Ker} i_{A'}$. Again, by general position and the above remark, we can represent a generator $\alpha\in\operatorname{Ker} i_{A'}$ by an embedding $(D^{k+1}, S^k)\to(A', F)$ such that S^k lies inside F and $S^k\cap TS^k=\phi$. Denote the tubular neighborhood of (D^{k+1}, S^k) by $N(\alpha)$ which does not meet $TN(\alpha)$. Put $A''=(A'-N(\alpha))\cup TN(\alpha)$. Then, $A''\cup TA''=M'$, $A''\cap TA''=F'$ and α is killed in $\operatorname{Ker} i_{A''}$. A finite number of iterations kills the generators of $\operatorname{Ker} i_{A'}$. Denote the resulting manifold (A_1, F_1) so that $M=A_1\cup TA_1$, $A_1\cap TA_1=F_1$, $(T, \Sigma)=\partial(T, M)$. It follows that $H_k(F_1)=H_k(A_1)=0$. Hence by induction we can show that (M, A, F) are $(n-2)$-connected.

We consider the exact sequence

$$\begin{array}{ccccccc}
\longrightarrow H_{n-1}(F^{2n-1}) & \longrightarrow & H_{n-1}(A^{2n})+H_{n-1}(TA^{2n}) & \longrightarrow & H_{n-1}(M^{2n}) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & & & \\
0\longrightarrow H_{n-1}(W^{2n-2}) & \longrightarrow & H_{n-1}(B^{2n-1})+H_{n-1}(TB^{2n-1}) & \longrightarrow & 0 . & &
\end{array}$$

Since $\dim A=2n$, we can perform framed surgery on A rel. boundary to make A $(n-1)$-connected. By the above remark we have an $(n-1)$-connected manifold with an involution (T, M') such that $\partial(T, M)=(T, \Sigma)$. This proves Lemma 4.1.2.

4.2. Under the situation, it follows by the Poincaré duality that $H_n(M)$ is a $Z[Z_2]$-module with a finite Z-basis. The structure of $Z[Z_2]$-modules with a Z-basis is determined in [8, § 74]. Then it implies that

(*) $H_n(M)$ is isomorphic as a $Z[Z_2]$-module to $mY+nA+l\Lambda$ for some integers m, n, l. Here Y is the trivial representation. A is the representation given by $T=-1$, and Λ is the regular representation.

THEOREM 4.2.1. *Assume $n\geqq 4$. There exists a standard involution (T', Σ'^{2n-1}) which is normally cobordant to (T, Σ^{2n-1}) and there is an $(n-1)$-connected $2n$-dimensional parallelizable manifold (T', M'^{2n}) with $(T', \Sigma')=\partial(T', M')$ such that $H_n(M)$ has no regular representation.*

Furthermore, we have the following result.

COROLLARY 4, 2, 2, *Under the above situation,*

$$\text{if } n=2k, \qquad H_{2k}(M')=mT$$

and

$$\text{if } n=2k+1, \qquad H_{2k+1}(M')=mA \text{ for some } m.$$

PROOF OF THEOREM 4.2.1.

From Lemma 4.1.2 the non-vanishing homology groups are those in the sequences

$$(1) \qquad 0\longrightarrow H_n(F)\longrightarrow H_n(A)+H_n(TA)\longrightarrow H_n(M)\longrightarrow H_{n-1}(F)\longrightarrow 0$$

$$(2) \qquad 0\longrightarrow H_n(F)\longrightarrow H_n(F, W)\longrightarrow H_{n-1}(W)\longrightarrow H_{n-1}(F)\longrightarrow$$
$$\longrightarrow H_{n-1}(F, W)\longrightarrow 0\,.$$

First we prove the following for the sequence (2).

We can perform equivariant surgeries on $H_{n-1}(\mathrm{F}, \mathrm{W})$ to yield the sequence,

$$(2)' \qquad 0\longrightarrow H_n(F', W')\longrightarrow H_{n-1}(W')\longrightarrow H_{n-1}(F')\longrightarrow 0\,.$$

Here all the groups are $Z[Z_2]$-modules with a finite Z-basis and the surgeries on W have the effect on trivial framed embedding of S^{n-2} in W, i.e., $W'=W\underset{Z_2}{\#}r(S^{n-1}\times S^{n-1})$ for some r.

For this, $H_{n-1}(F, W)$ is a finitely generated $Z[Z_2]$-module. For a generator $a\in H_{n-1}(F, W)$, take a $a'\in H_{n-1}(F)$ whose image is a. Reperesent a' by a framed embedding $S^{n-1}\rightarrow F$. Connecting S^{n-1} by a tube to W, we can represent a by a framed embedding $(D^{n-1}, S^{n-2})\rightarrow(F, W)$. Since $S^{n-2}\cap TS^{n-2}=\phi$, we have also $(D^{n-1}\times D^n)\cap T(D^{n-1}\times D^n)=\phi$. Hence, $H_{n-1}(F, W)$ is generated by framed disjoint embeddings

$$\bigcup_i \left\{ \begin{array}{l} (D_i^{n-1}, S_i^{n-2}) \longrightarrow (F, W) \\ T(D_i^{n-1}, S_i^{n-2}) \longrightarrow (F, W) \end{array} \right\}.$$

Let H be the union of the handles $:=\bigcup_i ((D^{n-1}\times D^n)\cup T(D^{n-1}\times D^n))\subset F^{2n-1}$ and $\bigcup_i ((S^{n-2}\times D^n)\cup T(S^{n-2}\times D^n))\subset W^{2n-2}$. Set $F'=F-\operatorname{int} H$. Then,

(4.2.3) (F, W) is normally cobordant to $(F', \partial F')$. Note that this cobordism is $F\times I$ and P, where $\partial(F\times I)=F\times 0\cup P\cup F'$, $\partial P=W\cup \partial F'$. Consider the exact sequence of the triad $(F, H\cup \partial F', W)$,

$$\longrightarrow H_{n-1}(H\cup\partial F', W)\longrightarrow H_{n-1}(F, W)\longrightarrow H_{n-1}(F, H\cup\partial F')\longrightarrow$$
$$\longrightarrow H_{n-2}(H\cup\partial F', W)\longrightarrow 0\,.$$

Here $(H, H\cup W)=$a collection of copies $\{(D^{n-1}\times D^n, S^{n-2}\times D^n)$ and $(T(D^{n-1}\times D^n), T(S^{n-2}\times D^n)\}$ and $(H, H\cap W)\to(H\cup\partial F', W)$ is an excision. Then, we have $H_i(H\cup\partial F', W)=0$ for $i\neq n-1$. Since $H_{n-1}(H\cup\partial F', W)\to H_{n-1}(F, W)$ is onto, it follows that $H_{n-1}(F, H\cup\partial F')=0$. So by excision, $H_{n-1}(F', \partial F')=0$. Since $H_i(F, W)=0$ for $i\neq n-1$, n, we have $H_i(F', \partial F')=0$ for $i\neq n$. Hence it follows that

$$0\longrightarrow H_n(F', \partial F')\longrightarrow H_{n-1}(\partial F')\longrightarrow H_{n-1}(F')\longrightarrow 0\,.$$

Put $W'=\partial F'$, then W' is obtained from W by performing surgery on trivial $(n-2)$-spheres equivariantly, i.e., $W'=W\underset{Z_2}{\#}r(S^{n-1}\times S^{n-1})$ for some r. This yields (2)$'$.

Define an involution T on $F\times I$ to be $T\times 1$, then the above surgery implies that the cobordism $(T, F\times I, P)$ between F and F' (W and W') is invariant under T. By the previous remarks, there is a (T, M') with $\partial(T, M')=(T, \Sigma)$ and $\Sigma=B'\cup TB'$, $B'\cap TB'=W'$, $M'=A'\cup TA'$ and $A'\cap TA'=F'$ (note that Σ is unaltered, because W' is obtained from W by trivial surgeries). The sequence (1) becomes

(1)$'$ $\qquad 0\longrightarrow H_n(A')+H_n(TA')\longrightarrow H_n(M')\longrightarrow H_{n-1}(F')\longrightarrow 0\,.$

Under the situations (1)$'$, (2)$'$, $H_{n-1}(W')$ is a free $Z[Z_2]$-module, while suppose that

$$H_n(F', W')=mY+nA+l\Lambda$$

Notice that $H_{n-1}(F')=mA+nY+l\Lambda$, i.e., $H_n(F', W')$ and $H_{n-1}(F')$ are mutually Λ^--isomorphic. Then we want to surgery on (F', W') so that $l\Lambda$-summand vanish in $H_n(F', W')$. By $n\geqq 4$, we can represent $l\Lambda$-summand in $H_n(F', W')$ by framed embeddings

$$\left\{ \begin{array}{l} (D^n, \partial D^n) \longrightarrow (F'^{2n-1}, W') \\ (TD^n, \partial TD^n) \longrightarrow (F'^{2n-1}, W') \end{array} \right\},$$

so that their boundaries define disjoint embeddings $S^{n-1}\to W'$ and $TS^{n-1}\to W'$ (see [33, p. 41-42]). And then attach corresponding n-handles to F'. If we denote the resulting manifold by (F_1^{2n-1}, W_1^{2n-2}), then it follows that

$$(3)\qquad 0\longrightarrow H_n(F_1)\longrightarrow H_n(F_1, W_1)\longrightarrow H_{n-1}(W_1)\longrightarrow H_{n-1}(F_1)\\ \longrightarrow H_{n-1}(F_1, W_1)\longrightarrow 0\,.$$

Here $H_n(F_1, W_1)\,(H_{n-1}(F_1))$ are unaltered. $H_n(F_1)$ is a free Λ-module with one base element corresponding to each handle (represented by the core of the dual handle so that $H_{n-1}(F_1, W_1)=l\Lambda$). Since the surgery on W_1 to return to W' is made on trivial $(n-2)$-spheres, we note that

$$(4)\qquad W' = W_1 \mathop{\#}_{Z_2} l(S^{n-1}\times S^{n-1})\,.$$

Choose a set of generators $\{e_i, Te_i\}$ in $H_{n-1}(F_1)$ corresponding to those of $H_{n-1}(F_1, W_1)$ and perform surgery on the elements $\{e_i, Te_i\}$. Let F_2 be the resulting manifold. Then, we have

$$(5)\qquad 0\longrightarrow H_n(F_2, W_1)\longrightarrow H_{n-1}(W_1)\longrightarrow H_{n-1}(F_2)\longrightarrow 0$$

and

$$H_{n-1}(F_2) = mA+nY\,.$$

We consider the geometry of W_1. It follows from Lemma 4. 1. 1 and (4) that

$$\text{(i)}\quad (T, W_1^{4k-2}) = (T, S^{4k-2}) \mathop{\#}_{Z_2} l_1(S^{2k-1}\times S^{2k-1})\,,\qquad n=2k$$

$$\text{(ii)}\quad (T, W_1^{4k}) = (T, W_c^{4k}) \mathop{\#}_{Z_2} l_2(S^{2k}\times S^{2k})\,,\qquad n=2k+1\,.$$

In each case $H_{n-1}(W_1)$ has an obvious basis $\{\alpha_i, T\alpha_i, i=1, \cdots, r\}$ such that the matrix $(\alpha_i\cdot T\alpha_j)$ is

$$\left[\begin{array}{c|c} 0 & \begin{matrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{matrix} \\ \hline \begin{matrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{matrix} & 0 \end{array}\right]\quad \left[\begin{array}{c|c} 0 & \begin{matrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{matrix} \\ \hline \begin{matrix} & & -1\\ & \cdot^{\cdot^{\cdot}} & \\ -1 & & \end{matrix} & 0 \end{array}\right]$$

with respect to $n=2k$ and $2k+1$. Hence, performing framed surgery on $\{\alpha_i\}$ and attaching disks to the boundary of the trace, we have a (T',Σ') which is normally cobordant to (T,Σ). (T,Σ) has a form such that $W_1= B_1\cap T'B_1$, $B_1\cup T'B_1=\Sigma'$ and $\alpha_i\in \operatorname{Ker} i_{B_1}=\operatorname{Ker}\{i_*: H_{n-1}(W_1)\to H_{n-1}(B_1)\}$. By the preceding remark there is a (T', M') with $\partial(T', M)=(T',\Sigma')$ such that $M'=A_1\cup T'A_1$, $A_1\cap T'A_1=F_2$. Then the non-vanishing homology groups are

$$(6) \qquad 0 \longrightarrow H_n(A_1) + H_n(T'A_1) \longrightarrow H_n(M') \longrightarrow H_{n-1}(F_2) \longrightarrow 0$$

$$(7) \qquad 0 \longrightarrow H_n(F_2, W_1) \xrightarrow{\partial} H_{n-1}(W_1) \longrightarrow H_{n-1}(F_2) \longrightarrow 0$$

Furthermore,
(*) We can take A_1 such that the elements corresponding to $H_n(A_1)$ (resp. $H_n(T'A_1)$) and $H_{n-1}(F_2)$ do not intersect in $H_n(M')$.

For (*), we can do surgery on B_1 and F_2 rel. W_1 to make them $(n-1)$-connected since the simply connected surgery obstruction groups are zero. Let B_1' and F_2' be the resultings respectively. Adding their traces along $W_1 \times I$, we have the manifold combined A_1 with their traces whose boundary is $B_1' \underset{W_1}{\cup} F_2'$. Then we can do surgery on it rel. boundary $B_1' \underset{W_1}{\cup} F_2'$ to obtain an $(n-1)$-connected manifold A_1'. If we consider the manifold A_1'' with boundary $B_1 \underset{W_1}{\cup} F_2$ which is obtained from A_1' and their traces along $(B_1' \underset{W_1}{\cup} F_2')$, then the sequence (6) holds for A_1'' and M'' in place of A_1 and M' respectively. And it is easily checked that the elements corresponding to $H_n(A_1'')$ (resp. $H_n(T'A_1'')$) and $H_{n-1}(F_2)$ do not intersect in $H_n(M'')$.

We consider the exact sequence for the pair $(A_1, \partial A_1)$, $A_1 = B_1 \underset{W_1}{\cup} F_2$,

$$0 \longrightarrow H_n(\partial A_1) \longrightarrow H_{n-1}(W_1) \xrightarrow{i_*} H_{n-1}(B_1) + H_{n-1}(F_2)$$
$$\longrightarrow H_{n-1}(\partial A_1) \longrightarrow 0 .$$

We show that i_* is injective. For let $x \in H_{n-1}(W_1)$ and $x = \sum_j a_{ij}\alpha_j + \sum_j b_{ij} T\alpha_j$. Suppose that $i_*(x) = (0, 0)$. Then, $x \in \operatorname{Ker} i_{B_1}$, so $x = \sum_j a_{ij}\alpha_j$ (i. e., all b_{ij} vanish). Since $H_{n-1}(B_1)$ and $H_{n-1}(F_2)$ have no torsion, we may assume that x is indivisible. From (5) and the fact that $H_n(F_2, W_1)$ and $H_{n-1}(F_2)$ are mutually Λ^--isomorphic, $H_n(F_2, W_1)$ consists of a $Z[Z_2]$-basis $\{a_i, b_i\}$ satisfying $Ta_i = a_i$, $Tb_i = -b_i$. There is an element $z \in H_n(F_2, W_1)$ in (7) such that $\partial z = x$. And so if we write $z = \sum_i t_i a_i + \sum_i s_i b_i$, then

$$x - Tx = \sum_j a_{ij}(\alpha_j - T\alpha_j) = \partial z - T\partial z$$
$$= \partial(z - Tz) = 2\Big(\partial\big(\sum_i s_i b_i\big)\Big) .$$

And thus, $2|a_{ij}$, i. e., $2|x$. This contradicts that x is indivisible. Hence $x = 0$. We have $H_n(\partial A_1) = 0$. Then, the non-vanishing homology of $(A_1, \partial A_1)$ lies in the sequnce

$$(8) \qquad 0 \longrightarrow H_n(A_1) \longrightarrow H_n(A_1, \partial A_1) \longrightarrow H_{n-1}(\partial A_1) \longrightarrow 0 .$$

Choose a basis $\{c_i\}$ in $H_n(A_1)$ so that $\{T'c_i\}$ form a basis of $H_n(T'A_1)$. We prove that the intersection matrix on $H_n(A_1)$ is unimodular, and hence that $H_{n-1}(\partial A_1)=0$ from (8). For this, let $\{\bar{c}_i, T'\bar{c}_i\}$ be the images of the map in $H_n(M')$ in (6). Since $H_n(M')$ is free abelian and by the above argument (*), we can take the basis corresponding to those of $H_{n-1}(F_2)$ of (6) such that they do not meet $\bar{c}_i$ and $T'\bar{c}_i$. Let $\{\bar{a}_i, \bar{b}_i\}$ be ones corresponding to those of $H_{n-1}(F_2)=mA+nY$ from (5). Clearly, $\{\bar{c}_i, T'\bar{c}_i, \bar{a}_i, \bar{b}_i\}$ form a Z-basis of $H_n(M')$. By the choice of our basis, the intersection matrix on $H_n(M')$ has the following form

$$\left(\begin{array}{c|c|c|c} (\bar{c}_i\cdot\bar{c}_j) & 0 & 0 & 0 \\ \hline 0 & (T\bar{c}_i\cdot T\bar{c}_j) & 0 & 0 \\ \hline 0 & 0 & (\bar{a}_i\cdot\bar{a}_j) & (\bar{a}_i\cdot\bar{b}_j) \\ \hline 0 & 0 & (\bar{b}_i\cdot\bar{a}_j) & (\bar{b}_i\cdot\bar{b}_j) \end{array}\right).$$

Since the intersection matrix on $H_n(M')$ is unimodular, so $\det(\bar{c}_i\cdot\bar{c}_j)=\pm 1$. By the following commutativity and $i_*(c_i)=\bar{c}_i$,

$$\begin{tikzcd} H_n(A_1)\times H_n(A_1) \arrow[d, "{(i_*, i_*)}"] \arrow[dr, "\cdot"] & \\ H_n(M')\times H_n(M') \arrow[r, "\cdot"'] & Z, \end{tikzcd}$$

it follows that $\det(c_i\cdot c_j)=\pm 1$. We have $H_{n-1}(\partial A_1)=0$ and hence that ∂A_1 is a homotopy sphere.

Let U be the tubular neighborhood of the union of the basis $\{c_i\}$ in A_1. Noting that U is viewed as a plumbing manifold, and adding a trace of 1-surgeries to the boundary, we have a manifold $M_1\subset \operatorname{int} A_1$. Then A_1 splits as $M_1\cup L$, where L is an h-cobordism between ∂M_1 and ∂A_1. Connecting M_1 to $B_1\subset\partial A_1$ by a tube (equivariantly, TM_1 to $TB_1\subset\partial TA_1$), we have $A_1=M_2\cup L_1$ as a splitting (see Figure 5). Then, put $M_2=L_1\cup TL_1$ along (T, F_2). Since $\partial(T, M_2)=(T, \Sigma_1)$ is a homotopy sphere (in fact, $\Sigma_1=\Sigma'\#-2\partial A_1$), Σ_1/T is normally cobordant to Σ'/T'. Consequently, it follows from the Mayer Vietoris sequence of the triad $(M_2, L_1\cup TL_1, F_2)$ that

$$0\longrightarrow H_n(M_2)\longrightarrow H_{n-1}(F_2)\longrightarrow 0,$$

i. e., $H_n(M_2)=mY+nA$. This proves the theorem 4. 2. 1.

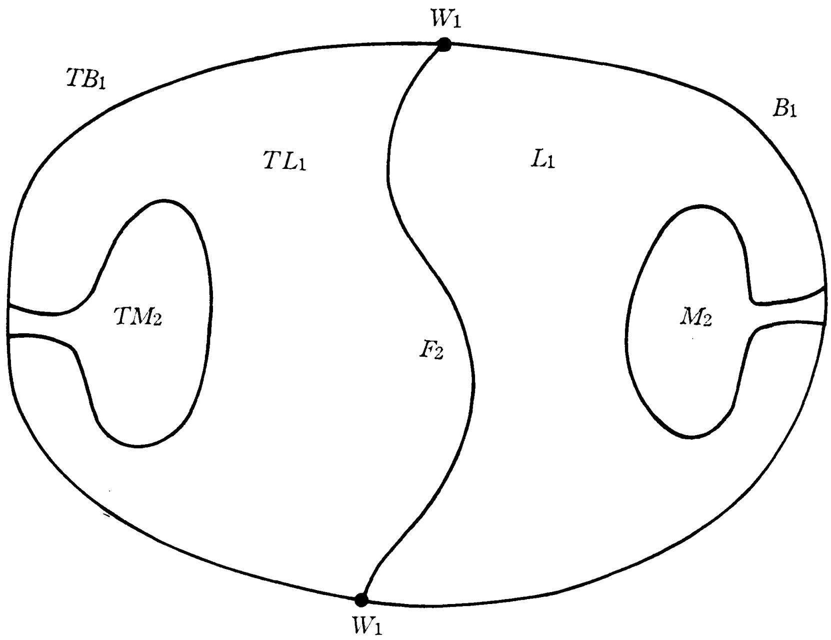


Fig. 5.

PROOF OF COROLLARY 4.2.2.

Suppose that $\partial(T, M)=(T, \Sigma)$ and $H_n(M)=mY+nA$. Let $\{a_i, b_i\}$ be a $Z[Z_2]$-basis of $H_n(M)$, i.e., $Ta_i=a_i$, $Tb_i=-b_i$. Then, $a_i\cdot b_j=Ta_i\cdot b_j=a_i\cdot Tb_j=-a_i\cdot b_j$, so $a_i\cdot b_j=0$. The intersection matrix on $H_n(M)$ is

$$\left(\begin{array}{c|c} (a_i\cdot a_j) & 0 \\ \hline 0 & (b_i\cdot b_j) \end{array}\right),$$

Hence $\det(a_i\cdot a_j)=\pm 1$ and $\det(b_i\cdot b_j)=\pm 1$. Let $n=2k$, then we will show that $H_{2k}(M^{4k})=mY$, i.e., consists of $\{a_i\}$. Represent b_i by an embedded sphere S_i^{2k}. Since $b_1\cdot b_j=Tb_1\cdot Tb_j$, if $S_1\cap S_j\ni x$, then $Tx\in TS_1\cap TS_j$. And so the intersection number $b_1\cdot b_j$ corresponds to such $\{(x, Tx)\}_{x\in S_1}$. Assume $x=Tx$ for some x in $\{(x, Tx)\}_{x\in S_1}$. $S_1\cap TS_1$ consists of pairs (y, Ty). Since $b_1\cdot Tb_1=-b_1\cdot b_1=$ even and $x=Tx$, $S_1\cap TS_1$ must contain some y such that $Ty=y$. Thus two fixed points x, y in M lie in S_1. Then, we have an invariant sphere S_1' which induces b_1 on homology. This follows from the following reason. Let $N(x)$ (resp. $N(y)$) be the tubular neighborhood of x (resp. y) in M which is diffeomorphic to (A, D^{4k}). Choose equivariant embeddings

$$(A, D_+^{2k})\longrightarrow(T, M),$$
$$(A, D_-^{2k})\longrightarrow(T, M)$$

which are homologous to $N(x)\cap S_1$ and $N(y)\cap S_1$ respectively. Then the obstruction to extending the equivariant maps $\partial D_{+}^{2k} \cup \partial D_{-}^{2k} \to M$ to an equivariant map $(A', S^{2k}) \to (T, M)$ lies in $H^2((S^{2k} - \{D_{+}^{2k} \cup D_{-}^{2k}\})/A', \{\partial D_{+}^{2k}, \partial D_{-}^{2k}\}/A'$; $\pi_1((M - \text{Fix } T)/T)) \cong H^2(P^{2k-1} \times I, P^{2k-1} \times \dot{I}\ ; Z_2) = Z_2$. Since T is an involution, we see that the obstruction is zero (cf. Lemma 4. 3. 1). By general position, we have an embedded sphere $f: (A', S^{2k}) \to (T, M)$. This invariant sphere is homologous to b_1 by construction. But we have $Tb_1 = Tf_* [S^{2k}] = f_* A'_* [S^{2k}] = f_* [S^{2k}] = b_1$. This yields a contradicition. Hence $x \neq Tx$. Therefore the intersection number $b_1 \cdot b_j$ which consists of pairs $\{(x, Tx)\}_{x \in S_1}$ is even for all j. It must be $2 | \det(b_i \cdot b_j)$. By the preceding remarks, it does not occur. Hence we conclude that $H_{2k}(M) = mY$. When $n = 2k+1$, the simular argument shows that $H_{2k+1}(M) = mA$.

4. 3. Now we will classify standard involutions.

Let (T, Σ^{2n-1}) be a standard one. Suppose that $\partial(T, M^{2n}) = (T, \Sigma)$. By Theorem 4. 2. 1 and Corollary 4. 2. 2, we can assume that M is $(n-1)$-connected and moreover,

(i) $H_{2k}(M^{4k}) = mY$,

(ii) $H_{2k-1}(M^{4k+2}) = mA$.

Applying the Smith theory to M^{2n}, it follows that $\chi(\text{Fix}(T, M)) = m+1$. $\text{Fix}(T, M) = \bigcup_{i=1}^{m+1} \{q_i\}$. The following result is a special case of [34, § 2] but the smooth version of which holds only in this case.

LEMMA 4. 3. 1. *Suppose that $n = 2k$ and $k \geqq 2$. Let S^{2k} be the unit sphere in R^{2k+1} with the Z_2-action, $A'(x_1, x_2, \cdots, x_{2k}, y) = (-x_1, -x_2, \cdots, -x_{2k}, y)$. Then, there exists an equivariant embedding of (A', S^{2k}) into (T, M). Moreover, $H_{2k}(M, Z)$ has a basis consisting of classes represented by invariant embedded spheres.*

SKETCH OF PROOF. This has been proved by several stages. Let D^{4k} be the unit disk in R^{4k} with the Z_2-action, $A(x_1, \cdots, x_{4k}) = (-x_1, \cdots, -x_{4k})$. Each fixed point has the neighborhood N which is equivariantly diffeomorphic to (A, D^{4k}). We can take an equivariant embedding of (A, D^{2k}) into the neighborhood N of each fixed point. Fix any two fixed points $\{q, q'\} \subset F$. And put $N' = (A, D_{+}^{2k}) \cup (A, D_{-}^{2k})$, the first factor representing the embedding into $N(q)$ and the second representing the embedding into $N(q')$. Put $U = S^{2k} - N'$. We would like to extend the embedding of $\partial U/A$ to a map of U/A into $M - F/T$. Since $\pi_i((M-F)/T) = 0$ for $1 < i < 2k$, there is an obstruction in $H^2(U/A, \partial U/A\ ; Z_2) = Z_2$. It is easily seen that the obstruction is zero (see Lemma 2. 3 [34]). Hence we have an equivariant map of (A', S^{2k})

into (T, M). By general position, we can make this map to an equivariant immersion. This immersion has no singularities other than in the free part of the action, therefore we can make this immersion equivariantly into an embedding.

Let $F=\left\{\bigcup_{i=1}^{m+1} q_i\right\}$. Let $S_{i,j}^{2k}$ be an invariant embedded sphere constructed as above which contains q_i and q_j. Then, it follows by [Theorem 2.4, [34]] that $[S_{i,j}^{2k}]_2\in H_{2k}(M, Z_2)$ is well defined and $[S_{i,j}^{2k}]\in H_{2k}(M, Z)$ may be varied by any element of Kernel $\{H_{2k}(M, Z)\rightarrow H_{2k}(M, Z_2)\}$. That is, if $x\in H_{2k}(M)$ and $2|x$, then we obtain an invariant embedded sphere representing the homology class $[S_{i,j}^{2k}]+x$.

Let $S_i^{2k}=S_{i,1}^{2k}$, $i=2, \cdots, m+1$. S_1^{2k}/T is the suspension of P_1^{2k-1}, i.e., $c(P_1^{2k-1})\cup\bar{c}(P_1^{2k-1})$. Then, we have the following commutative diagram.

$$\begin{tikzcd}
 & H_{2k}\left(\bigcup_{i=2}^{m+1} S_i^{2k}/T\right) \arrow[d, equal] & & \\
0 \arrow[r] \arrow[d] & H_{2k}\left(\bigcup_{i=2}^{m+1} c(P_i)\cup\bigcup_{i=2}^{m+1}\bar{c}(P_i)\right) \arrow[r] \arrow[d, "i_*"] & H_{2k-1}\left(\bigcup_{i=2}^{m+1} P_i\right) \arrow[r] \arrow[d] & 0 \\
H_{2k}\left(M-F/T\cup c(\partial N(q_1)/T)\right) \arrow[r] \arrow[d, equal] & H_{2k}\left(M-F/T\cup\bigcup_{i=1}^{m+1} c(\partial N(q_i)/T)\right) \arrow[r] \arrow[d, equal] & H_{2k-1}\left(\bigcup_{i=2}^{m+1} c(\partial N(q_i)/T)\right) \arrow[r] & 0. \\
H_{2k}\left(M-F/T, \partial N(q_1)/T\right) & H_{2k}\left(M-F/T, \bigcup_{i=1}^{m+1}\partial N(q_i)/T\right). & &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Z^m \arrow[r, "id"] \arrow[d, "i_*"] & Z^m \arrow[r] \arrow[d, "\bmod 2"] & 0 \arrow[d] \\
Z^m \arrow[r, "(2\times)^m"] & Z^m \arrow[r, "\bmod 2"] & Z_2^m \arrow[r] & 0.
\end{tikzcd}$$

Here $P_i=P_i^{2k-1}$.

Hence, i_* is congruent mod 2 to the identity matrix: $Z^m\rightarrow Z^m$. But then, by using the fact that the map $SL_m(Z)\rightarrow SL_m(Z_n)$ for any integers m and n, induced by reduction mod n, is onto and the above remark of changing the invariant spheres by arbitrary 2-divisible elements of $H_{2k}(M)$, we can make i_* into an isomorphism. Therefore, lifting to M, $H_{2k}(M)$ has a basis cosisting of invariant spheres.

COROLLARY 4.3.2. *Suppose that* $n=2k+1$ *and* $k\geqq 2$. *Let* S^{2k+1} *be the* *unit sphere in R^{2k+2} with the Z_2-action, $A'(x_1, \cdots, x_{2k+1}, y)=(-x_1, \cdots, -x_{2k+1}, y)$. Then, there exists an embedding of (A', S^{2k+1}) into (T, M^{4k+2}). $H_{2k+1}(M, Z)$ has a basis cosisting of classes represented by invariant embedded spheres.*

Proof. As in the proof of Theorem 4. 2. 1, we have the following exact sequence

$$0 \longrightarrow H_{2k+1}(M^{4k+2}) \xrightarrow{\partial} H_{2k}(F^{4k+1}) \longrightarrow 0\,,$$

where $F^{4k+1}=A\cap TA$, $A\cup TA=M$. F is $(2k-1)$-connected. We notice that if $H_{2k+1}(M)=mA$, then $H_{2k}(F)=mY$. We apply Lemma 4. 3. 1 to F. Then, $H_{2k}(F)$ has a basis consisting of invariant embedded spheres. Let $f:(A', S^{2k})\to(T, F)$ be an embedding. f extends to an embedding $\bar{f}: D^{2k+1}\to A$ from the above sequence. Then, there is an embedding $g:(A', S^{2k+1})\to(T, M^{4k+2})$ glued on (A', S^{2k}). Since $\partial g_*[S^{2k+1}]=f_*[S^{2k}]$ and ∂ is an isomorphism, $H_{2k+1}(M, Z)$ consists of a basis represented by invariant embedded spheres.

4. 4. We will state our classification theorems.

Theorem 4. 4. 1. *Let T be a free involution on a homotopy sphere $\Sigma^{4k-1}\in bP_{4k}(k\geqq 2)$. Suppose that T extends to an involution with isolated fixed points on a $(2k-1)$-connected $4k$-dimensional parallelizable manifold M^{4k} and such that the induced action is trivial on the homology $H_{2k}(M)$. Then, (T, Σ^{4k-1}) is equivariantly diffeomorphic to the unique representative element of the table 2.*

Remark 4. 4. 2. The effect on the spin invariant. It will be shown that the spin invariant for (T, Σ^{4k-1}) must have the formula $\sigma(T, \Sigma^{4k-1})=\pm(8m\pm 1)$ mod 2^{2k}. If $\sigma(T, \Sigma^{4k-1})=h$, then choosing (T_h, Σ_h^{4k-1}) within the classes of the form $\pm(8m\pm 1)$ in the table 2, it is proved that $(T, \Sigma)\cong(T_h, \Sigma_h)$.

We have the main theorem for the classification of standard involutions.

Theorem 4. 4. 3. *Let T be a standard involution on a homotopy sphere $\Sigma^{4k-1}\in bP_{4k}(k\geqq 2)$. Then, (T, Σ) is equivariantly diffeomorphic to the equivariant connected sum of the definite element $\Sigma'\in bP_{4k}$ with the unique representative (T_h, Σ_h) in the table 2, i. e., the quotient $\Sigma/T\cong\Sigma_h/T_h\#\Sigma'$.*

The proof of (4. 4. 1) is carried out by the following steps.

Step 1. Geometry of M.
Step 2. Normal cobordism class of (T, Σ).
Step 3. Determination of (T, Σ).

4. 5. Step 1.

4. 5. 1. Suppose that $H_{2k}(M)=mY$, where Y is the trivial representation and m is a non-negative integer.

Let $\Phi : H_{2k}(M) \times H_{2k}(M) \to Z$ be the intersection form. It follows by the Poincaré duality that Φ is unimodular, even, symmetric and Z_2-invariant. We have a basis consisting of invariant embedded spheres $\{a_i, i=1, \cdots, m\}$ from Lemma 4.3.1. Let U be an invariant tubular neighborhood of the union of the spheres $\{a_i\}$. U is the sum of components, $U = U_1 \cup \cdots \cup U_h$. Then Φ is written as a sum of blocks $(\Phi | U_i)$, i.e.,

$$\Phi \cong \left(\begin{array}{c|c|c} \Phi | U_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \Phi | U_h \end{array} \right).$$

We notice that if one forgets the action, then each U_i is viewed as a plumbing manifold and $\Phi | U_i$ as its plumbing matrix. In our case, it may be impossible, because not only two spheres but several spheres meet in one point. That is, suppose that two spheres meet at a point and let us assume that another sphere meets at that point. If it is a free point of the action, then by general position, we can make it into a double point. We can do this equivariantly when intersection points are free. While, an intersection point is a fixed point of the action, we cannot make it into a double point equivariantly. So, it may be possible for several spheres to meet at the fixed point. However in this case, it has the same homotopy type as a join of spheres. Thus, in each case it follows similarly to the plumbing theory that for each j,

(1) (i) $\pi_1(\partial U_j) \cong \pi_1(U_j)$ is free.

(ii) $H_i(\partial U_j) = H_i(U_j) = 0,\ 1 < i < 2k-1$ and $H_{2k-1}(U_j) = 0$.

Put

$$(2) \qquad X_l = M - \bigcup_{i=1}^{l} \operatorname{int} U_i \ (1 \leqq l \leqq h).$$

Then T acts freely on X_h.

LEMMA (3). *X_h is a trace of "equivariant 1-surgeries". For any generator $\alpha \in \pi_1(\partial U_j)$ it follows that $T\alpha \neq \alpha$ for each j.*

PROOF of (3). We may prove that the generators of $\pi_1(\partial U_j)$ are killed equivariantly in X_h. The generators of $\pi_1(\partial U_j)$ correspond to the components of the intersections for the union which gives ∂U_j. If α is a generator of $\pi_1(\partial U_j)$, we represent α by an embedded sphere S^1 in ∂U_j. Then it is sufficient to prove that

(*) S^1 and TS^1 do not belong to same component of intersections.

Suppose that S^1 and TS^1 belong to the same component. Since a generator arises in the components of intersections, so S^1 and TS^1 induce the same generator such that $\alpha = T\alpha$ in $\pi_1(\partial U_j)$. On the other hand, we have $X_1 \cup U_1 = M$ by (2). It follows by the Van Kampen theorem and (i) that $\pi_1(X_1)=0$. We can show inductively that $\pi_1(X_l)=0$ for $1 \leqq l \leqq h$. From the Mayer-Vietoris sequence of the triad (M, X_1, U_1) and (ii), it is easily seen that $H_i(X_1)=0$ for $0<i<2k$, and inductively $H_i(X_l)=0$ for $0<i<2k$, $1 \leqq l \leqq h$. Hence X_h is a trace of 1-surgeries (see [4, p. 119]). We notice that the surgery on ∂U_j does not affect any homology groups other than the first homology group. Now, S^1 is killed in X_h, so S^1 bounds a 2-disk $D^2 \subset X_h$. Assume that $TS^1 \neq S^1$ (i. e., S^1 is not an invariant sphere). Since X_h is invariant under T, TS^1 must bound $TD^2 \subset X_h$. Put

$$W = \partial U_j \times I \cup (D^2 \times F^{4k-2}) \cup T(D^2 \times D^{4k-2}) \subset X_h$$

which is a trace of S^1 and TS^1. Noting the above remark, $H_2(W)$ is a summand in $H_2(X_h)$, so that $H_2(W)=0$. W has the homotopy type of $(\partial U_j \cup D^2 \cup TD^2)$. By the hypothesis of $\alpha = T\alpha$, $(\partial U_j \cup D^2 \cup TD^2)$ has the homotopy type of $\partial U_j \cup S^2$. Hence, $H_2(W) = H_2(\partial U_j \cup S^2) \cong Z$. This yields a contradiction. While S^1 is an invariant sphere, i. e., $TS^1 = S^1$, then S^1 bounds an invariant disk $D^2 \subset X_h$. But T acts freely on X_h, hence this case does not occur. Therefore, (*) is proved.

It follows from (*) and (1) that $\pi_1(\partial U_j)$ is a free $Z[T]$-module for each j. Then, performing "equivariant 1-surgeries" on ∂U_j for each j, we obtain X_h. This proves (3).

4. 5. 2. We investigate the geometry of U_j.

Put $U = U_j$ for convenience. Suppose that two invariant spheres a_1, a_2 are contained in U. If a_1 and a_2 meet only at free points of the action, then they meet transversely at even points. So, there exists actually an odd number of generators in $\pi_1(\partial U)$. Within them, we can find a generator α such that $T\alpha = \alpha$. But by (3) it does not occur. If they meet at the two fixed points on each, we have also a generator α such that $T\alpha = \alpha$. Then, by the same reason there exists no such α. Hence, if a_1 and a_2 meet, then they meet transversely at only one fixed point on each. Once they meet at the fixed point, they may meet at free points of the action on each. If we assume further, that $a_3 \in U$ and a_2 meets a_3, then only two cases occur : the first is that a_2 and a_3 meet at the used fixed point of a_2, and the second is that they meet at the unused fixed point of a_2. a_1 and a_3 do not meet at the unused fixed point on each, otherwise we can find a generator α such that $T\alpha=\alpha$, so it contradicts (3). a_1, a_2 and a_3 may meet at free points of the action on each (see Figure 6, for example $m=4$).

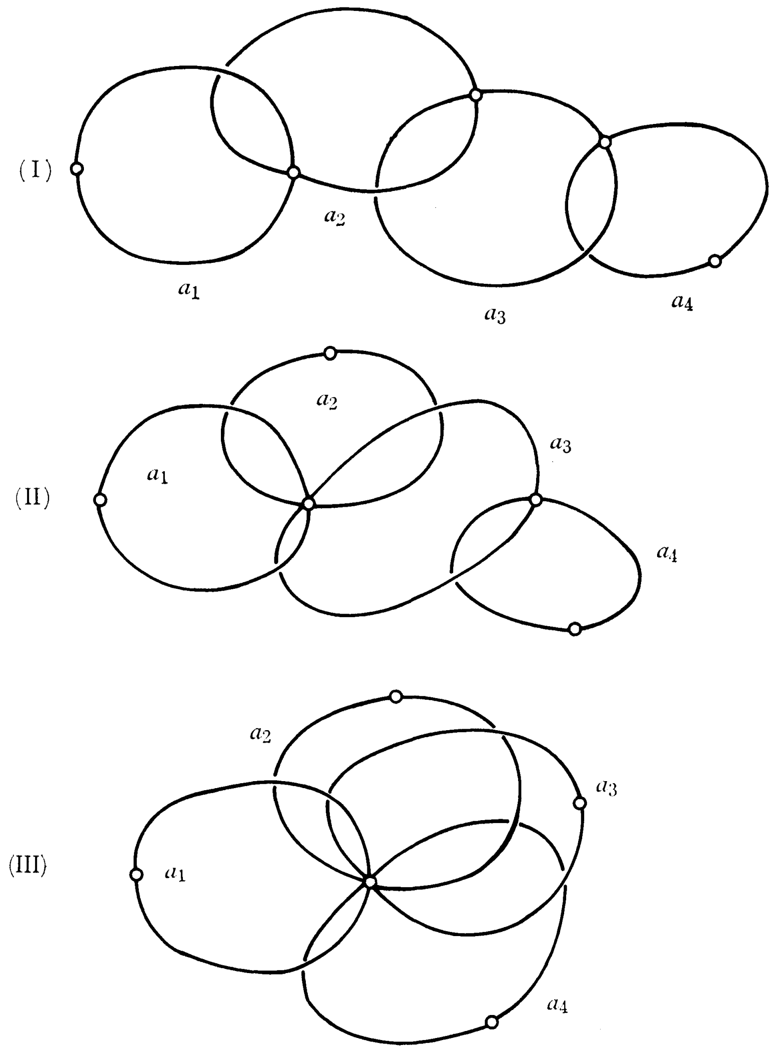


Fig. 6.

When we continue in this way around the fixed points of the action on these embedded spheres, if $a_1, \cdots, a_i \in U$ and a_{i+1} does not meet the preceding a_j's, then $a_{i+1} \notin U$. Because in this care if $a_{i+1} \in U$, a_{i+1} meets some a_j only at free points of the action. So it does not occur as above by the same reason. Thus we can start with a_{i+1} in a different component. Eventually we arrive at h-components, consisting of U_1 containing $\{a_1, a_2, \cdots, a_{i_1}\}$, U_2 containing $\{a_{i_1+1}, \cdots, a_{i_2}\}, \cdots$, and U_h containing $\{a_{i_{h-1}+1}, \cdots, a_{i_h}\}$ so that $a_{i_h}=a_m$.

On the other hand, it follows that $\mathrm{Fix}(T, U_1)=i_1+1$, $\mathrm{Fix}(T, u_2)=(i_2-i_1+1), \cdots, \mathrm{Fix}(T, U_h)=(m-i_{h-1}+1)$. Since $\bigcup_{i=1}^{h} \mathrm{Fix}(T, U_i)=\mathrm{Fix}(T, M)$, it follows that $m+1=m+h$, i. e., $h=1$. Hence, $U=U_1 \cup \cdots \cup U_h=U_1$ is the equivariant connected tubular neighborhood containing all a_i's, $i=1, \cdots, m$ in M.

4.6. Step 2.

4.6.1. Let $(T, U) \subset (T, M)$ be the one as above. It follows by (3) that $M - \operatorname{int} U = X$ is a trace of "equivariant 1-surgeries". Denote by (T_{a_i}, E_{a_i}) the equivariant tubular neighborhood of the invariant sphere a_i in U for each i. Then, by the equivariant tubular neighborhood theorem, (T_{a_i}, E_{a_i}) has, up to (equivariant) isotopy, the following form,

$$(T_{a_i}, E_{a_i}) = D^{2k} \times D^{2k} \underset{b_{f_{a_i}}}{\cup} D^{2k} \times D^{2k}.$$

Here the action on $D^{2k} \times D^{2k}$ is the diagonal action $A \times A$ and $b_{f_{a_i}} : S^{2k-1} \times D^{2k} \to S^{2k-1} \times D^{2k}$ is an equivariant map defined by $b_{f_{a_i}}(x, y) = (x, f_{a_i}(\pi(x))(y))$, where $f_{a_i} : P^{2k-1} \to SO(2k)$ is a map and $\pi : S^{2k-1} \to P^{2k-1}$ is the projection.

4.6.2. We then wish to identify these bundles (T_{a_i}, E_{a_i}) with the bundles with the Z_2-actions introduced in chapter III. First to do so, we write the bundles in Lemma 3.2. of chapter III by the above form.

We shall recall the characteristic map on the tangent bundle of S^n. Let u_n be the map of S^{n-1} into $SO(n)$ defined by

$$u_n(x) = (\delta_{ij} - 2x_i x_j) \begin{pmatrix} I_{n-1} & \\ & -1 \end{pmatrix},$$

$x = (x_1, \cdots, x_n) \in S^{n-1}$. Then, $u_n \in \pi_{n-1}(SO(n))$ represents the tangent bundle τ_{S^n} over S^n. u_n is invariant under the action, i.e., $u_n(Ax) = u_n(x)$, where $A(x_1, \cdots, x_n) = (-x_1, \cdots, -x_n)$. So, u_n factors through the projection $\pi : S^{n-1} \to P^{n-1}$,

$$\begin{tikzcd} S^{n-1} \arrow[r, "u_n"] \arrow[d] & SO(n) \\ P^{n-1} \arrow[ur, "v_n"'] & \end{tikzcd}$$

Define a map $v_n^h : P^{n-1} \to SO(n)$ for each h by setting $v_n^h([x])(y) = (v_n([x]))^h(y)$ for $x \in S^{n-1}$ and $y \in D^n$. We have a stably trivial bundle over S^n with a Z_2-action T_h,

$$N_{v_n^h} = D^n \times D^n \underset{b_{v_n^h}}{\cup} D^n \times D^n,$$

where $b_{v_n^h} : S^{n-1} \times D^n \to S^{n-1} \times D^n$, $b_{v_n^h}(x, y) = (x, v_n^h([x])\, y)$ and the action on $D^n \times D^n$ is the diagonal action $A \times A$.

If one forgets the action, then it is known (for example, see [23], see also § 4.1 of chapter IV) that

(i) $n=$ even, $N_{v_n^h}$ is a stably trivial bundle over S^n with the euler class $2h$.

(ii) $n=$ odd, and $n\neq 3$, 7, $N_{v_n^h}$ is isomorphic to the tangent disk bundle $E(\tau_{S^n})$ if h is odd and to the trivial bundle of S^n if h is even.

By construction of E in chapter III, the bundles $N_{v_n^h}$ with Z_2-actions are same as E with Z_2-actions. From now on, we identify (T, E) with $(T_h, N_{v_n^h})$ to our need.

4.6.3. We note the following to determine such (T_{a_i}, E_{a_i}). Since M is parallelizable, E_{a_i}'s are stably trivial bundles. So,

(i) if one forgets the action, then E_{a_i} is classified by its euler class. When E_{a_i} and $N_{v_{2k}^h}$ have the same euler class, the difference between them lies in the effect on the action around the two fixed points on each.

(ii) Then, the spin invariants ([1], [2]) make a contribution to distinguish them.

We quote the results of [6], [10].

LEMMA 4.6.4. ([10]). $[P^{n-2}, SO(n)]=KO^{-1}(P^{n-2})$

$$=\begin{cases} Z_2 & n-2\not\equiv -1\,(4) \\ Z_2+Z & n-2\equiv -1\,(4)\,. \end{cases}$$

Consider the cofibration, $S^{n-2}\xrightarrow{\pi} P^{n-2}\xrightarrow{i} P^{n-1}$ and the exact sequence for $SO(n)$,

$$(4.6.5)\quad \pi_{n-1}(SO(n))\xrightarrow{c^*}[P^{n-1}, SO(n)]\xrightarrow{i^*}[P^{n-2}, SO(n)]\xrightarrow{\pi^*}\pi_{n-2}(SO(n))\,,$$

where $c: P^{n-1}\rightarrow S^{n-1}$ is the collapsing map.

LEMMA 4.6.6. ([6, LEMMA (5.4)]). (1) *If we represent a generator of Z_2-summand in $[P^{n-2}, SO(n)]$ by β, then for the map $v_n: P^{n-1}\rightarrow SO(n)$ it follows that $i^*[v_n]=\beta$. In particular, $i^*[v_n^h]=\beta^h$, i.e., $\beta^h=\beta$ (h odd) and $\beta^h=1$ (h even).*

(2) *If $n-2\equiv -1\,(4)$, Z-summand comes from the image c^* of $\pi_{n-2}\,(SO(n))$, $c: P^{n-2}\rightarrow S^{n-2}$. Hence* $\mathrm{Im}\, i^*=\langle\beta\rangle=Z_2\subset[P^{n-2}, SO(n)]$.

For the last assertion, when $f\in \mathrm{Im}\, i^*$ and if f lies in the Z-summand so that $f=C^*(g)$ for some $g: S^{n-2}\rightarrow SO(n)$. Since $n-2\equiv -1\,(4)$, $c\pi: S^{n-2}\rightarrow S^{n-2}$ is of degree 2 and $\pi_{n-2}(SO(n))=Z$, we have by (4.6.5) that $\pi^*(f)=\pi^*c^*(g)=2g=0$, so $g=0$. Hence, $f=0$.

4.6.7. We now identify the bundles (T_{a_i}, E_{a_i}) with $(T_h, N_{v_{2k}^h})$. In (4.6.1) it follows by the equivariant tubular neighborhood theory that

$$(T_{a_i}, E_{a_i})=N_{f_{a_i}}=D^{2k}\times D^{2k}\underset{bf_{a_i}}{\cup} D^{2k}\times D^{2k}\,,$$

where $f_{a_i}\in[P^{2k-1}, SO(2k)]$.

We have by (4.6.5) and (4.6.6) that $i^*(f_{a_i})=\beta^t=i^*(v_{2k}^t)$ for $t=0$ or 1. So, f_{a_i} differs from v_{2k}^t by a map $c^*(g)\in[P^{2k-1}, SO(2k)]$ for some $g\in\pi_{2k-1}(SO(2k))$. Hence it follows as Z_2-bundles that $N_{f_{a_i}}\cong N_{c^*(g)\cdot v_{2k}^t}$. Since $N_{f_{a_i}}$ is stably trivial, $S(f_{a_i}\cdot\pi)=0$ in the following exact sequence of the fibration $SO(2k)\to SO(2k+1)\to S^{2k}$, $\pi_{2k}(S^{2k})\xrightarrow{\partial}\pi_{2k-1}(SO(2k))\xrightarrow{S}\pi_{2k-1}(SO(2k+1))\to$, and hence that $S(c^*(g)\cdot\pi)=0$. Noting that $c\pi$ is of degree 2, then $S(2g)=0$. Thus we have $S(g)=0$ if $2k\not\equiv 2\,(8)$. In this case g is generated by u_{2k}^m for some m by the above exact sequence. Therefore it follows from the diagram below that $c^*(g)=c^*(u_{2k}^m)=v_{2k}^{2m}$,

$$\begin{tikzcd}
S^{2k-1} \arrow[d] \arrow[ddr, "u_{2k}^{2m}"] & \\
P^{2k-1} \arrow[d] \arrow[dr, "v_{2k}^{2m}"] & \\
S^{2k-1} \arrow[r, "u_{2k}^{m}"'] & SO(2k)\,.
\end{tikzcd}$$

And hence if $2k\not\equiv 2\,(8)$, we conclude that $N_{f_{a_i}}\cong N_{v_{2k}^{2m+t}}$ for $t=0$ or 1 and some m. On the other hand, assume $2k\equiv 2\,(8)$. Then $\pi_{2k-1}(SO(2k))=Z+Z_2$ and $\pi_{2k-1}(SO(2k+1))=Z_2$. We can write $N_{f_{a_i}}\cong N_{c^*(g)\cdot v_{2k}^{2m+t}}$ ($t=0$ or 1), where g is the generator of the Z_2-summand in $\pi_{2k-1}(SO(2k))$. Since $c^*(g)\cdot\pi=2g=0$, $N_{c^*(g)}$ is a trivial bundle with a Z_2-action. $N_{f_{a_i}}$ is isomorphic as bundles to $N_{v_{2k}^{2m+t}}$. In order to regard $N_{f_{a_i}}$ as $N_{v_{2k}^{2m+t}}$ with Z_2-actions, we must show that the effect on the action by the map $c^*(g)$ which makes $N_{v_{2k}^{2m+t}}$ into $N_{c^*(g)\cdot v_{2k}^{2m+t}}$ is unaltered. By note (ii) of (4.6.3), it is sufficient to prove that the spin invariant

(4.6.8) $\alpha(T, N_{c^*(g)})=0$, i.e.,

(4.6.9) $\alpha(T, N_{c^*(g)\cdot v_{2k}^{2m+t}})=\alpha(T, N_{v_{2k}^{2m+t}})$.

PROOF OF (4.6.8). c is the collapsing map, so we have a map h,

$$\begin{array}{ccccc}
P^{2k-1} & \xrightarrow{\ c\ } & S^{2k-1} & \xrightarrow{\ g\ } & SO(2k)\\
\cup & & \cup & & \cup\\
h: P^{2k-2} & \longrightarrow & pt & \longrightarrow & SO(2k-1)\,,
\end{array}$$

which is the trivial map of P^{2k-2} into $SO(2k-1)$. Then, there is an equivariant embedding

$$\alpha:(T, N_h^{4k-2})\subset(T, N_{c^*(g)}^{4k})$$

which has the trivial normal bundle. Choosing a specific field of normal

2-frames, we have an embedding of the spaces of oriented orthonomal $4k(4k-2)$-frames,

$$F(\alpha): FN_h^{4k-2} \subset FN_{c^*(g)}^{4k}.$$

Then the embedding induces an isomorphism of Z_2 into itself

$$F(\alpha)_* : \pi_1(FN_h^{4k-2}) \longrightarrow \pi_1(FN_{c^*(g)}^{4k}).$$

Since h is the trivial map, it follows that

$$(T, N_h^{4k-2}) \cong (A' \times A,\ D^{2k-1} \times D^{2k-1}).$$

Thus, we have $\sigma(T, N_h^{4k-2})=0$, i. e., the two fixed points have the different sign. It follows from the proposition 8. 44 [1] that the certain loop c joining the two fixed points is non-trivial in $\pi_1(FN_h^{4k-2})=Z_2$. The above isomorphism $F(\alpha)_*$ maps the loop c to the corresponding loop in $FN_{c^*(g)}^{4k}$. So, $c \neq 0$ in $\pi_1(FN_{c^*(g)}^{4k})$. Again by proposition 8. 44 [1] it follows that $\sigma(T, N_{c^*(g)}^{4k})=0$. Thus,

(4. 6. 10) In each case we can identify as Z_2-bundles (T_{a_i}, E_{a_i}) with $(T_h, N_{v_{2k}^h})$ for some h. Here we notice that $(T_h, N_{v_{2k}^h})$ is the bundle with the Z_2-action (T, E) introduced in chapter III (see (4. 6. 2)).

4. 6. 11. We turn to the situation (4. 6. 1). When we identify the bundle (T_{a_i}, E_{a_i}) with the bundle (T, E) of chapter III by the above remark, we are able to apply the corresponding Lemma 3. 3, 4. 5 of chapter III to (T_{a_i}, E_{a_i}). Let $(T, \Sigma)=\partial(T, M)$ and (T, U), (T, M) be as above. Then we have

LEMMA 4. 6. 12. *There exists a normal cobordism $F: V \rightarrow P^{4k-1}$ between $\partial U/T$ and $\{m_1(P^{4k-1}, id) \cup m_2(P^{4k-1}, c \times 1)\}$ for some m_1, m_2 such that $m_1+m_2=m+1$, where $V=(U-\text{int}\, N((m+1)\, pts))/T$ so that $U=\tilde{V} \cup N((m+1)\, pts))$, and $c \times 1$ is defined in Lemma 3. 3 of chapter* III.

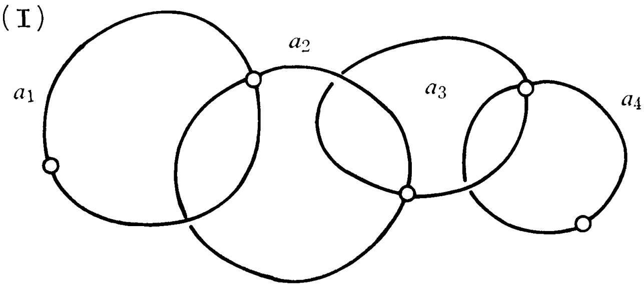
PROOF. We have already determined the structure of U in step 1. So, according to that of U and the preceding remarks, the proof goes by ad-hoc argument just as in Lemma 3. 4 of chapter III. We use the same notation of (3. 4). a_1 meets a_2 at a fixed point on each, i. e., E_{a_1} is equivariantly plumbed with E_{a_2} at the fixed point. Denote the resulting manifold M' when E_{a_1} and E_{a_2} are plumbed. Then, the resulting cobordism

$$V' = M' - \text{int}\,\{N_1 \cup N_2 \cup N_2'\}/T$$

defines a normal cobordism. Here the first N_1, N_2 are in E_{a_1} and the last N_2' in E_{a_2}, and N_1' in E_{a_2} is identified with N_2 in E_{a_1}. Assume that a_3 meets a_2 at a fixed point. There are two possibilities: they meet at the

EXAMPLES. (I), (II), (III) are examples of standard involutions which are all diffeomorphic to (A, S^{4k-1}).

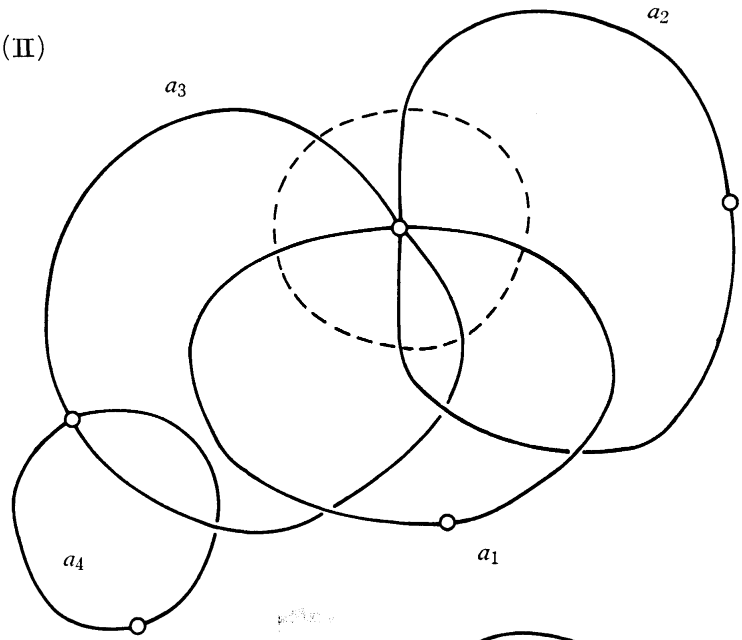
(I)



all a_i's are trivial bundles
plumbing matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

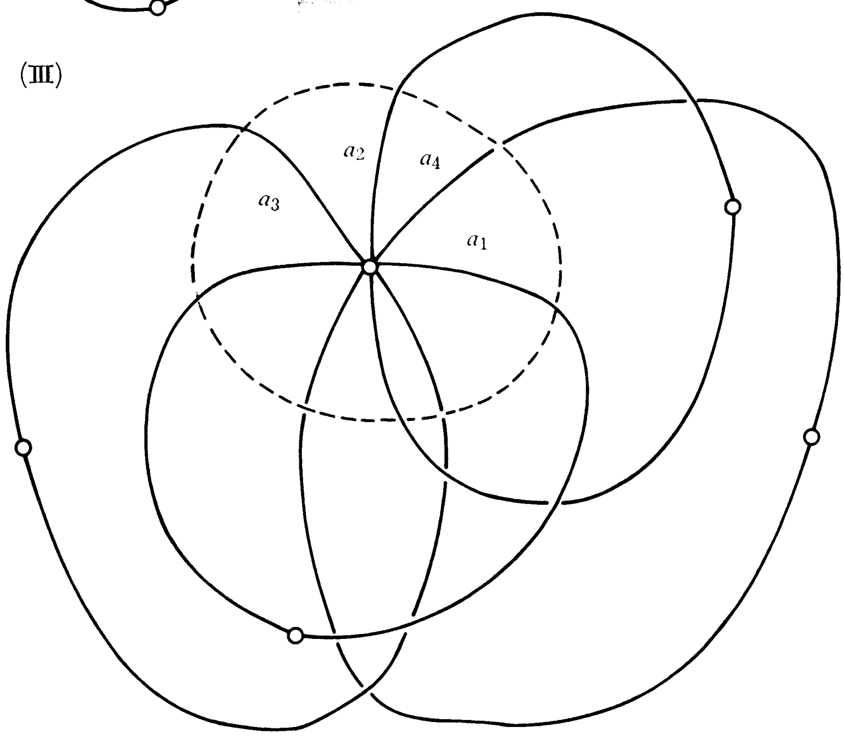
(II)



a_1, a_3, a_4 trivial
a_2 tangent bundle $\tau_{S^{2k}}$
intersection matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(III)



a_1, a_2 trivial
a_3, a_4 tangent bundle $\tau_{S^{2k}}$
intersection matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

Fig. 7.

unused fixed point of a_2, i.e., around N_2'. They meet at the used fixed point of a_2, i.e., $N_1'(=N_2)$.

In the first case, the same argument of Lemma 3.4 shows that the resulting cobordism (which is plumbed further with E_{a_3}) defines a normal cobordism. In the second case, take $(D^{2k}\times D^{2k}-\operatorname{int} N_1'')/T$ from $E_{a_3}-\operatorname{int}(N_1''\cup N_2'')/T$. And then, the identified space by h' between E_{a_1} and E_{a_2}, i.e., $D^{2k}\times D^{2k}-\operatorname{int} N_2/T(=D^{2k}\times D^{2k}-\operatorname{int} N_1'/T)$ is again identified with the above $(D^{2k}\times D^{2k}-\operatorname{int} N_1'')/T$ by the map which is modified (up to isotopy) from h' compatibly. The compatibility (1) of Lemma 3.4 holds around these spaces. Thus the resulting cobordism defines a normal cobordism. The above argument holds when a_4 meets a_3 at the used or unused point of a_3. Hence when we continue in this way about the fixed points for all a_i's, the resulting cobordism V' defines a normal cobordism (see Figure 7). Next, if a_i's meet at free points, then we can apply Lemma 3.5 to V' so that the resulting V also defines a normal cobordism. By construction of U and Lemma 3.5, the cover $(\tilde{V}\cup N(m+1)\,pts))$ is just the equivariant neighborhood U. Therefore, as in Lemma 3.5, V is a normal cobordism between $\partial U/T$ and $\{(m_1)(P^{4k-1}, id)\cup(m_2)(P^{4k-1}, c\times 1),\ m_1+m_2=m+1\}$.

LEMMA 4.6.13. *Σ^{4k-1}/T is normally cobordant to $(8l\pm 1)(P^{4k-1}, id)$ for some l.*

PROOF. Let $F: V\to P^{4k-1}$ be a normal cobordism between $\partial U/T$ and $\{m_1(P^{4k-1}, id)\cup m_2(P^{4k-1}, c\times 1)\}$ as in Lemma 4.6.12. Put $F|\partial U/T=f': \partial U/T\to P^{4k-1}$. Clearly, $\pi_1(f')=0$. By the preceding argument of (4.6.1), equivariant 1-surgeries on ∂U inside $(T, M)\supset(T, U)$ is equivalent to perform normal surgery on $\pi_2(f')=\operatorname{Ker}\{f'_*:\pi_1(\partial U/T)\to\pi_1(P^{4k-1})\}$. Hence,

(4.6.14) There is a trace W and a normal map $H: W\to P^{4k-1}$ between $(\partial U/T, f')$ and (Q^{4k-1}, f) such that f is 2-connected (note that in the cover, we can do this inside $(M-\operatorname{int} U)$).

Since $\Phi|U$ is unimodular and by the plumbing theory, it follows that $H_i(\tilde{Q})=0$ for $i\neq 4k-1$. Hence Q is a homotopy projective space. If we put $U'=U\cup\widetilde{W}$ along ∂U, then $(M-\operatorname{int} U')/T$ is an h-cobordism between Q and $\Sigma/T(=\partial M/T)$. Therefore, Σ/T is diffeomorphic to Q. The normal map H and (4.6.12) show that Σ/T is normally cobordant to $\{m_1(P^{4k-1}, id)\cup m_2(P^{4k-1}, c\times 1)\}$. For convenience, put this $P^{4k-1}(m_1, m_2)$. Since $m_1+m_2=m+1=$odd (m is the rank of the unimodular, even, symmetric matrix), we may assume $m_1>m_2$. Then, $P^{4k-1}(m_1, m_2)$ is normally cobordant to $(m_1-m_2)(P^{4k-1}, id)$, because $c\times 1$ is the orientation reversing diffeomorphism. Hence we have

(4.6.15) Σ/T is normally cobordant to $(m_1-m_2)(P^{4k-1}, id)$.

On the other hand, it follows by Proposition 2.22 of chapter II that Σ/T is normally cobordant to S^{4k-1}/T' which has a sequence of desuspensions until 5 dimension with $\sigma(T', S^5)=0$. Then, S^5/T' is normally cobordant to $(m_1-m_2)(P^5, id)$ from (4.6.15). When we recall Lemma 2.17 that the Brieskorn involutions (T^d, Σ_d^5) generate $hS(P^5)$ and table 3 in (3, 8) that Σ_d^5/T_d is normally cobordant to $d(P^5, id)$, we conclude that S^5/T' is diffeomorphic to Σ_d^5/T_d for $d=m_1-m_2$. Hence, as is well known (see Lemma 2.17), $\sigma(T_d, \Sigma_d^5/T_d)=\sigma(T', S^5)=0$ if and only if $d=m_1-m_2=\pm 1$ mod 8. This completes the proof of Lemma 4.6.13.

4.7. Step 3.

4.7.1. We give a proof of Theorem 4.4.1. We shall recall in chapter I that the Browder-Livesay invariant and the Atiyah-Singer invariant agree for free involutions, i. e.,

$$\sigma(T, \Sigma^{4k-1}) = 1/8\big(\mathrm{Sinh}\,(T, M) - L\,(\mathrm{Fix}\, T \cdot \mathrm{Fix}\, T)\big).$$

M has isolated fixed points and the action is trivial on homology. It follows that

(4.7.2) $\sigma(T, \Sigma^{4k-1}) = 1/8\sigma(M)$.

It follows from Lemma 4.6.13 that Σ^{4k-1}/T is normally cobordant to

$$(8l\pm 1)\ (P^{4k-1}, id)$$

for some l. Then we note that the spin invariant

(4.7.3) $\alpha(T, \Sigma^{4k-1}) = \pm(8l\pm 1) \mod 2^{2k}$.

We can take $(M-N(m+1)\,pts))/T$ as a normal cobordism between them (see Figure 8).

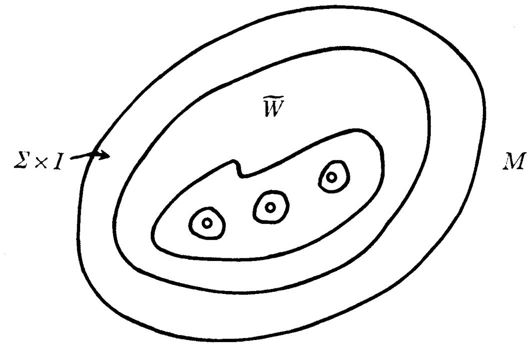


Fig. 8.

Put $X=M-N((m+1)\,pts)/T$. Then the index of $\tilde{X}$ is same as that of M. Suppose that $\sigma(T,\Sigma)=h$. Then we can find the unique representative (T_h, Σ_h) out of the class of $(8l\pm 1)$ in table 2. Furthermore, it follows from (2) and (3) of Lemma 3.7 in chapter III that

(4.7.4) Σ_h bounds M_h with $\sigma(T_h, \Sigma_h)=1/8\sigma(M)=h$, and

$$M_h-N((8l\pm 1)pts)/T_h$$

is a normal cobordism between Σ_h/T_h and $(8l\pm 1)(P^{4k-1}, id)$. Put $Y=M_h-N((8l\pm 1)\,pts)/T_h$. If we set $Z=X\cup -Y$ along $(8l\pm 1)(P^{4k-1}, id)$, then Z is a normal cobordism between Σ/T and Σ_h/T_h. The surgery obstruction for Z to making it homotopy equivalent to $P^{4k-1}\times I$ lies in $L_{4k}(Z_2)(=Z+Z)$, i.e.,

(4.7.5) $\theta(Z)=(\sigma(Z)/8, \sigma(\tilde{Z})/8)\in L_{4k}(Z_2)$.

$$\begin{aligned}\text{Since } \sigma(T_h, \Sigma_h)-\sigma(T,\Sigma) &= (2\sigma(Z)-\sigma(\tilde{Z}))/8\\ &=0 \quad \text{by the assumption,}\end{aligned}$$

and

$$\begin{aligned}\sigma(\tilde{Z}) = \sigma(\tilde{X})-\sigma(\tilde{Y}) &= \sigma(M)-\sigma(M_h)\\ &=0 \quad \text{by (4.7.2) and (4.7.4),}\end{aligned}$$

we have

$$\sigma(Z)=\sigma(\tilde{Z})=0\,.$$

Hence the surgery obstruction $\theta(Z)=0$. Therefore, there is an h-cobordism between Σ/T and Σ_h/T_h and hence Σ/T is diffeomorphic to Σ_h/T_h. This completes the proof of Theorem 4.4.1.

4.8. Proof of Theorem 4.4.3.

It follows from Theorem 4.2.1 that Σ/T is normally cobordant to some Σ'/T' such that $\partial(T', M')=(T', \Sigma')$ and $H_{2k}(M)=mY$ for some m. We can apply Theorem 4.4.1 to Σ'/T' so that Σ'/T' is diffeomorphic to the unique representative (T'_h, Σ'_h) in table 2. When we assume that $\sigma(T,\Sigma)=h$, we can choose (T_h, Σ_h) with $\sigma(T_h, \Sigma_h)=h$ within the class $[T'_h, \Sigma'_h]$. Thus Σ/T is normally cobordant to Σ_h/T_h such that they have same Browder-Livesay invariant. Then, considering the surgery obstruction of the cobordism between Σ/T and Σ_h/T_h, we conclude that Σ/T is diffeomorphic to $\Sigma_h/T_h\#\Sigma'$ for some $\Sigma'\in bP_{4k}$. Since $\Sigma_h=h\Sigma_1$ and by taking 2-fold cover so that $\Sigma\cong\Sigma_h\#2\Sigma'$, Σ' is uniquely determined with respect to the differentiable structure of Σ.

NOTE. The effect to taking Σ' is to arrange the differentiable structure for a choice of the representative Σ_h.

4.9. Similar results for $(4k+1)$-dimensional case.

When we consider $(4k+1)$-dimensional standard involutions, we have the similar results to $(4k-1)$-dimensional case. The corresponding results also hold under the assurance of Theorem 4.4.1 and Corollary 4.2.2, Corollary 4.3.2.

Given (T, Σ^{4k+1}), Σ/T is normally cobordant to Σ'/T' such that $\partial(T', M') =(T', \Sigma')$ and $H_{2k+1}(M')=mA$. Then we can show that Σ'/T' is normally coborant to $(m_1-m_2)(P^{4k+1}, id)$, where $m_1+m_2=m+1=$odd (skew-symmetric matrices have even ranks). Assume $m_1>m_2$ and put $d=m_1-m_2$. Then, d is odd. Since Brieskorn involutions Σ_d^{4k+1}/T_d is normally cobordant to $d(P^{4k+1}, id)$, Σ/T is normally cobordant to Σ_d^{4k+1}/T_d. And hence, Σ/T is diffeomorphic to $\Sigma_d^{4k+1}/T_d \# \Sigma_k^{4k+1}$. Summarizing up, we get

THEOREM 4.9.1. *Let T be a free involution on a homotopy sphere $\Sigma^{4k+1}\in bP_{4k+2}(k\geqq 1)$. Suppose that T extends to an involution with isolated fixed points on a $(4k+2)$-dimensional parallelizable manifold M^{4k+2}. Then, (T, Σ) is equivariantly diffeomorphic to the equivariant connected sum of some $\Sigma'\in bP_{4k+2}$ with the unique representative (T_d, Σ_d^{4k+1}) in table 3, i.e., the quotient*

$$\Sigma^{4k+1}/T \cong \Sigma_d^{4k+1}/T_d \# \Sigma' .$$

REMARK 4.9.2. (1) The Kervaire sphere $\Sigma_{\mathscr{K}}$ generates bP_{4k+2}. It is unknown whether $\Sigma_{\mathscr{K}}$ acts freely on P^{4k+1}, provided that $\Sigma_{\mathscr{K}} \not\cong S^{4k+1}$. From this point of view, it is unknown to be allowable to avoid the ambiguity $\Sigma' \in bP_{4k+2}$.

(2) The above argument cannot apply to the 5-dimensional case, but the result is true using the result on $hS(P^5)$.

5. Characterization on low dimensional free involutions

5.1. We relate the models in table 1 with the classical well known examples due to Bredon [3]. We recall again [13] which is introduced in the previous section. Let

$$u_n : S^{n-1} \longrightarrow SO(n) \quad \text{be the map defined by}$$

$$u_n(x) = (\delta_{ij} - 2x_i x_j) \begin{pmatrix} I_{n-1} & \\ & -1 \end{pmatrix}, \text{ where}$$

$x=(x_1, \cdots, x_n)\in S^{n-1}$ and I_{n-1} is the identity matrix with rank $n-1$. u_n is the characteristic map of the tangent bundle τ_{S^n} with the structure group $SO(n)$. Put $u_n^k(x)=u_n(x)^k$, the k^{th}-power of u_n, $: S^{n-1}\rightarrow SO(n)$ for $x\in S^{n-1}$. We denote the bundle of S^n induced from u_n^k by $k\tau_{S^n}$. It is known that

(1) If n is even, $k\tau_{S^n}$ is a stably trivial bundle whose euler class is $2k$.

(2) If n is odd and $n\neq 3$, 7, $k\tau_{S^n}$ is isomorphic to τ_{S^n} when k is odd, or to the trivial bundle when k is even.

(3) If $n=3$, 7, $k\tau_{S^n}$ is isomorphic to the trivial bundle.

Since $SO(n)\subset O(n)$, we can view $O(n)$ as the structure group of τ_{S^n}. When we set $\alpha_n: S^{n-1}\rightarrow O(n)$ by $\alpha_n(x)=(\delta_{ij}-2x_ix_j)$, the inclusion $i: SO(n)\rightarrow O(n)$ induces an isomorphism $i_*: \pi_{n-1}(SO(n))\rightarrow\pi_{n-1}(O(n))$ whose image of u_n is α_n. We note that both u_n and α_n are are invariant with respect to the action A on S^{n-1}, $A(x_1, \cdots, x_n)=(-x_1, \cdots, -x_n)$. Moreover, we have the commutative diagram of characteristic maps,

$$\begin{tikzcd} S^{n-1} \arrow[r, "u_n"] & SO(n) \\ S^{n-2} \arrow[u, "i"] \arrow[r, "u_{n-1}"] & SO(n-1) \arrow[u, "j"'] \end{tikzcd}.$$

Here $i: S^{n-2}\rightarrow S^{n-1}$, $i(x_2, \cdots, x_n)=(0, x_2, \cdots, x_n)$ and $j: SO(n-1)\rightarrow SO(n)$, $j(H)=\begin{pmatrix}1 & \\ & H\end{pmatrix}$.

5.2. We summarize here the main ideas and results of [3] to our necessity.

If θ_x is the refrection through the line Rx, i.e., $\theta_x(y)=2(x, y)\,x-y$, x, $y\in S^{n-1}$, then $x\rightarrow\theta_x$ defines a smooth map

$$\theta: S^{n-1}\longrightarrow O(n).$$

(5.2.1) It follows easily that $-\theta=\alpha_n: S^{n-1}\rightarrow O(n)$ and $\theta^2=1$ (i.e., $\theta_x^2(y)=y$).

NOTE. The fact that $-\theta=\alpha_n$ implies that $(-1)^n\det\theta_x=\det\alpha_n(x)$ for each $x\in S^{n-1}$. Since $\det\alpha_n(x)=-1$, so $\det\theta_x=1$ (n odd), $\det\theta_x=-1$ (n even). $\theta: S^{n-1}\rightarrow O(n)$ reduces to a map into $SO(n)$ if n is odd.

Consider the maps

$\Psi_r: S^{n-1}\times S^{n-1}\rightarrow S^{n-1}\times S^{n-1}$ defined by $\Psi_r(x, y)=((\theta_x\theta_y)^r x, (\theta_x\theta_y)^r y)$.

Ψ_r is equivariant with respect to the diagonal action of $O(n)$. It is easily checked that $\Psi_{-r}=(\Psi_r)^{-1}$, so that Ψ_r is an equivariant diffeomorphism for each r. Attaching $D^n\times S^{n-1}$ to $S^{n-1}\times D^n$ by means of Ψ_r, we have smooth $O(n)$-manifolds.

(5.2.2) $M_r^{2n-1}=(D^n\times S^{n-1})\underset{\Psi_r}{\cup}(S^{n-1}\times D^n)$.

(5. 2. 3) M_r^{2n-1} is equivariantly diffeomorphic to M_{-r-1}^{2n-1} for negative values.

Let $O(n)$ act, as a subgroup of $O(n+1)$, on the unit tangent disk bundle of S^n via the differential.

Let J_r be the result of equivariant plumbing of $2r$-copies of the unit tangent disk bundles of S^n at the fixed points. Then, Hirzebruch has shown that

(5. 2. 4) M_r^{2n-1} is equivariantly diffeomorphic to ∂J_r. When we concentrate on the boundary of J_r, this follows from the fact (5. 2. 1).

Let Σ_d^{4k+1} be the Brieskorn sphere in C^{2k+2} given by two equations

$$z_0^d+z_1^2+\cdots+z_{2k+1}^2=0$$

$$z_0\bar{z}_0+\cdots+z_{2k+1}\bar{z}_{2k+1}=1\,.$$

Let $O(2k+1)$ act on C^{2k+2} by the natural complex representation on $(z_1, \cdots, z_{2k+1})$. Then, it is clear that Σ_d^{4k+1} is an invariant submanifold. By the classification of [13], we obtain

(5. 2. 5) M_r^{4k+1} is equivariantly diffeomorphic to Σ_{2r+1}^{4k+1}.

Hereafter, we shall consider the manifolds M_r^{2n-1} with involutions.

5. 3. Let Z_2 be the subgroup of order 2 generated by $-I_n\in O(n)$. Denote by T the action of Z_2 in $O(n)$-manifold M_r^{2n-1} and by T_d the action of Z_2 in Σ_d^{4k+1} which is the Brieskorn involution. We obtain from (5. 2. 5) that

(5. 3. 1) $(T, M_r^{4k+1})\cong(T_{2r+1}, \Sigma_{2r+1}^{4k+1})$. ("$\cong$" stands for equivariantly diffeomorphic).

Suppose $n\geqq3$. The action of T on $D^n\times S^{n-1}$ is the diagonal antipodal map $A\times A$. There are invariant submanifolds

$$(T, D^n\times S^{n-1})\subset(T, D^{n+1}\times S^n)$$

$$(T, S^{n-1}\times D^n)\subset(T, S^n\times D^{n+1})\,.$$

The normal bundles of these inclusions are $D^n\times S^{n-1}\times D^2$ (resp. $S^{n-1}\times D^n\times D^2$) and the action on the fiber D^2 is the antipodal map A. We have equivariant embeddings

(5. 3. 2) $(T, (D^n\times S^{n-1}\underset{\Psi_r}{\cup} S^{n-1}\times D^n))\subset(T, (D^{n+1}\times S^n\underset{\Psi_r}{\cup} S^n\times D^{n+1})$. It induces an embedding of quotient spaces

(5. 3. 3) $M_r^{2n-1}/T\subset M_r^{2n-1}/T$, the normal bundle of the embedding being $M_r^{2n-1}\underset{Z_2}{\times} D^2$.

LEMMA 5. 3. 4. *Suppose* $n\geqq3$. *There is a chain of codimension 2-characteristic submanifolds of* (T, M_r^{2n-1}) *for each* r. *That is, there exists a normal map* $f_{2n-1}: M_r^{2n-1}/T\to P^{2n-1}$ *which is transverse on* $P^{2n-3}\subset P^{2n-1}$,

$M_r^{2n-3}/T=f_{2n-1}^{-1}(P^{2n-3})$, and $f_{2n-1}|(M_r^{2n-3}/T)=f_{2n-3}$, such as f_{4k+1} is a homotopy equivalence, and so on.

REMARK. This has been proved more generally, which holds for codimension 1-characteristic submanifolds with help of the equations which describe the Brieskorn sphere Σ_d^{4k+1}.

SKETCH OF PROOF. Since $(T, M_r^5)=(T_{2r+1}, \Sigma_{2r+1}^5)$ by (5. 3. 1), there is a homotopy equivalence $f_5: M_r^5/T\to P^5$. By (5. 3. 3), f_5 is covered by a bundle map $b_5: E(\nu)\to E(\eta\oplus\eta)$, where η is the canonical line bundle over P^5. put $b_5'=b_5|\partial E(\nu): \partial E(\nu)\to E(\eta\oplus\eta)=S^5\underset{Z_2}{\times}S^1$. The obstruction to extending b_5' to a map $M_r^7/T-E(\nu)\to p^7-E(\eta\oplus\eta)\cong S^1$ is $\theta(b_5')\in H^2(M_r^7/T-E(\nu), \partial E(\nu); \pi_1(S^1))\overset{\text{exc.}}{\cong} H^2(M_r^7/T, M_r^5/T; Z)$. So, we prove in the following exact sequence that i^* is an isomorphism for $i=1, 2$,

$$(*)\quad H^i(M_n^{2n+1}/T, M_r^{2n-1}/T)\longrightarrow H^i(M_r^{2n+1}/T)\xrightarrow{i^*} H^i(M_r^{2n-1}/T)$$
$$\longrightarrow H^{i+1}(M_r^{2n+1}/T, M_r^{2n-1}/T)\quad (n\geqq 3)\,.$$

Since $i_*: \pi_1(M_r^{2n-1}/T)\to\pi_1(M_r^{2n+1}/T)\cong Z_2$ is isomorphic by (5. 3. 3), hence i^*: $H^1(M_r^{2n+1}/T)\to H^1(M_r^{2n-1}/T)$ is isomorphic. On the other hand,

$$H^2(M_r^{2n+1}/T)=\mathrm{Hom}\left(H_2(M_r^{2n+1}/T), Z\right)+\mathrm{Ext}\left(H_1(M_r^{2n-1}/T), Z\right)$$
$$=\mathrm{Ext}\left(H_1(M_r^{2n+1}/T), Z\right)=Z_2\,.$$

Hence $i^*: H^2(M_r^{2n+1}/T)=\mathrm{Ext}(H_1(M_r^{2n+1}/T), Z)\to H^2(M_r^{2n-1}/T)=$
$\mathrm{Ext}\,(H_1(M_r^{2n-1}/T)\,,\ Z)$

is isomorphic. Therefore, $H^2(M_r^{2n+1}/T, M_r^{2n-1}/T)=0$.

b_5' can be extended to a map $f_7: M_r^7/T\to P^7$ which extends f_5. Starting with f_7 and applying the above argument to it, we have a map $f_9: M_r^9/T\to P^9$ extending f_7. f_9 is transverse on $P^7\subset P^9$ and has degree 1. Since $M_r^9/T=\Sigma_{2r+1}^9/T_{2r+1}$, f_9 is a homotopy equivalence. Iterating in this way, we have a chain of maps f_{2n-1} $(n\geqq 3)$. $f_{4k+1}: M_r^{4k+1}/T\to P^{4k+1}$ determines a normal map, by taking $\xi=g^*(\nu_{M/T})$, where $\nu_{M/T}$ is the stable normal bundle of M_r^{4k+1}/T and g is a homotopy inverse of f_{4k+1}. Hence, $f_{4k-1}: M_r^{4k-1}/T\to P^{4k-1}$ is a restricted normal map. Thus we have the desired result.

As to the normal cobordism classes, the relationship between our models (table 1) and (T, M_r^{4k-1}) is as follows.

COROLLARY 5. 3. 5. *Assume $k\geqq 2$.*

(1) *M_r^{4k-1}/T is normally cobordant to $(2r+1)(P^{4k-1}, id)$.*

(2) *When r moves among $\{0, -1\ mod\ 4\}$, M_r^{4k-1}/T is normally cob-*

ordant to the elements of table 1 *as below.*

r	$8l-1$	$8l$	$8l-4$	$8l-5$
M_r^{4k-1}/T	$\Sigma^{4k-1}_{(2l,+)}/T^{+}_{2l}$	$\Sigma^{4k-1}_{(2l,-)}/T^{-}_{2l}$	$\Sigma^{4k-1}_{(2l-1,+)}/T^{+}_{2l-1}$	$\Sigma^{4k-1}_{(2l-1,-)}/T^{-}_{2l-1}$

PROOF. Since $M_r^{4k+1}/T=\Sigma^{4k+1}_{2r+1}/T_{2r+1}$, M_r^{4k+1}/T is normally cobordant to $(2r+1)(P^{4k+1}, id)$. By Lemma 5.3.4, M_r^{4k-1}/T is normally cobordant to $(2r+1)(P^{4k-1}, id)$. Then comparing with the table, the result follows.

5.4. General setting and examples

We shall recall the definition of standard. Let T be a free involution on a homotopy sphere Σ^n.

DEFINITION 5.4.1. *We say that T is standard if there exists a parallelizable manifold M^{n+1} which Σ bounds such that T extends to an involution with isolated fixed points on M. If there exists no such parallelizable manifold, we call T non-standard.*

To make clear the notion of non-standard, we need some preparations. Let $bspin_n$ be the group of homotopy $(n-1)$-spheres which bound spin manifolds. In [9], Eells-Kuiper have defined an invariant μ for certain $(4k-1)$-manifolds. μ is available how to distinguish differentiable structures of topological manifolds. When a homotopy sphere Σ^{4k-1} bounds a parallelizable manifold M^{4k}, it follows that

$$\mu(\Sigma^{4k-1}) = -\sigma(M^{4k})/a_k\left(2^{2k+1}(2^{2k-1}-1)\right) \bmod 1 .$$

Here a_k is 1 if k is even, 2 if k is odd. Let $|bP_n|$ be the order of the cyclic group bP_n.

Examples of low values n

n	5	7	9	11	13	15	17	19
$\lvert bP_{n+1}\rvert$	1	$2^2.7$	2	$2^5.31$	1	$2^6.127$	2	$2^9(2^9-1)$

Put $\mu'=a_k(2^{2k-2}(2^{2k-1}-1))\,\mu$. In the above case, $\mu'(\Sigma^{4k-1})=-\sigma(M)/2^3$. Following Kervaire and Milnor [18], bP_{4k} is generated by Σ_1^{4k-1} with $\mu'(\Sigma_1)=1$. We note that $a_k(2^{2k-2}(2^{2k-1}-1))=|bP_{4k}|$ for $1<k\leqq 5$ and $a_k(2^{2k-2}(2^{2k-1}-1))/|bP_{4k}|$ in general.

We consider the following cases for non-standard involutions. Let T be a free involution on a homotopy sphere Σ^{4k-1}.

DEFINITION 5.4.2. (i) *If there exists a spin manifold W^{4k} which Σ*

bounds such that

$$\sigma(T, \Sigma) \neq \mu'(\partial W) \ mod\ 2\,,$$

we call that T "curious".

(ii) *If there exists a spin manifold W^{4k} which Σ bounds such that T extends to an involution with isolated fixed points on W, then we call T "semi-standard".*

(iii) *If* (T, Σ^{4k-1}) *satisfies* (i) *and* (ii), *we call T "spin involution".*

REMARK 5. 4. 3. By definition, standard involutions are semi-standard. On the other hand, Porposition 2. 1 of chapter II shows that standard involutions are not curious. So, from defintion 5. 4. 2 we can settle a general question.

PROBLEM 5. 4. 4.

$$\{\text{semi-standard involutions}\} = \{\text{standard involutions}\} \oplus \{\text{spin involutions}\}\ ?$$

5. 4. 5. Examples of standard involutions

(1) Weintraub's actions, which are obtained by applying an equivariant plumbing (see [15], [34]).

(2) López De Medrano's involutions, which are obtained from the antipodal map on the sphere.

(3) Brieskorn's involutions.

5. 4. 6. Examples of curious involutions

(1) The restrictions of some of free S^1-actions on homotopy 7-spheres constructed by Montgomery-Yang [22].

(2) Some of Hirsch-Milnor involutions [12].

(3) When we apply the López's construction to curious involutions, then we obtain new involutions which are again curious.

Now we shall consider here Hirsch-Milnor involutions. 'curious' is used there originally.

5. 4. 7. Hirsch-Milnor involutions

Let N_h^7 be the Milnor sphere which is the boundary of a certain D^4-bundle over S^4. Taking the antipodal map on each fiber, we obtain a smooth involution $\alpha_h: N_h^7 \to N_h^7$. Then they showed that (α_h, N_h^7) has a double desuspension, i. e., $\sigma(\alpha_h, N_h^7)=0$. On the other hand, N_h^7 bounds the spin manifold $E(\xi_{h,1-h})$ (see [12]). Since the Pontrjagin class of $E(\xi_{h,1-h})$ is $2(2h-1)\,\iota$, it follows that $\mu'(N_h^7)=h(h-1)/2$. By definition we have

LEMMA 5.4.8. *If $h\equiv 2$ or 3 mod 4, then (α_h, N_h^7) is a curious involution.*

N_h^7 is described as follows. Let H be the quaternion field.

$$D^4=\left\{u\in H \middle| \|u\|^2\leqq 1\right\},$$

$$S^3=\left\{u\in D^4 \middle| \|u\|^2=1\right\},$$

$$D^3=\left\{u\in D^4 \middle| \mathrm{Re}(u)=0\right\},$$

$$S^2=\left\{u\in D^3 \middle| \|u\|^2=1\right\}.$$

Let $f_{h,j}: S^3\to SO(4)$ be the map defined, using quaternion multiplication, by

$$f_{h,j}(u)(v)=u^h v u^j$$

for $u\in S^3$, $v\in D^4$.
Put $f_h=f_{h,1-h}$. The Milnor sphere N_h^7, $h\in Z$ is obtained by attaching $D^4\times S^3$ to $D^4\times S^3$ by the map f_h,

$$N_h^7=(D^4\times S^3)\underset{b_h}{\cup}(D^4\times S^3).$$

Here $b_h: S^3\times S^3\to S^3\times S^3$ is defined by $b_h(u, v)=(u, f_h(u)v)$. Set $b_h(u, v)=(u', v')$, i.e., (u, v) stands for a point in the first factor and (u', v') in the second. Then $(u, u^h v u^{1-h})=(u', v')$.

Define an involution on $D^4\times S^3$ to be $(u, v)\to(u, -v)$. It is easily checked that b_h is equivariant under the action. We obtain a free involution α_h on N_h^7. They showed that (α_h, N_h^7) has the double desuspension (α_h, N_h^6), (α_h, N_h^5) as follows,

$$N_h^6=N_h^7\cap\left\{\mathrm{Re}(uv)=\mathrm{Re}(v')=0\right\},$$

$$N_h^5=N_h^6\cap\left\{\mathrm{Re}(v)=\mathrm{Re}(u' v'^{-1})=0\right\}.$$

N_h^5 is written more explicitly.

$$N_h^5=\left\{D^4\times S^2\cap\left\{\mathrm{Re}(uv)=0\right\}\right\}\underset{b_h}{\cup}\left\{D^4\times S^2\cap\left\{\mathrm{Re}(u' v'^{-1})=0\right\}\right\}, \tag{1}$$

where $b_h: \{S^3\times S^2\cap\{\mathrm{Re}(uv)=0\}\}\to\{S^3\times S^2\cap\{\mathrm{Re}(u' v'^{-1})=0\}\}$, $b_h(u, v)=(u', v')=(u, u^h v u^{1-h})=(u, u^{2h-1}v)$, is a diffeomorphism. The last equality follows since $\overline{uv}=-uv$, $u^{-1}=\bar{u}$ and $v^{-1}=-v$ imply that $uv=vu^{-1}$. Here $\bar{u}$ is the conjugation of u.

We shall classify N_h^5 by the Brieskorn involutions. First we recall the results of 5.2. Using the above notations, the reflection θ is reformed as

$$\theta : S^2 \longrightarrow O(3)\,, \qquad \theta_u(v) = uvu^{-1}\,.$$

Hence it follows by (5. 2. 2) that

(2) $$M_h^5 = (D^3 \times S^2) \underset{\Psi_h}{\cup} (S^2 \times D^3)\,,$$

$$\Psi_h(u, v) = \Big((\theta_u \theta_v)^h u, (\theta_u \theta_v)^h v\Big)\,.$$

Denote by β_h the action T on M_r^5 of (5. 3). Since $(uv)^{-1} = \overline{uv} = vu$, it is easily seen that

$$(\theta_u \theta_v)^h u = (uv)^h u (uv)^{-h} = (uv)^{2h} u\,,$$

$$(\theta_u \theta_v)^h v = (uv)^{2h} v\,.$$

Thus we have $\Psi_h(u, v) = ((uv)^{2h} u, (uv)^{2h} v)$. We note by (5. 3. 1) that

(3) $$(\beta_h, M_h^5) \cong (T_{2h+1}, \Sigma_{h+1}^5)\,.$$

LEMMA 5. 4. 9. *$(\alpha_{h+1}, N_{h+1}^5) \cong (\beta_h, M_h^5)$ for each $h \in Z$. Hence by* (3), $(\alpha_{h+1}, N_{h+1}^5) \cong (T_{2h+1}, \Sigma_{2h+1}^5)$.

REMARK. The result of [36] is an easy mistake.

PROOF. We define maps

$$\lambda : \Big\{D^4 \times S^2 \cap \Big(\mathrm{Re}\,(uv) = 0\Big)\Big\} \longrightarrow D^3 \times S^2$$

and

$$\mu : \Big\{D^4 \times S^2 \cap \Big(\mathrm{Re}\,(u' v'^{-1}) = 0\Big)\Big\} \longrightarrow S^2 \times D^3$$

by setting

$$\lambda(u, v) = (uv, v)$$

$$\mu(u', v') = (v', v' u')\,.$$

It is easily checked that they are equivariant diffeomorphisms. Furthermore,

$$\lambda\Big(\Big\{S^3 \times S^2 \cap \Big(\mathrm{Re}\,(uv) = 0\Big)\Big\}\Big) = S^2 \times S^2\,,$$

$$\mu\Big(\Big\{S^3 \times S^2 \cap \Big(\mathrm{Re}\,(u' v'^{-1}) = 0\Big)\Big\}\Big) = S^2 \times S^2\,.$$

Then we show that the following diagram is commutative,

$$\begin{array}{ccc} \Big\{S^3 \times S^2 \cap \Big(\mathrm{Re}\,(uv) = 0\Big)\Big\} & \xrightarrow{\ b_{h+1}\ } & \Big\{S^3 \times S^2 \cap \Big(\mathrm{Re}\,(u' v'^{-1}) = 0\Big)\Big\} \\ \Big\downarrow \lambda & & \Big\downarrow \mu \\ S^2 \times S^2 & \xrightarrow{\qquad \Psi_h \qquad} & S^2 \times S^2\,. \end{array}$$

When $(u, v) \in \{S^3 \times S^2 \cap (\mathrm{Re}\,(uv)=0)\}$, we note that

(i) $v^{-1} = -v$ (i. e., $v^2 = -1$)

$u^{-1}v^{-1} = (vu)^{-1} = \overline{vu} = -vu$ $(\mathrm{Re}\,(uv) = \mathrm{Re}\,(vu) = 0)$, so by (i)

(ii) $u^{-1}v = vu$.

Then, $\mu b_{h+1}(u, v) = \mu(u, u^{2h+1}v) = (u^{2h+1}v, u^{2h+1}vu)$

$$= (u^{2h+1}v, u^{2h}v) \quad ((\mathrm{ii}))\,,$$

$$\Psi_h \lambda(u, v) = \Psi_h(uv, v) = \left((uvv)^{2h}uv, (uvv)^{2h}v\right)$$

$$= (u^{2h+1}v, u^{2h}v) \quad ((\mathrm{i}))\,.$$

Hence the above compatibility defines an equivariant diffeomorphism

$$\nu : (\alpha_{h+1}, N^5_{h+1}) \longrightarrow (\beta_h, M^5_h)\,.$$

By periodicity of $\{\Sigma^5_d, d\,\text{odd}\}$ of [16] and Lemma 5. 4. 9, we have

COROLLARY 5. 4. 10 (Periodicity) *For each $h \in Z$,*

$$(\alpha_h, N^5_h) \cong (\alpha_{h+8}, N^5_{h+8}) \cong (\alpha_{9-h}, N^5_{9-h})\,.$$

PROOF. It follows by [16] that $(T_d, \Sigma^5_d) \cong (T_{d+16}, \Sigma^5_{d+16}) \cong (T_{-d+16}, \Sigma^5_{-d+16})$ for $d>0$. For $h \leqq -1$, we have by (5. 2. 3) and (3) that

$$(T, M^5_h) \cong (T, M^5_{-h-1}) \cong (T_{-2h-1}, \Sigma^5_{-2h-1})\,.$$

COROLLARY 5. 4. 11 (Desuspension). *$(\alpha_{h+1}, N^7_{h+1})$ has $(T_{2h+1}, \Sigma^5_{2h+1})$ for $h \geqq 0$ and $(T_{-2h-1}, \Sigma^5_{-2h-1})$ for $h \leqq -1$ as a desuspension.*

5. 5. Reestablishment of the classification of free involutions on homotopy 7-spheres

We will classify free involutions of homotopy 7-spheres by standard involutions and spin involutions.

REMARK. There were two steps. The first is to use the normal cobordism theory due to López and Wall, and the second is to use the spin invariants due to Mayer [21].

The set of equivalence classes of free involutions on homotopy spheres is denoted by Φ^n_+ (in chapter I, p. 346, we write $\prod_n$), and Φ^n_+ is in one-to-one correspondence with $hS(P^n)$. It has been shown in [20] that $hS(P^7) \cong Z_4 + Z + Z_{28}$ and $[P^7, G/O] = Z_4 + Z_2$. The summand Z_2 is identified with the surgery obstruction group $L_3(Z_2)$. So, we may ignore so far as are concerned with Φ^n_+. Taking characteristic submanifolds, the restriction $[P^7, G/O] \to [P^5, G/O] = Z_4$ is onto. We quote the result of [16].

LEMMA 5.5.1. *$hS(P^5)\cong[P^5, G/O]$ consists of $\{\Sigma_d^5/T_d,\ d=1,3,5,7\}$ and $\Sigma_d^5/T_d\cong\Sigma_{-d+16}^5/T_{-d+16}\cong\Sigma_{d+16}^5/T_{d+16}$ for $d>0$.*

The last equality implies that

(5.5.2) $\Sigma_d^5/T_d\cong\Sigma_{-d+16i}^5/T_{-d+16i}\cong\Sigma_{d+16j}^5/T_{d+16j}$ for $d,\ i,\ j>0$.

We first determine standard involutions in table 1.

LEMMA 5.5.3. *In the table 1, there are two distinct normal cobordism classes of standard involutions, which are classified by the spin invariant α.*

(i) $\alpha=\pm 1 \bmod 2^4$, $(A, S^7)\sim(T_{2l}^-, \Sigma_{(2l,-)}^7)\sim(T_{2l}^+, \Sigma_{(2l,+)}^7)$

(ii) $\alpha=\pm 7 \bmod 2^4$, $(T_{2l-1}^-, \Sigma_{(2l-1,-)}^7)\sim(T_{2l-1}^+, \Sigma_{(2l-1,+)}^7)$.

'$\sim$' stands for "normally cobordant".

PROOF. We notice from the table that

(5.5.4) $\Sigma_{(2l,-)}^7/T_{2l}^-\sim(16l+1)(P^7, id)$

$\Sigma_{(2l,+)}^7/T_{2l}^+\sim(16l-1)(P^7, id)$

(5.5.5) $\Sigma_{(2l-1,-)}^7/T_{2l-1}^-\sim(16l-9)(P^7, id)\left(=\left(16(l-1)+7\right)\right)(P^7, id)$

$\Sigma_{(2l-1,+)}^7/T_{2l-1}^+\sim(16l-7)(P^7, id)$,

and the spin invariants are $\pm 1 \bmod 2^4$ for (i) and $\pm 7 \bmod 2^4$ for (ii). On the other hand, since Σ_d^5/T_d is normally cobordant to $d(P^5, id)$, it follows from (5.5.2) that

(5.5.6) $d(P^5, id)\sim(-d+16i)(P^5, id)\sim(d+16j)(P^5, id)$ in $[P^5, G/O]$.

Hence, by taking $d=1$ and $d=7$, (i) and (ii) follow accordingly.

REMARK 5.5.7. By the result [5] of Browder, θ^7 acts freely on $hS(P^7)$. If follows that $\Sigma_{(2l,-)}^7/T_{2l}^-\cong\Sigma_{(2l,+)}^7/T_{2l}^+\#\Sigma'$ such that $2\Sigma'=0$ for (i) and

$$\Sigma_{(2l-1,-)}^7/T_{2l-1}^-\cong\Sigma_{(2l-1,+)}^7/T_{2l-1}^+\#\Sigma''$$

such that $2\Sigma''=0$ for (ii). (they have the same Browder-Livesay invariants). We cannot get rid of the ambiguity of Σ', Σ''.

REMARK 5.5.8. Denote by $[A, S^7]$ the equivalence classes of (i) and by $[T_1^-, \Sigma_{(1,-)}^7]$ those of (ii)

	Normal cobordism class	Spin invariant	μ'	σ
$[A, S^7]$	(P^7, id)	± 1	0	0
$[T_1^-, \Sigma_{(1,-)}^7]$	$7(P^7, id)$	± 7	1	1

On the other hand, we have by [16] that

LEMMA 5.5.9. *There exists a free involution T_d on a homotopy sphere Σ^7_d with (T_d, Σ^5_d) (= Brieskorn involution) as a desuspension (i.e., $\sigma(T_d, \Sigma^7_d)=0$). Furthermore, (T_d, Σ^7_d) satisfies that*

1. $\alpha(T_d, \Sigma^7_d)=\pm d \mod 2^4$.
2. *If we set $d=2h+1$, then* $\mu'(\Sigma^7_{2h+1})=\begin{cases} h/2, & h \text{ even} \\ (h+1)/2, & h \text{ odd}. \end{cases}$

We now state our classification.

PROPOSITION 5.5.10. *Every free involution on homotopy 7-sphere is semi-standard.*

PROOF. Let T be a free involution on a homotopy sphere Σ^7. Then by proposition 1.25, there is a free involution (T', S^7) which is normally cobordant to (T, Σ^7) such that (T', S^7) has a double desuspension, $(T', S^6)\supset(T', S^5)$. Since $S^5/T' \in hS(P^5)$, it follows that $S^5/T' \cong \Sigma^5_d/T_d$ for some d. Using the supension construction and by the fact that $\theta^6=0$, we have $S^6/T' \cong \Sigma^6_d/T_d$, where (T_d, Σ^6_d) is the desupension of (T_d, Σ^7_d). Again, the suspension construction yields that $S^7/T' \cong \Sigma^7_d/T_d \# \Sigma'$ for some $\Sigma' \in \theta^7$. Hence, Σ^7/T is normally cobordant to $\Sigma^7_d/T_d \# \Sigma'$. Let W^8 be a normal cobordism between them. Since $\theta^7=bP_8$, Σ' bounds a parallelizable manifold M'^8. We know in Lemma 5.5.9 that Σ^7_d bounds the spin manifold M^8_d on which T_d extends to an involution with isolated fixed points. Set $V^8_d=\widetilde{W}^8 \cup M^8_d \# 2M'$, glued on $\Sigma^7_d \# 2\Sigma'$ equivariantly. Then we see that $\Sigma^7=\partial V^8_d$ and T extends to an involution with isolated fixed points on a spin manifold V^8_d (see Figure 9).

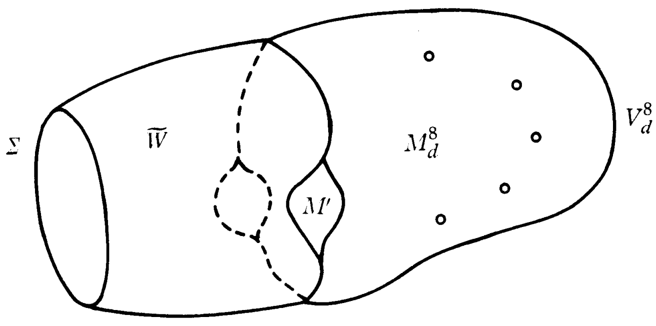


Fig. 9.

For convenience we put $\Pi_d=\Sigma^5_d/T_d$. Then $\{\Pi_d, d=1, 3, 5, 7\}$ represent the normal cobordism classes.

THEOREM 5.5.11 *Free involutions on homotopy 7-spheres are classified by standard involutions and spin involutions. Moreover,*

(i) *standard if and only if* $\alpha=\pm 1,\ \pm 7$ *mod* 2^4
spin if and only if $\alpha=\pm 3,\ \pm 5$ *mod* 2^4

(ii) *Characteristic normal cobordism classes of standard involutions move over* $\{\Pi_1, \Pi_7\}$, *while those of curious involutions move over* $\{\Pi_3, \Pi_5\}$.

COMPLEMENT. $\sigma=\mu'$ mod 2 if and only if standard.

PROOF. Let T be a free involution on a homotopy sphere Σ^7. As above, Σ^7 bounds V_d^8. We calculate μ' of Σ^7. Write $\mu'(\Sigma^7)=\mu'(V_d^8)$ for our necessity. In our case, μ' is additive from the definition, $\mu'(V_d^8)=\mu'(\widetilde{W}^8)+\mu'(M_d^8\#2M')=\mu'(\widetilde{W}^8)+\mu'(M_d^8)+2\mu'(M')$. Since $\widetilde{W}^8$ is parallelizable, it follows that $\mu'(\widetilde{W}^8)=\sigma(\widetilde{W}^8)/8$. If we put $d=2h+1$, then it follows from Lemma 5.5.9 that

$$\mu'(V_d^8)=\begin{cases}\sigma(\widetilde{W})/8+h/2\,, & h \text{ even} \\ \sigma(\widetilde{W})/8+(h+1)/2\,, & h \text{ odd}\,.\end{cases}$$

On the other hand we have $\sigma(T_d\underset{z_2}{\#}1, \Sigma_d^7\#2\Sigma')-\sigma(T, \Sigma^7)=(2\sigma(W)-\sigma(\widetilde{W}))/8$. Since $\sigma(T_d\underset{z_2}{\#}1, \Sigma_d^7\#2\Sigma')=\sigma(T_d, \Sigma_d^7)=0$ by Lemma 5.5.9, we conclude that

$$(5.5.12)\qquad \begin{array}{lll}\text{(I)} & \mu'(\Sigma^7)=\sigma(T, \Sigma^7)+h/2 \text{ mod } 2 & \text{if } h \text{ is even,} \\ \text{(II)} & \mu'(\Sigma^7)=\sigma(T, \Sigma^7)+(h+1)/2 \text{ mod } 2 & \text{if } h \text{ is odd.}\end{array}$$

Now there occur exactly two cases, the first is $\sigma(T, \Sigma)\neq\mu'(V_d^8)$ and the second is $\sigma(T, \Sigma)=\mu'(V_d^8)$. From definition 5.4.2, T is curious (i.e., $\mu'\neq\sigma$, the second case occurs) if and only if $h\equiv 2$, $h\equiv 1$ mod 4 for (I), (II) respectively. Then the spin invariant for (T, Σ) is $\alpha(T, \Sigma^7)=\alpha(T_{2h+1}, \Sigma_{2h+1}^7)=\pm(2h+1)$ mod 2^4. Hence, T is 'spin' if and only if

h	(I)		(II)	
	$8l+2$	$8l+6$	$8l+1$	$8l+5$
α	± 5	± 3	± 3	± 5
characteristic normal cobordism cless	Π_5	Π_3	Π_3	Π_5

On the other hand, suppose the first case, i.e., $\mu'(\Sigma^7)=\sigma(T, \Sigma^7)$ (2), then we show that T is 'standard'. This case occurs if and only if $h\equiv 0$, $h\equiv 3$ mod 4 with respect to (I) and (II). Thus we have

h	(I)		(II)	
	$8l$	$8l+4$	$8l+3$	$8l+7$
$\mathscr{a}$	± 1	± 7	± 7	± 1
characteristic normal cobordism class	Π_1	Π_7	Π_7	Π_1

(T, Σ) has the same characteristic cobordism as one of the elements in Remark 5.5.8 since $\Pi_d \sim d(P^5, id)$. Hence as in the proof of Proposition 5.5.10, (T, Σ) is normally cobordant to (A, S^7) or to $(T_1^-, \Sigma^7_{(1,-)})$. Let W be its normal cobordism between them. (A, S^7) (resp. $(T_1^-, \Sigma^7_{(1,-)})$ is standard, so bounds a parallelizable manifold with an involution. Adding $\widetilde{W}$ to it along (A, S^7) (resp. $(T_1^-, \Sigma^7_{(1,-)})$, we have a parallelizable manifold with boundary Σ such that T extends to an involution with isolated fixed points on it. Therefore, (T, Σ) is a standard involution.

SUMMARY 5.5.13 *Let Φ^n (resp. Φ^n_+) be the set of equivalence classes of free (resp. orientation preserving free) involutions on homotopy n-spheres. Then,*

$$\Phi^5 = \Phi^5_+ = \{\textit{standard involutions}\}.$$

$$\begin{aligned}\Phi^7_+ &= \{\textit{semi-standard involutions}\} \\ &= \{\textit{standard involutions}\} \oplus \{\textit{spin involutions}\}\end{aligned}$$

Problem 5.4.4 is true for $n=5, 7$.

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