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Convexity of nodes of discrete Sturm-Liouville functions

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1. Let f be a real function defined on a set of consecutive integers $\{a, a+1, \cdots, b\}$. If the points $(k, f(k))$, $a\leq k\leq b$, are joined by straight line segments to form a broken line, then this broken line gives a representation of a continuous function, henceforth denoted by $f^{*}(t)$, such that $f^{*}(k)=f(k)$ for $a\leq k\leq b$. The zeros of $f^{*}(t)$ are called the nodes of $f(k)$. This paper is concerned with the convexity of nodes of functions which satisfy the following second order difference equation

$$(1)\qquad \Delta^{2}x(k-1)+q(k)x(k)=0$$

where $q(k)$ is a real function defined on a set of consecutive integers to be considered. Specifically, if a nontrivial solution of (1) has three consecutive nodes t_{1}, t_{2} and t_{3}, we shall be interested in the relation between the two distances $t_{2}-t_{1}$ and $t_{3}-t_{2}$ when $q(k)$ decreases over a set of consecutive integers $\{a, a+1, \cdots, b\}$ such that $a\leq t_{1}\leq t_{3}\leq b$.

Our work is motivated by a result of Makai [1] which states that if $x(t)$ is a nontrivial solution of the differential equation

$$(2)\qquad x''+q(t)x=0,\ a<x<b$$

with three consecutive zeros t_{1}, t_{2} and t_{3} in (a, b) and if $q(t)$ is positive, continuous and decreasing in (a, b), then

$$(3)\qquad |x(t_{2}-s)|\leq|x(t_{2}+s)|$$

for $0\leq s\leq t_{2}-t_{1}$. As can easily be seen, (3) implies the well known convexity of the zeros, *i.e.*

$$(4)\qquad t_{2}-t_{1}\leq t_{3}-t_{2}.$$

In view of the obvious similarity between equations (1) and (2), one is tempted to conjecture that the inequality (4) also holds for the nodes of a nontrivial solution of (1) if $q(k)$ decreases over $\{a, \cdots, b\}$. This, however,

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is not true in general as can be seen from the following example.

EXAMPLE. Let $q(1)=10$ and $q(2)=9$. Then the function $x(k)$ defined by

$$x(0)=31,\ x(1)=-4,\ x(2)=1,\ x(3)=-3$$

satisfies (1) for $k=1, 2$. The nodes of $x(k)$ are

$$t_1=31/35,\ t_2=9/5,\ t_3=9/4.$$

Thus

$$32/35=t_2-t_1>t_3-t_2=9/20\ ,$$

which shows that $t_2-t_1 \leq t_3-t_2$ does not hold.

In spite of the above example, we can, however, show that

$$t_3-t_2>t_2-t_1-1$$

and other results of similar nature. These shall be illustrated in the last section after we have developed the necessary tools in the following section.

2. In the sequel, the smallest integer which is larger than or equal to the real number t will be denoted by t^+. We first state the following lemma, the proof of which is extracted from the proof of a result of Moulton [2].

LEMMA 1. *Let f and g be real functions defined on a set of consecutive integers $\{a, a+1, \cdots, b\}$. If $f(c+1)>0, f(c)\leq 0$ and $g(c+1)>0$ for some c in $\{a, \cdots, b-1\}$, and if $W(k)=f(k+1)\,g(k)-g(k+1)f(k)\leq 0$ at $k=c$, then $g(k)$ has a node in $[\alpha, c+1)$, where α is the node of $f(k)$ in $[c, c+1)$.*

PROOF. Assume to the contrary that $g(k)$ does not have a node in $[\alpha, c+1)$, then either $g(c)>0$ or $g(k)$ has a node δ in $[c, \alpha)$. If $g(c)>0$, then clearly $W(c)=f(c+1)\,g(c)-g(c+1)f(c)>0$ which is a contradiction. If $g(k)$ has a node δ in $[c, \alpha)$, then

$$\begin{aligned} W(c) &= \left[\frac{g(c)}{g(c+1)}-\frac{f(c)}{f(c+1)}\right] g(c+1)f(c+1) \\ &= \frac{c-\delta}{c+1-\delta}+\frac{\alpha-c}{c+1-\alpha}\, g(c+1)f(c+1) \\ &= \frac{(\alpha-\delta)\, g(c+1)f(c+1)}{(c+1-\delta)(c+1-\alpha)} \\ &> 0\,. \end{aligned}$$

This contradiction concludes the proof.

Similarly, we can show that if f and g are real functions defined on a set of consecutive integers $\{a, \cdots, b\}$, if $f(d)>0$, $g(d)>0$ and $f(d+1)\leq 0$ for some d in $\{a, \cdots, b-1\}$ and if $W(k)=f(k+1)\,g(k)-g(k+1)f(k)\geq 0$ at

$k=d$, then $g(k)$ has a node in $(d, \beta]$ where β is the node of $f(k)$ in $(d, d+1]$.

LEMMA 2. *Suppose $x(k)$ and $y(k)$, $a \leq k \leq b$, are respectively nontrivial solutions of the equations*

$$\Delta^2 x(k-1)+f(k)\, x(k)=0 \tag{5}$$

and

$$\Delta^2 y(k-1)+g(k) y(k)=0 \tag{6}$$

for $a+1 \leq k \leq b-1$. If $x(k)$ has two consecutive nodes α and β in $[a, b]$ and if $g(k) \geq f(k)$ for $a+1 \leq k \leq b-1$, then $y(k)$ has a node in $(\alpha, \beta]$.

PROOF. We may assume without loss of generality that $a \leq \alpha < a+1$, $b-1 < \beta \leq b$ and that $x^*(t) > 0$ for $\alpha < t < \beta$. If we assume to the contrary that $y^*(t) > 0$ for $\alpha < t \leq \beta$ then clearly $x(k) y(k) > 0$ for $a+1 \leq k \leq b-1$. Moreover, in view of Lemma 1, the function $W(k)=x(k+1)\, y(k)-y(k+1)\, x(k)$ is non-negative at $k=a$ and is negative at $k=b-1$. However, as can be verified easily, $W(k)$ satisfies

$$\Delta W(k)=\left(g(k+1) f-(k+1)\right) x(k+1)\, y(k+1) \geq 0$$

for $a \leq k \leq b-2$. But then

$$0 > W(b-1)=W(a)+\sum_{k=a}^{b-2} \Delta W(k) \geq 0$$

which is the desired contradiction.

LEMMA 3. *Suppose $x(k)$ and $y(k)$, $a \leq k \leq b$, are respectively positive solutions of the equations (5) and (6). Suppose $g(k) \geq f(k)$ for $a+1 \leq k \leq b-1$. If $x(a) \geq y(a)$ and $x(a+1)\, y(a)-y(a+1)\, x(a) \geq 0$, then $x(k) \geq y(k)$ for $a \leq k \leq b$.*

PROOF. As already pointed out in the proof of Lemma 2, the function $W(k)=x(k+1)\, y(k)-y(k+1)\, x(k)$ satisfies

$$\Delta W(k)=\left(g(k+1)-f(k+1)\right) x(k+1)\, y(k+1)$$

for $a \leq k \leq b-2$. Since $x(k)$ and $y(k)$ are positive for $a \leq k \leq b$, $\Delta W(k) \geq 0$ for $a \leq k \leq b-2$. Moreover, since $W(a) \geq 0$,

$$W(k)=W(a)+\sum_{j=a}^{k=1} \Delta W(j) \geq 0$$

for $a+1 \leq k \leq b-1$. It follows that

$$\Delta\left(\frac{x(k)}{y(k)}\right)=\frac{x(k+1)\,y(k)-y(k+1)\,x(k)}{y(k)\,y(k+1)}$$
$$=W(k)/y(k)\,y(k+1)$$
$$\geq 0$$

for $a+1\leq k\leq b-1$. But then

$$1\leq\frac{x(a)}{y(a)}\leq\frac{x(a+1)}{y(a+1)}\leq\cdots\leq\frac{x(b)}{y(b)}$$

as required.

The final result which we shall need is the following comparison theorem.

LEMMA 4. *Suppose $x(k)$ and $y(k)$, $a\leq k\leq b$, are nontrivial solutions of* (5) *and* (6) *respectively. Suppose $g(k)\geq f(k)$ for $a+1\leq k\leq b-1$. Suppose $x(a+1)\geq y(a+1)>0$, $x(a+1)\,y(a)-y(a+1)\,x(a)\geq 0$ and that β is the first node of $x(k)$ in $(a+1, \beta]$. Then $y(k)$ has a node in $(a+1, \beta]$ and $x^*(t)\geq y^*(t)$ for $a+1<t<\delta$ where δ is the first node of $y(k)$ in $(a+1, \beta]$.*

PROOF. The proof of the fact that $y(k)$ has a node in $(a+1, \beta]$ is similar to the proof of Lemma 2 and is thus omitted. Since $x(a+1)\geq y(a+1)>0$, if δ is the first node of $y(k)$ in $(a+1, \beta]$, then $x(k)$ and $y(k)$ are positive for $a+1\leq k\leq\delta^+-1$. Consequently, by Lemma 3, $x(k)\geq y(k)$ for $a+1\leq k\leq\delta^+-1$. Furthermore, since $\delta\leq\beta$ and since $x(\delta^+-1)\geq y(\delta^+-1)$, $x^*(t)\geq y^*(t)$ for $\delta^+-1\leq t\leq\delta$. This concludes the proof.

3. We have shown by an example that the convexity of nodes of solutions of (1) does not hold in general. We can, however, establish the convexity in certain special cases.

THEOREM 1. *Suppose $x(k)$, $a\leq k\leq b$, is a nontrivial solution of* (1) *which has three consecutive nodes t_1, t_2 and t_3 in $[a, b]$. Suppose $q(k)\geq q(k+1)$ for $a+1\leq k\leq b-2$. If t_2 is an integer or midway between two consecutive itegers in $\{a, \cdots, b\}$, then $|x^*(t_2-s)|\leq|x^*(t_2+s)|$ for $0\leq s\leq t_2-t_1$.*

PROOF. The geometrical interpretation of the assertion of Theorem 1 is as follows. Let C_1 be the graph of $x^*(t)$ lying between t_1 and t_2, and let C_2 be the graph of $x^*(t)$ lying between t_2 and t_3. Now turn the graph C_1 by 180° around its endpoint $(t_2, 0)$ and denote its new position by C_1'. Then the graph C_1' lies in the area bounded by the t axis and the graph C_2.

We shall assume $t_2\in\{a, \cdots, b\}$. The proof of the other case is similar and will be omitted. Without loss of generality, we may assume that $a\leq t_1<a+1<t_2<b-1<t_3\leq b$ and that $x^*(t)<0$ for $t_1<t<t_2$. For $t_2\leq j\leq 2t_2-a$, let $y(j)=-x(2t_2-j)$. Then the function $y(k)$ satisfies

$$\Delta^2 y(k-1)+q(2t_2-k)\,y(k)=0$$

for $t_2+1 \leq k \leq 2t_2-a-1$. Moreover, the point $\delta=2t_2-t_1$ is the first node of $y(k)$ in $(t_2, 2t_2-a]$. We assert that $\delta \leq t_3$. Suppose to the contrary that $\delta > t_3$. Note first that $x(t_2)=y(t_2)=0$, furthermore, since $x(t_2)=0$,

$$0=\Delta^2 x(t_2-1)+q(t_2)\,x(t_2)=x(t_2-1)+x(t_2+1)\,,$$

so that $y(t_2+1)=x(t_2+1)$. Finally, we note that

$$q(k) \leq q(2t_2-\mathrm{k})$$

for $t_2 \leq k \leq t_3^+$ since $q(k)$ decreases over $\{a, \cdots, b\}$. According to Lemma 4, $y(k)$ would have a node in $(t_2, t_3]$ which is a contradiction. We may now apply Lemma 4 again to conclude that $x^*(t) \geq y^*(t)$ for $t_2 \leq t \leq \delta$. Consequently, letting $t=t_2-s$, we have

$$\left|x^*(t_2-s)\right| \leq \left|x^*(t_2+s)\right|$$

for $0 \leq s \leq t_2-t_1$ as required.

We now establish the main result of our investigation.

THEOREM 2. *Suppose $x(k)$, $a \leq k \leq b$, is a nontrivial solution of* (1) *which has three consecutive nodes t_1, t_2 and t_3 in $[a, b]$. Suppose $q(k) \geq q(k+1)$ for $a+1 \leq k \leq b-2$. Then*

(7) $\quad t_3-2t_2+t_1 > -1\,.$

PROOF. If t_2 is an integer or midway between two consecutive integers in $\{a, \cdots, b\}$, then Theorem 1 implies $t_3-t_2 \geq t_2-t_1$. We may therefore assume that $t_2 \in \left(c, c+\frac{1}{2}\right)$ or $t_2 \in \left(c+\frac{1}{2}, c+1\right)$ for some integer c in $\{a, \cdots, b-1\}$. Suppose $t_2 \in \left(c+\frac{1}{2}, c+1\right)$ first. For $c \leq k \leq 2c-a+1$, let $y(k)=-x(2c+1-k)$. Then the function $y(k)$ satisfies

$$\Delta^2 y(k-1)+\mathrm{q}(2c+1-k)\,y(k)=0$$

for $c+1 \leq k \leq 2c-a$. Moreover, the point $\delta=2c+1-t_1$ is the first node of $y(k)$ in $[t_2, b]$. We assert that $\delta \leq t_3$. Assume to the contrary that $\delta > t_3$. Since $q(k)$ decreases over $\{a, \cdots, b\}$,

$$q(2c+1-k) \geq q(k)$$

for $c \leq k \leq t_3^+-1$. According to Lemma 2, $y(k)$ would have a node in $[t_2, t_3]$ which is a contradiction. Now that

$$2c+1-t_1=\delta \leq t_3\,,$$

hence

$$2c+1-t_2-t_1\leq t_3-t_2\,.$$

But since

$$t_2<c+1\,,$$

thus

$$t_2-1<2c+1-t_2$$

so that

$$t_2-1-t_1<2c+1-t_2-t_1\leq t_3-t_2\,.$$

Next suppose $t_2\in\left(c,c+\frac{1}{2}\right)$. For $c\leq k\leq 2c-a$, let $y(k)=-x(2c-k)$. The function $y(k)$ satisfies

$$\Delta^2 y(k-1)+q(2c-k)\,y(k)=0$$

for $c+1\leq k\leq 2c-a-1$. The point $\beta=2c-t_1$ is the first node of $y(k)$ in $(c, b]$. We may now proceed as in the proof of first case to show that

$$2c-t_1=\beta\leq t_3\,.$$

Futhermore, since

$$t_2<c+\frac{1}{2}\,,$$

thus

$$t_2-1<2c-t_2\,.$$

Accordingly,

$$t_3-t_2\geq 2c-t_1-t_2>t_2-1-t_1$$

as required. Q. E. D.

We remark that the constant -1 in (7) is the best possible. For suppose $0<\delta<\frac{1}{2}$ and let $x(k)$, $0\leq k\leq 3$, be defined by

$$x(0)=2(1-\delta)/\delta^2-1$$
$$x(1)=(\delta-1)/\delta$$
$$x(2)=1$$
$$x(3)=(\delta-1)/\delta\,.$$

Then the function $x(k)$ satisfies equation (1) for $1\leq k\leq 2$ where

$$q(1)=q(2)=2/\delta\,.$$

Futhermore, $x(k)$ has three consecutive nodes

$$t_1 = \frac{2-4\delta+\delta^2}{2-3\delta}, \quad t_2 = 2-\delta, \quad t_3 = 2+\delta\,.$$

It can easily be checked that

$$t_3 - 2t_2 - t_1 = \frac{-8\delta^2+8\delta-2}{2-3\delta}$$

is greater than -1 and can be made arbitrarily close to -1 by taking δ sufficiently small.

References

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