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On holomorphic vector fields on compact non-degenerate real hypersurfaces of complex manifolds

By Yutaka SE-ASHI

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Introduction

Let M be a real hypersurface of a complex manifold M'. As usual we say that M is non-degenerate (resp. strongly pseudo-convex) if the Levi form L_x at each point x of M is non-degenerate (resp. definite).

Now let E be the restriction of the holomorphic tangent bundle $T^{1,0}(M')$ of M' to M. By a holomorphic vector field on M, we mean a cross section u of E, which satisfies the so-called tangential Cauchy-Riemann equation. It is clear that if X is a holomorphic vector field on M', then the restriction $u=X|M$ of X to M is a holomorphic vector field on M. Conversely we know the following facts:

1) Let u be a holomorphic vector field on M. If both M and u are real analytic, then there is a unique holomorphic vector field X defined on a neighborhood of M such that $X|M=u$ (cf. [12]).

2) If M is non-degenerate and is not strongly pseudo-convex, then there is a neighborhood U of M such that any holomorphic vector field u on M can be extended to a unique holomorphic vector field X on U (cf. [9]). (A similar fact is also known even in the case where M is strongly pseudo-convex.)

These facts show that the study of the holomorphic vector fields on M is closely related to the study of the complex manifold M' itself.

Let $\mathfrak{g}(M)$ be the Lie algebra of all holomorphic vector fields on M, and let $\mathfrak{a}(M)$ be the Lie algebra of all infinitesimal automorphisms of the real hypersurface M, which may be considered as a real subalgebra of $\mathfrak{g}(M)$. It is well known that if M is non-degenerate, then $\mathfrak{a}(M)$ is finite dimensional (cf. [12]), and that if M is compact and if M is non-degenerate and is not strongly pseudo-convex, then $\mathfrak{g}(M)$ is finite dimensional (cf. [5]). It can be also shown that if M is non-degenerate, then the natural homomorphism $\boldsymbol{C}\mathfrak{a}(M)\rightarrow\mathfrak{g}(M)$, $\boldsymbol{C}\mathfrak{a}(M)$ being the complexification of $\mathfrak{a}(M)$, is injective, and hence $\boldsymbol{C}\mathfrak{a}(M)$ may be considered as a subalgebra of $\mathfrak{g}(M)$.

In the present paper, we prove a series of structure theorems on the Lie algebras $\mathfrak{g}(M)$ and $C\mathfrak{a}(M)$, assuming the following conditions: 1) M is compact 2) M is non-degenerate, 3) M is normal in the sense of [14], *i.e.*, M admits an infinitesimal automorphism ξ such that, at each point x of M, ξ_x is transversal to the maximal complex subspace of the tangent space $T(M)_x$, and 4) Certain conditions on the pair (M, ξ), M being equipped with the induced PC (or CR) structure. The theorems consist of decomposition theorems and vanishing theorems, and are stated in different forms according as M is strongly pseudo-convex or not. See Theorems 3.2, 3.5, 3.6, 5.4, 5.6 and 5.7. We also exihibit some examples, and apply the theorems to some problems on real hypersurfaces and complex manifolds. In particular, we obtain a theorem (Theorem 6.5) characterizing the hyperplane bundles over the complex projective spaces. See also Theorems 4.5, 4.7 and 6.1.

In § 1 we first recall several known facts on PC structures. Then we introduce a space $F(M)$ of functions on M satisfying a certain differential equation, and construct a linear isomorphism of the Lie algebra $\mathfrak{g}(M)$ onto the space $F(M)$. Thus the study of $\mathfrak{g}(M)$ is reduced to that of $F(M)$. § 2 is a preliminary to the subsequent sections. We define differential operators N, $\square_1$ and $\square_2$ on $F(M)$, and then describe $F(M)$ in terms of these operators. In § 3 and § 5, we state and prove the structure theorems. The proofs are based on the decompositions $F(M)$ into the eigenspaces of N, $\square_1$ and $\square_2$. § 4 and § 6 are devoted to the examples and applications.

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Preliminary remarks

1) Throughout this paper we always assume the differentiability of class C^∞, and assume that the manifolds to be considered are connected.

2) Given a manifold M, $C^\infty(M)$ denotes the space of all complex valued differentiable functions on M. Let E be a vector bundle over M. We denote by E^* the dual vector bundle of E and by $\Gamma(E)$ the space of all differentiable cross sections of E. For $\phi \in \Gamma({}^{p+1}\wedge E^*)$ and $X \in \Gamma(E)$, $X \lrcorner \phi \in \Gamma({}^{p}\wedge E^*)$ is defined by

$$(X \lrcorner \phi)(Y_1, \cdots, Y_p) = \phi(X, Y_1, \cdots, Y_p)$$

for $Y_1, \cdots, Y_p \in \Gamma(E)$.

3) In the case where M is a complex manifold and E is a holomorphic vector bundle over M, we denote by $\Gamma_{hol}(E)$ the space of all holomorphic

cross sections of E, and by $\Omega^p(E)$ the sheaf of germs of local holomorphic E-valued p-forms. We denote by $H^q(M, \Omega^p(E))$ the q-th cohomology group of the sheaf $\Omega^p(E)$. In particular if $p=0$, we use the notation $H^q(M, E)$ in stead of $H^q(M, \Omega^0(E))$.

§ 1. Non-degenerate PC manifolds and the Lie algebras $\mathfrak{g}(M)$ and $\mathfrak{a}(M)$

1.1. PC manifolds and the holomorphic tangent bundles (cf. [14]). Let M be a differentiable manifold of dimension $2n-1$. A partially complex structure (or briefly a PC structure) on M is a subbundle S of the complexified tangent bundle $CT(M)$ of M which satisfies the following conditions:

(PC. 1) $\dim_c S=n-1$, and $S\cap\bar{S}=0$;

(PC. 2) $[\Gamma(S), \Gamma(S)]\subset\Gamma(S)$.

The manifold M equipped with the partially complex structure S is called a partially complex manifold (or briefly a PC manifold).

Let M be a PC manifold. Let us recall the definition of the holomorphic tangent bundle of M. We define a complex vector bundle $\hat{T}(M)$ over M by

$$\hat{T}(M)=CT(M)/\bar{S} \qquad \text{(factor bundle)}$$

and define a differential operator

$$\bar{\partial}: \Gamma\left(\hat{T}(M)\right) \longrightarrow \Gamma\left(\hat{T}(M)\otimes\bar{S}^*\right)$$

as follows. Let $\pi: CT(M)\rightarrow\hat{T}(M)$ be the natural projection. For any cross section u of $\hat{T}(\mathrm{M})$ and any cross section $\bar{Y}$ of $\bar{S}$, we define a cross section $(\bar{\partial}u)(\bar{Y})$ of $\hat{T}(M)$ by

$$(\bar{\partial}u)(\bar{Y})=\pi\left([\bar{Y}, X]\right),$$

where X is a cross section of $CT(M)$ such that $\pi(X)=u$. Then it is easy to see that $(\bar{\partial}u)(\bar{Y})$ does not depend on the choice of X and the assignment $\bar{Y}\rightarrow(\bar{\partial}u)(\bar{Y})$ gives a cross section $\bar{\partial}u$ of $\hat{T}(M)\otimes\bar{S}^*$.

Here we notice that the complex vector bundle $\hat{T}(M)$ together with the operator $\bar{\partial}$ becomes a holomorphic vector bundle in the sense of Tanaka [14], that is, the following hold:

(HV. 1) $\qquad \bar{Y}(fu)=(\bar{Y}f)\,u+f(\bar{Y}u)$;

(HV. 2) $\qquad [\bar{Y}, \bar{Z}]\,u=\bar{Y}(\bar{Z}u)-\bar{Z}(\bar{Y}u)$,

where $u\in\Gamma(\hat{T}(M))$, $f\in C^\infty(M)$, $\bar{Y}$, $\bar{Z}\in\Gamma(\bar{S})$, and $\bar{Y}u$ denotes $(\bar{\partial}u)(\bar{Y})$.

The holomorphic vector bundle $\hat{T}(M)$ thus obtained is called the holomorphic tangent bundle of M, and a cross section u of $\hat{T}(M)$ is called a holomorphic cross section or preferably a holomorphic vector field on M, if it satisfies the (tangential Cauchy-Riemann) equation

$$\bar{\partial} u = 0 .$$

Let M' be a complex manifold of dimension n and let M be a real hypersurface of M'. Let $T^{1,0}(M')$ be the holomorphic tangent bundle of the complex manifold M' or the vector bundle of complexified tangent vectors of type $(1, 0)$ to M'. For each point x of M, we define a subspace S_x of $\boldsymbol{C}T(M)_x$ by

$$S_x = \boldsymbol{C}T(M)_x \cap T^{1,0}(M')_x ,$$

and set $S = \cup S_x$. Then we see that S defines a PC structure on M, which is called the induced PC structure.

Let i (resp. p') denote the natural injection $\boldsymbol{C}T(M) \to \boldsymbol{C}T(M')$ (resp. the natural projection $\boldsymbol{C}T(M') \to T^{1,0}(M')$). Let $E(M)$ denote the restriction of $T^{1,0}(M')$ to M. Since $\operatorname{Ker}(p' \circ i) = \bar{S}$, we see that the map $p' \circ i$ induces an isomorphism j of $\hat{T}(M)$ onto $E(M)$ as differentiable vector bundles in a natural manner. Hereafter we will identify the two vector bundles $\hat{T}(M)$ and $E(M)$ by this isomorphism.

Let us now consider the Cauchy-Riemann operator of the holomorphic vector bundle $T^{1,0}(M)$, which is the differential operator

$$\bar{\partial}' : \Gamma\Big(T^{1,0}(M')\Big) \longrightarrow \Gamma\Big(T^{1,0}(M) \otimes \overline{T^{1,0}(M')^*}\Big)$$

defined by

$$\bar{\partial}' X(\bar{Y}) = p'\Big([\bar{Y}, X]\Big) ,$$

where X, $Y \in \Gamma(T^{1,0}(M'))$. Now let X be any cross section of $T^{1,0}(M')$, it is easy to verify that

$$(\bar{\partial} u)(\bar{Y}) = (\bar{\partial}' X)(\bar{Y}) \qquad \text{for all } \bar{Y} \in \bar{S} ,$$

where u denotes the restriction $X|M$ of X to M.

It follows that if X is a holomorphic vector field on M' or a holomorphic cross section of $T^{1,0}(M')$, then $u = X|M$ is a holomorphic vector field on M or a holomorphic cross section of $\hat{T}(M) = E(M)$.

We define a differential operator $\bar{\partial} : C^\infty(M) \to \Gamma(\bar{S}^*)$ by

$$(\bar{\partial} f)(\bar{Y}) = \bar{Y} f , \qquad \bar{Y} \in \bar{S}$$

Then a function f on M is called a holomorphic function if it satisfies the (tangential Cauchy-Riemann) equation

$$\bar{\partial} f=0 .$$

Now let u be a cross section of $\hat{T}(M)=E(M)$, and let $z^1, \cdots, z^n$ be any local complex coordinate system of M'. The restrictions u_i, $1 \leqq i \leqq n$, of the vector fields $\partial/\partial z^i$ to M form a local (holomorphic) frame of $\hat{T}(M)$, and hence u can be expressed as follows:

$$u=\sum_i f^i u_i$$

f^i being local functions on M. Then we remark that u is a holomorphic vector field if and only if all the components f^i of u are local holomorphic functions on M.

1.2. The Lie algebras $\mathfrak{g}(M)$ and $\mathfrak{a}(M)$. Let M be a PC manifold. We denote by $\mathfrak{g}(M)$ the space of holomorphic vector fields on M. We show that $\mathfrak{g}(M)$ is endowed with the structure of a complex Lie algebra. Let $u_i \in \mathfrak{g}(M)$, $i=1,2$, and let us choose cross sections X_i such that $u_i=\pi(X_i)$. Then we define a cross section $[u_1, u_2]$ of $\hat{T}(M)$ by

$$[u_1, u_2]=\pi\left([X_1, X_2]\right) .$$

Since $[X_i, \Gamma(\bar{S})] \subset \Gamma(\bar{S})$ and $[\Gamma(\bar{S}), \Gamma(\bar{S})] \subset \Gamma(\bar{S})$, we see that $[u_1, u_2]$ does not depend on the choices of X_1 and X_2. By using the Jacobi identity for vector fields on M, we also see that $[u_1, u_2] \in \mathfrak{g}(M)$ and that the vector space $\mathfrak{g}(M)$ equipped with this bracket operation becomes a complex Lie algebra.

Let X be a real vector field on M, and let ϕ_t be the local 1-parameter group of local transformations generated by X. Then X is called an infinitesimal automorphism of M if each ϕ_t is a local automorphism, *i.e.*, preserves the PC structure S of M. Note that X is an infinitesimal automorphism if and only if $[X, \Gamma(S)] \subset \Gamma(S)$. We denote by $\mathfrak{a}(M)$ the Lie algebra of infinitesimal automorphisms of M.

Let $X \in \mathfrak{a}(M)$. Then it is easy to see that $\pi(X) \in \mathfrak{g}(M)$ and that the assignment $X \rightarrow \pi(X)$ gives an injective homomorphism of $\mathfrak{a}(M)$ to $\mathfrak{g}(M)$ as real Lie algebras. Thus we may regard $\mathfrak{a}(M)$ as a real subalgebra of $\mathfrak{g}(M)$.

Finally let us consider a real hypersurface M of a complex manifold M'. Let X be a holomorphic vector field on M'. Then we see that $X|M$ is an infinitesimal automorphism of M if and only if the real part of X is tangent to M at each point of M.

1.3. Non-degenerate PC manifolds. Let θ be a real valued 1-form defined on a neighborhood of x which satisfies the following conditions:

1) $\theta_x \neq 0$,

2) θ annihilates S.

Then we define a hermitian form $L_x = L_x^\theta$ by

$$L_x^\theta(X, Y) = -\sqrt{-1}\,(d\theta)(X, \bar{Y}), \qquad X, Y \in S_x,$$

which is called the Levi form of M at x (corresponding to the 1-form θ). Let θ' be another 1-form satisfying the condition above. Then θ' can be expressed as $\theta' = f\theta$ with a function f defined on a neighborhood of x, and we can easily see that

$$L_x^{\theta'} = f(x)\, L_x^\theta .$$

It follows that neither the dimension of the null space of L_x nor the signature of L_x (up to sign) depends on the choice of θ. This being said, we define a non-negative integer $\lambda(x)$ $\left(\leqq \frac{n-1}{2}\right)$ by

$$\lambda(x) = \mathrm{Min}\left(\lambda_+(x),\ \lambda_-(x)\right),$$

where $\lambda_+(x)$ (resp. $\lambda_-(x)$) stands for the number of positive (resp. negative) eigenvalues of the hermitian form L_x.

We say that the PC manifold M is non-degenerate and of index r, if the Levi form L_x is non-degenerate and $\lambda(x) = r$ at each point x of M. In particular, we say that M is a strongly pseudo-convex (or briefly s. p. c.) manifold if it is non-degenerate of index 0 or equivalently the Levi form L_x is definite at each point x of M.

PROPOSITION 1. 1. *Let M be a non-degenerate PC manifold. Then the subspaces $\mathfrak{a}(M)$ and $\sqrt{-1}\,\mathfrak{a}(M)$ of $\mathfrak{g}(M)$ satisfy $\mathfrak{a}(M) \cap \sqrt{-1}\,\mathfrak{a}(M) = 0$, and hence the subalgebra $\mathfrak{a}(M) + \sqrt{-1}\,\mathfrak{a}(M)$ of $\mathfrak{g}(M)$ may be considered as the complexification $\boldsymbol{C}\mathfrak{a}(M)$ of $\mathfrak{a}(M)$.*

PROOF. We define a subbundle T_1 of $\hat{T}(M)$ by

$$T_1 = \left(T(M) + \bar{S}\right)/\bar{S}.$$

Then we have

$$T_1 \cap \sqrt{-1}\, T_1 = (S + \bar{S})/\bar{S}.$$

Take any element u of $\mathfrak{a}(M) \cap \sqrt{-1}\,\mathfrak{a}(M)$. From the remark above, we can find a cross section X of S such that $u = \pi(X)$. Since $u \in \mathfrak{a}(M)$, we obtain $[X, \Gamma(\bar{S})] \subset \Gamma(\bar{S})$, and hence

$$\begin{aligned} L_x(X_x, Y_x) &= -\sqrt{-1}\,(d\theta)(X_x, \bar{Y}_x) \\ &= -\sqrt{-1}\left\{X_x\theta(\bar{Y}) - \bar{Y}_x\theta(X) - \theta\left([X, \bar{Y}]_x\right)\right\} = 0, \end{aligned}$$

where $Y\in\Gamma(S)$. Since L_x is non-degenerate at each point x of M, it follows that $X=0$. q. e. d.

1.4. Condition (C. 1) and the canonical affine connections (cf. [14]). Let M be a PC manifold. We assume the following condition:

(C. 1) There exists an infinitesimal automorphism ξ such that $\xi_x\notin(S+\bar{S})_x$ for any point x of M.

From now on we will be concerned with the pair (M, ξ). We denote by P the 1-dimensional complex subbundle of $\boldsymbol{C}T(M)$ spanned by ξ:

$$P_x=\boldsymbol{C}\xi_x\,, \qquad x\in M\,.$$

Then we have

$$\boldsymbol{C}T(M)=S+\bar{S}+P \qquad \text{(direct sum)}\,.$$

We define a real valued 1-form θ by

$$\begin{aligned}&\theta(\xi)=1\,,\\ &\theta(X)=0\,, \qquad X\in(S+\bar{S})_x\,,\end{aligned}$$

and consider the Levi form $L_x=L_x^\theta$ corresponding to the 1-form θ.

Proposition 1.2 (cf. [14]). *Let M be a non-degenerate PC manifold. Assume that M satisfies condition* (C. 1), *then there is a unique affine connection*

$$\nabla:\Gamma\big(T(M)\big)\longrightarrow\Gamma\big(T(M)\otimes T(M)^*\big)$$

on M satisfying the following conditions:

1) *S is parallel with respect to ∇.*

2) *ξ, θ, and $d\theta$ are all parallel.*

3) *The torsion tensor T of ∇ has the following properties*:

$$\begin{aligned}&T(X, Y)=0\,,\\ &T(X, \bar{Y})=(d\theta)(X, \bar{Y})\,\xi_x\big(=\sqrt{-1}\,L_x(X, Y)\,\xi_x\big)\,,\\ &T(\xi_x, Y)\in\bar{S}_x\,,\end{aligned}$$

where X, $Y\in S_x$.

The connection ∇ in Proposition 1.2 is called the canonical affine connection of (M, ξ). We denote by R the curvature tensor of ∇.

Proposition 1.3 (cf. [14]). *Let X, Y, Z, $W\in\Gamma(S)$.*

(1) $T(\xi, X)=0$.

(2) $\nabla_\xi X=\mathcal{L}_\xi X$.

(3) $(d\theta)(R(X,\bar{Y})Z,\bar{W})+(d\theta)(Z,R(X,\bar{Y})\bar{W})=0$.

(4) $R(X,\bar{Y})Z=R(Z,\bar{Y})X\in\Gamma(S)$.

(5) $R(X,Z)=0$.

(6) $R(\xi,X)=0$.

PROOF. Since $\nabla_\xi X$ and $[\xi,X]$ are cross sections of S, and since $\nabla_X\xi=0$, we have $T(\xi,X)\in\Gamma(S)$. On the other hand, we know that $T(\xi,X)\in\Gamma(\bar{S})$ by Proposition 1. 2. Hence we have $T(\xi,X)=0$, proving (1). (2) follows from (1). (3) follows immediately from the fact that $\nabla d\theta=0$. From Bianchi's first identity and (1) together with the fact that $\nabla T=0$, we have

$$R(X,\bar{Y})Z+R(\bar{Y},Z)X+R(Z,X)\bar{Y}=0\,. \tag{1.1}$$

Since the subbundle S is parallel with respect to the canonical affine connection ∇, we have $R(X,\bar{Y})Z\in\Gamma(S)$, $R(\bar{Y},Z)X\in\Gamma(S)$ and $R(Z,X)\bar{Y}\in\Gamma(\bar{S})$. By taking the S-component of (1. 1), we obtain

$$R(X,\bar{Y})Z+R(\bar{Y},Z)X=0\,,$$

implying(4). In the same manner, we obtain

$$R(Z,X)\bar{Y}=0\,.$$

Since $\nabla d\theta=0$, we have

$$(d\theta)\big(R(X,Z)W,\bar{Y}\big)+(d\theta)\big(W,R(X,Z)\bar{Y}\big)=0\,.$$

Hence it follows that

$$L\big(R(X,Z)W,Y\big)=-\sqrt{-1}\,(d\theta)\big(R(X,Z)W,\bar{Y}\big)=0\,.$$

Since the Levi form L is non-degenerate, it follows that

$$R(X,Z)W=0\,.$$

Hence we have proved (5). Since the vector field ξ leaves invariant the PC structure S and the vector field ξ, it follows that ξ is an infinitesimal affine transformation. Hence ξ satisfies the following

$$[\xi,\nabla_X\bar{Y}]-\nabla_X\big([\xi,\bar{Y}]\big)-\nabla_{[\xi,X]}\bar{Y}=0\,. \tag{1.2}$$

By (1) we have $\nabla_\xi(\nabla_X\bar{Y})=[\xi,\nabla_X\bar{Y}]$ and $\nabla_\xi\bar{Y}=[\xi,\bar{Y}]$. Hence the left hand side of (1. 2) is equal to $R(\xi,X)\bar{Y}$, which implies $R(\xi,X)\bar{Y}=0$. Similarly we have $R(\xi,X)Y=0$, proving (6). q. e. d.

1. 5. The space $F(M)$. We define a subbundle T of $\boldsymbol{C}T(M)$ by $T=S+P$. Then we have

$$\boldsymbol{C}T(M)=T+\bar{S}\qquad\text{(direct sum)}$$

and the projection $\pi: CT(M)\to\hat{T}(M)$ induces a bundle isomorphism P of T onto $\hat{T}(M)$. Let u be a cross section of $\hat{T}(M)$, and let the capital letter U denote the corresponding cross section of T. Then u is a holomorphic cross section of $\hat{T}(M)$ if and only if the cross section U satisfies the condition

$$(1.3)\qquad \left[U, \Gamma(\bar{S})\right]\subset\Gamma(\bar{S}).$$

Let us denote by U^S (resp. by U^P) the S-component of U (resp. the P-component of U). We interpret (1.3) in terms of U^S and U^P. Then, for any $\bar{Y}\in\Gamma(\bar{S})$, we have

$$(1.4)\qquad [\bar{Y}, U^P]^P+[\bar{Y}, U^S]^P=0,$$

$$(1.5)\qquad [\bar{Y}, U^S]^S=0,$$

where we use the fact that $[\bar{Y}, U^P]^S=0$,

We define a function f_u by

$$f_u=\theta(U).$$

Then we have $U^P=f_u\xi$. By using the canonical affine connection ∇, we obtain

$$(1.4)'\qquad \nabla_{\bar{Y}}f_u+(d\theta)(U^S, \bar{Y})=0,$$

$$(1.5)'\qquad \nabla_{\bar{Y}}U^S=0,$$

where $\bar{Y}\in\bar{S}_x$. It follows from (1.4)′ and (1.5)′ that

$$(1.6)\qquad \nabla_{\bar{Y}}\nabla_{\bar{Z}}f_u=0, \qquad \bar{Y}, \bar{Z}\in\bar{S}_x.$$

Here we define a subspace $F(M)$ of $C^\infty(M)$ by

$$F(M)=\left\{f\in C^\infty(M)\middle|\nabla_{\bar{Y}}\nabla_{\bar{Z}}f=0 \quad \text{for any } \bar{Y},\ \bar{Z}\in\bar{S}_x \text{ and } x\in M\right\}.$$

PROPOSITION 1.4. *The assignment $u\to f_u$ gives a linear isomorphism of $\mathfrak{g}(M)$ onto $F(M)$.*

PROOF. Let $u\in\mathfrak{g}(M)$. Suppose that $f_u=0$. By (1.4)′, we have

$$L(U^S, Y)=-\sqrt{-1}(d\theta)(U^S, \bar{Y})=0 \qquad \text{for any } \bar{Y}\in\bar{S}_x.$$

Since the Levi form L_x is non-degenerate at each point x of M, it follows that $U^S=0$, which implies $u=0$. Conversely let f be a function contained in $F(M)$. Since the Levi form L_x is non-degenerate at each point x of M, we can take a unique cross section U^S of S satisfying (1.4)′. Since f satisfies (1.6), it follows that (1.5)′ holds. If we put $U=f\xi+U^S$, then we see that U satisfies (1.3). Therefore the cross section u of $\hat{T}(M)$ corresponding to U is a holomorphic vector field satisfying $f_u=f$. q. e. d.

We define a subspace $\bar{F}(M)$ of $C^{\infty}(M)$ by

$$\bar{F}(M)=\left\{\bar{f}\middle| f\in F(M)\right\}.$$

Then we have the following

PROPOSITION 1.5. *Let $u\in\mathfrak{g}(M)$. Then $u\in\mathfrak{a}(M)$ if and only if f_u is a real valued function. Hence the assignment $u\to f_u$ gives a linear isomorphism of $\boldsymbol{C}\mathfrak{a}(M)$ onto $F(M)\cap\bar{F}(M)$.*

PROOF. First assume that u is contained in $\mathfrak{a}(M)$. Then we have an infinitesimal automorphism X such that $\pi(X)=u$. Let X^T be the T-component of X. Then it follows that $U=X^T$ and

$$f_u=\theta(U)=\theta(X^T)=\theta(X),$$

implying that f_u is a real valued function.

Conversely assume that f_u is a real valued function. We define a real vector field X by

$$X=f_u\xi+U^S+\bar{U}^S.$$

Then we see that $\pi(X)=u$, and X is an infinitesimal automorphism.

q. e. d.

§ 2. The fundamental equalities

2.1. Condition (C. 2), and the operators $\square_i$ and N. Let M be a non-degenerate PC manifold of index r satisfying condition (C. 1). In this and the subsequent sections, we assume that M is compact and satisfies the following condition:

(C. 2) There exist subbundles S^1 and S^2 satisfying the following:

1) $\dim_c S^1=r$, and $\dim_c S^2=s$, where $s=n-r-1$.

2) $S=S^1+S^2$ (direct sum).

3) Both S^1 and S^2 are parallel with respect to the canonical affine connection ∇.

4) At each point x of M, the Levi form L_x is negative definite (resp. positive definite) on S_x^1 (resp. on S_x^2), and S_x^1 and S_x^2 are mutually orthogonal with respect to L_x.

First of all, let us define a Riemannian metric g on M by

[1] $g_x(X,\bar{Y})=-L_x(X,Y),\quad X,\ Y\in S_x^1;$

[2] $g_x(X,\bar{Y})=L_x(X,Y),\quad X,\ Y\in S_x^2;$

[3] $g_x(\xi_x,\xi_x)=1;$

[4] The other components of g_x vanish.

We easily see that the Riemannian metric g thus defined is parallel with respect to the canonical affine connection ∇. Let us denote by dV the volume element $\frac{1}{(n-1)!}\theta\wedge(d\theta)^{n-1}$, which is nothing but the volume element associated with g. We define a hermitian inner product (,) in the space $C^{\infty}(M)$ by

$$(f, g)=\int_{M}\langle f, g\rangle dV$$

where $\langle f, g\rangle$ is the function defined by

$$\langle f, g\rangle(x)=f(x)\overline{g(x)}, \qquad x\in M.$$

In the following, the three indices a, b, c range over the integers $1, \cdots, r$ while the three indices α, β, γ range over the integers $r+1, \cdots, n-1$. Let x be any point of M, and let $e_1, \cdots, e_r$ (resp. $e_{r+1}, \cdots, e_{n-1}$) be a base of S_x^1 (resp. of S_x^2) such that $g(e_a, \bar{e}_b)=\delta_{ab}$ (resp. $g(e_\alpha, \bar{e}_\beta)=\delta_{\alpha\beta}$). Then the $2n-1$ vectors $\xi, e_1, \cdots, e_r, e_{r+1}, \cdots, e_{n-1}, \bar{e}_1, \cdots, \bar{e}_r, \bar{e}_{r+1}, \cdots, \bar{e}_{n-1}$ form a base of $CT(M)_x$. By using these bases, we will express various tensor fields in terms of their components.

We define a bilinear form $R^*: S_x\times\bar{S}_x\to \boldsymbol{C}$ by

$$R^*(X, \bar{Y})=\sum_{i=1}^{n-1} g\left(R(X, \bar{Y})e_i, \bar{e}_i\right), \qquad X, Y\in S_x.$$

It is easy to see that

$$R^*(X, \bar{Y})=\overline{R^*(Y, \bar{X})}.$$

The tensor R^* thus defined will be called the Ricci tensor.

PROPOSITION 2.1.

(1) $$R^*(X, \bar{Y})=\sum_a g\left(R(X, \bar{Y})e_a, \bar{e}_a\right) \quad \textit{for} \quad X, Y\in S_x^1.$$

$$R^*(X, \bar{Y})=\sum_\alpha g\left(R(X, \bar{Y})e_\alpha, \bar{e}_\alpha\right) \quad \textit{for} \quad X, Y\in S_x^2.$$

(2) $$R^*(X, \bar{Y})=0 \quad \textit{for} \quad X\in S_x^1 \textit{ and } Y\in S_x^2.$$

PROOF. Let $X\in S_x^1$. By Proposition 1.3 and the fact that S^1 and S^2 are parallel with respect to ∇, we have

$$R(X, \bar{Y})e_\alpha=R(e_\alpha, \bar{Y})X\in S_x^1\cap S_x^2=0,$$

which implies the first assertion of (1). The second assertion of (1) is proved in the same manner. Now let $X\in S_x^1$ and $Y\in S_x^2$. By Proposition 1.3 and the fact that $\nabla g=0$, we have

$$g\left(R(X, \bar{Y})\, e_i, \bar{e}_i\right) = g\left(R(e_i, \bar{e}_i)\, X, \bar{Y}\right).$$

Since S^1 is parallel with respect to ∇, it follows that

$$R(e_i, \bar{e}_i)\, X \in S^1_x.$$

Therefore we have

$$R^*(X, \bar{Y}) = \sum_i g\left(R(e_i, \bar{e}_i)\, X, \bar{Y}\right) = 0,$$

proving (2). q. e. d.

We define the scalar curvatures σ_1 and σ_2 by

$$\sigma_1 = \frac{1}{r(r+1)} \sum_a R^*(e_a, \bar{e}_a),$$

$$\sigma_2 = \frac{1}{s(s+1)} \sum_\alpha R^*(e_\alpha, \bar{e}_\alpha).$$

We also define differential operators $\Box_1$, $\Box_2$ and N on $C^\infty(M)$ respectively by

$$\Box_1 f = -\sum_a \nabla_a \nabla_{\bar{a}} f, \qquad \Box_2 f = -\sum_\alpha \nabla_\alpha \nabla_{\bar{\alpha}} f, \qquad Nf = \sqrt{-1}\, \xi f,$$

for $f \in C^\infty(M)$.

PROPOSITION 2. 2. *The operators $\Box_1$, $\Box_2$ and N are self-adjoint operators, moreover $\Box_1$ and $\Box_2$ are positive semi-definite.*

PROOF. Let $f, f' \in C^\infty(M)$. Then we have

$$\sum_a \nabla_a \langle \nabla_{\bar{a}} f, f' \rangle = \sum_a \langle \nabla_a \nabla_{\bar{a}} f, f' \rangle + \sum_a \langle \nabla_{\bar{a}} f, \nabla_{\bar{a}} f' \rangle.$$

Define a cross section Z of S^1 by

$$Z = \sum_a \langle \nabla_{\bar{a}} f, f' \rangle\, e_a.$$

and a (complexified) tensor field A_Z of type $\binom{1}{1}$ by

$$A_Z(X) = \nabla_X Z + T(Z, X), \qquad X \in CT(M)_x.$$

Then we have

$$d(Z \lrcorner\, dV) = Trace\,(A_Z)\, dV.$$

We know that $\nabla_X Z \in S^1$ for $X \in CT(M)_x$ and $T(Z, X) \in P_x$, for $X \in (S + \bar{S})_x$, and $T(Z, \xi) = 0$. Hence we have

$$Trace\,(A_Z) = \sum_a (\nabla_a Z)^a.$$

Since the subbundle S^1 is parallel with respect to ∇, we have

$$\sum_a (\nabla_a Z)^a = \sum_a \nabla_a \langle \nabla_{\bar{a}} f, f' \rangle .$$

Hence, by using Stokes' theorem, we obtain

$$(\square_1 f, f') = -\sum_a (\nabla_a \nabla_{\bar{a}} f, f') = \sum_a (\nabla_{\bar{a}} f, \nabla_{\bar{a}} f') .$$

Similarly, we obtain

$$(\square_2 f, f') = -\sum_\alpha (\nabla_\alpha \nabla_{\bar{\alpha}} f, f') = \sum_\alpha (\nabla_{\bar{\alpha}} f, \nabla_{\bar{\alpha}} f') .$$

Therefore we see that $\square_1$ and $\square_2$ are positive semi-definite self-adjoint operators.

Finally since $\mathscr{L}_\xi g=0$, we have

$$\xi\big(\langle f, f' \rangle\big) = \langle \xi f, f' \rangle + \langle f, \xi f' \rangle .$$

Since $\xi(\langle f, f' \rangle)\, dV = d(\langle f, f' \rangle\, \xi \rfloor dV)$, it follows from Stokes' theorem that

$$(\xi f, f') + (f, \xi f') = 0 ,$$

and hence

$$(Nf, f') = (f, Nf') ,$$

implying that N is a self-adjoint operator. q. e. d.

We need the following lemma.

LEMMA 2.3 (The Ricci formula c f. [14]). *Let $f \in C^\infty(M)$ and let X, Y, $Z \in T(M)_x$.*

(1) $\nabla_X \nabla_Y f = \nabla_Y \nabla_X f - \nabla_{T(X,Y)} f$.

(2) $\nabla_X \nabla_Y \nabla_Z f = \nabla_Y \nabla_X \nabla_Z f - \nabla_{T(X,Y)} \nabla_Z f - \nabla_{R(X,Y)Z} f$.

PROPOSITION 2.4. *Let $\bar{\square}_1$, $\bar{\square}_2$ and $\bar{N}$ denote the conjugate operators of $\square_1$, $\square_2$ and N respectively.*

(1) $\square_1 N = N \square_1$, $\square_2 N = N \square_2$, *and* $\square_1 \square_2 = \square_2 \square_1$.

(2) $\bar{\square}_1 - \square_1 = rN$, $\bar{\square}_2 - \square_2 = -sN$.

(3) $\bar{N} = -N$.

PROOF. By Proposition 1.3, we have $T(\xi, X)=0$ and $R(\xi, X)=0$ for $X \in T(M)$. It follows from the Ricci formula that $\xi \square_1 = \square_1 \xi$ and $\xi \square_2 = \square_2 \xi$, which implies the first and the second equalities of (1). Similarly by using the Ricci formula and the facts that $T(e_a, \bar{e}_\alpha)=0$ and $R(e_a, \bar{e}_\alpha)=0$, we have $\square_1 \square_2 = \square_2 \square_1$, implying the third equality of (3).

By using the Ricci formula again, we see that

$$\overline{\square}_1 = -\sum \nabla_{\bar{a}} \nabla_a = -\sum \nabla_a \nabla_{\bar{a}} - \sum T(e_a, \bar{e}_a) = \square_1 + rN.$$

In the same manner, We have the second equality of (2).

Finally (3) is clear from the definition of N. q. e. d.

2.2. The space $F(M)$, and the operators A_i. We define a cross section W_1 (resp. W_2) of S^1 (resp. of S^2) by

$$W_1 = \sum_{a,b} \nabla_{\bar{a}} R^*_{a\bar{b}} e_b, \qquad W_2 = \sum_{\alpha,\beta} \nabla_{\bar{a}} R^*_{\alpha\bar{\beta}} e_\beta,$$

LEMMA 2.5. *Let $f, f' \in C^\infty(M)$.*

(1)
$$\sum_{a,b} (\nabla_{\bar{a}} \nabla_{\bar{b}} f, \nabla_{\bar{a}} \nabla_{\bar{b}} f') = (\square_1^2 f - \square_1 Nf + \sum_{a,b} R^*_{a\bar{b}} \nabla_b \nabla_{\bar{a}} f + \overline{W}_1 f, f').$$

(2)
$$\sum_{\alpha,\beta} (\nabla_{\bar{\alpha}} \nabla_{\bar{\beta}} f, \nabla_{\bar{\alpha}} \nabla_{\bar{\beta}} f') = (\square_2^2 f + \square_2 Nf + \sum_{\alpha,\beta} R^*_{\alpha\bar{\beta}} \nabla_\beta \nabla_{\bar{\alpha}} f + \overline{W}_2 f, f').$$

(3)
$$\sum_{a,\alpha} (\nabla_{\bar{a}} \nabla_{\bar{\alpha}} f, \nabla_{\bar{a}} \nabla_{\bar{\alpha}} f') = (\square_2 \square_1 f, f') = (\square_1 \square_2 f, f').$$

(4)
$$\sum_{a,\alpha} (\nabla_a \nabla_{\bar{\alpha}} f, \nabla_a \nabla_{\bar{\alpha}} f') = (\square_2 \overline{\square}_1 f, f') = (\overline{\square}_1 \square_2 f, f'),$$

$$\sum_{a,\alpha} (\nabla_\alpha \nabla_{\bar{a}} f, \nabla_\alpha \nabla_{\bar{a}} f') = (\overline{\square}_2 \square_1 f, f') = (\square_1 \overline{\square}_2 f, f').$$

PROOF. First we have

$$\sum_{a,b} \nabla_a \langle \nabla_{\bar{a}} \nabla_{\bar{b}} f, \nabla_{\bar{b}} f' \rangle = \sum_{a,b} \langle \nabla_a \nabla_{\bar{a}} \nabla_{\bar{b}} f, \nabla_{\bar{b}} f' \rangle + \sum_{a,b} \langle \nabla_{\bar{a}} \nabla_{\bar{b}} f, \nabla_{\bar{a}} \nabla_{\bar{b}} f' \rangle.$$

Define a cross section Z of S^1 by

$$Z = \sum_{a,b} \langle \nabla_{\bar{a}} \nabla_{\bar{b}} f, \nabla_{\bar{b}} f' \rangle e_a.$$

As in the proof of Proposition 2.2, we have

$$\sum_{a,b} \nabla_a \langle \nabla_{\bar{a}} \nabla_{\bar{b}} f, \nabla_{\bar{b}} f' \rangle dV = d(Z \lrcorner dV).$$

It follows from Stokes' theorem that

$$\sum_{a,b} \langle \nabla_{\bar{a}} \nabla_{\bar{b}} f, \nabla_{\bar{a}} \nabla_{\bar{b}} f' \rangle = -\sum_{a,b} (\nabla_a \nabla_{\bar{a}} \nabla_{\bar{b}} f, \nabla_{\bar{b}} f').$$

Now we see from the Ricci formula that

$$\begin{aligned} \nabla_a \nabla_{\bar{a}} \nabla_{\bar{b}} f &= \nabla_a \nabla_{\bar{b}} \nabla_{\bar{a}} f \\ &= \nabla_{\bar{b}} \nabla_a \nabla_{\bar{a}} f + \delta_{ab} \nabla_{\bar{a}} (Nf) + \big(R(\bar{e}_b, e_a) \bar{e}_a\big) f. \end{aligned}$$

By Proposition 1.3, we have

$$R(\bar{e}_b, e_a)\,\bar{e}_a = R(\bar{e}_a, e_a)\,\bar{e}_b\,.$$

Therefore we obtain

$$\sum_a \nabla_a \nabla_{\bar{a}} \nabla_{\bar{b}} f = -\nabla_{\bar{b}}(\Box_1 f) + \nabla_{\bar{b}}(Nf) + \sum_a R^{*}_{a\bar{b}} \nabla_{\bar{a}} f\,.$$

It follows from Stokes' theorem that

$$\begin{aligned}\sum_{a,b} &(\nabla_{\bar{a}} \nabla_{\bar{b}} f,\ \nabla_{\bar{a}} \nabla_{\bar{b}} f') \\ &= (\Box_1^2 f - \Box_1 Nf + \sum_{a,b} R^{*}_{a\bar{b}} \nabla_b \nabla_{\bar{a}} f + \overline{W}_1 f, f')\,,\end{aligned}$$

proving (1). The proof of (2) is quite similar.

Next, by using Stokes' theorem, we obtain

$$\sum_{a,\alpha} (\nabla_{\bar{a}} \nabla_{\bar{\alpha}} f,\ \nabla_{\bar{a}} \nabla_{\bar{\alpha}} f') = -\sum_{a,\alpha} (\nabla_a \nabla_{\bar{a}} \nabla_{\bar{\alpha}} f,\ \nabla_{\bar{\alpha}} f')\,.$$

From the Ricci formula and the fact that $T(e_a, \bar{e}_\alpha)=0$, $T(\bar{e}_a, \bar{e}_\alpha)=0$ and $R(e_a, \bar{e}_\alpha)\,\bar{e}_a=0$, we obtain

$$\sum_a \nabla_a \nabla_{\bar{a}} \nabla_{\bar{\alpha}} f = \sum_a \nabla_a \nabla_{\bar{\alpha}} \nabla_{\bar{a}} f = \sum_a \nabla_{\bar{\alpha}} \nabla_a \nabla_{\bar{a}} f\,.$$

It follows that

$$\sum_{a,\alpha} (\nabla_{\bar{a}} \nabla_{\bar{\alpha}} f,\ \nabla_{\bar{a}} \nabla_{\bar{\alpha}} f') = (\Box_2 \Box_1 f, f')\,,$$

proving (3). In the same manner we obtain (4). q. e. d.

We define differential operators A_i, $i=1, 2, 3$, on $C^\infty(M)$ respectively by

(2.1) $$A_1 f = \Box_1^2 f - \Box_1 Nf + \sum_{a,b} R^{*}_{a\bar{b}} \nabla_b \nabla_{\bar{a}} f + \overline{W}_1 f\,,$$

(2.2) $$A_2 f = \Box_2^2 f + \Box_2 Nf + \sum_{\alpha,\beta} R^{*}_{\alpha\bar{\beta}} \nabla_\beta \nabla_{\bar{\alpha}} f + \overline{W}_2 f\,,$$

(2.3) $$A_3 f = \Box_1 \Box_2 f = \Box_2 \Box_1 f\,,$$

for $f \in C^\infty(M)$.

By (1), (2) and (3) of Lemma 2.5, we have the following proposition.

PROPOSITION 2.6. *Let $f \in C^\infty(M)$. Then the following conditions are mutually equivalent*:

(1) $f \in F(M)$.

(2) $A_i f=0$, $i=1, 2, 3$.

(3) $(A_i f, f)=0$, $i=1, 2, 3$.

PROPOSITION 2.7. *The operators A_i, $i=1, 2, 3$, are positive semi-definite self-adjoint operators and satisfy $A_i N = N A_i$.*

PROOF. By Lemma 2. 5, we see that A_i are positive semi-definite self-adjoint operators. By Lemma 1. 3 and condition (C. 2), we have

$$\mathcal{L}_\xi\big(\Gamma(S^i)\big)\subset\Gamma(S^i) \quad \text{and} \quad \mathcal{L}_\xi g=0\,.$$

Hence it follows that

$$\sum_{a,b}(\nabla_{\bar a}\nabla_{\bar b}Nf,\ \nabla_{\bar a}\nabla_{\bar b}f')=\sum_{a,b}(\nabla_{\bar a}\nabla_{\bar b}f,\ \nabla_{\bar a}\nabla_{\bar b}Nf')$$

for any $f, f'\in C^\infty(M)$, which implies $A_1N=NA_1$. The other assertions can be proved in the same manner. q. e. d.

Let us denote by $\bar A_i$ the conjugate operator of A_i. Then we have the following

PROPOSITION 2. 8. *The operators $\bar A_i$, $i=1, 2, 3$, are positive semi-definite self-adjoint operators and satisfy the following*

(1) $\bar A_1=A_1+(r+1)(rN^2-r\sigma_1N+2\square_1N)+W_1-\overline{W}_1$;

(2) $\bar A_2=A_2+(s+1)(sN^2+s\sigma_2N-2\square_2N)+W_2-\overline{W}_2$;

(3) $\bar A_3=A_3-s\square_1N+r\square_2N-rsN^2$.

PROOF. It follows from the Ricci formula that

$$\begin{aligned}\overline{\sum_{a,b}R^*_{a\bar b}\nabla_b\nabla_{\bar a}}&=\sum_{a,b}R^*_{b\bar a}\nabla_{\bar b}\nabla_a=\sum_{a,b}R^*_{b\bar a}\big(\nabla_a\nabla_{\bar b}+T(e_a,\bar e_b)\big)\\&=\sum R^*_{b\bar a}\nabla_a\nabla_{\bar b}-(r+1)\,r\sigma_1N\,.\end{aligned}$$

Hence (1) follows from Proposition 2. 3 and (2. 1). The other equalities can be proved quite similarly. q. e. d.

§ 3. The structures of the Lie algebras $\mathfrak{g}(M)$ and $C\mathfrak{a}(M)$ (the non-degenerate case)

3. 1. A general structure theorem on the Lie algebras $\mathfrak{g}(M)$ and $C\mathfrak{a}(M)$. Let M be a non-degenerate PC manifold of index r satisfying conditions (C. 1) and (C. 2). First of all, let us recall the following fact.

THEOREM A (cf. [5]). *Let M be a compact non-degenerate PC manifold of index r. If $r\geqq 1$, then $\mathfrak{g}(M)$ is finite dimensional.*

Let us assume that $r\geqq1$. By Proposition 2. 4 and Theorem A, we see that $F(M)$ is a finite dimensional vector space. For each $\nu\in\boldsymbol{R}$, we define a subspace $F_{(\nu)}$ of $F(M)$ and a subspace $\tilde F_{(\nu)}$ of $F(M)\cap\bar F(M)$ respectively by

$$F_{(\nu)}=\big\{f\in F(M)\,\big|\,Nf=\nu f\big\}\,,$$

$$\tilde F_{(\nu)}=\big\{f\in F(M)\cap\bar F(M)\,\big|\,Nf=\nu f\big\}\,.$$

PROPOSITION 3.1. *Let M be a compact non-degenerate PC manifold of index r. Assume that $r\geqq 1$ and M satisfies conditions* (C. 1) *and* (C. 2).

(1) $F(M)=\sum_{\nu}F_{(\nu)}$ *(direct sum)*, $F(M)\cap\bar{F}(M)=\sum_{\nu}\tilde{F}_{(\nu)}$ *(direct sum) and* $\dim\tilde{F}_{(\nu)}=\dim\tilde{F}_{(-\nu)}$.

Assume further that the scalar curvatures σ_1 and σ_2 are equal to real constants c_1 and c_2.

(2) *The case where* $c_1=c_2$ $(=c)>0$: $F_{(\nu)}=\tilde{F}_{(\nu)}=0$ *for* $\nu\neq 0,\ -c,\ c$, $F_{(0)}=\tilde{F}_{(0)}$, $F_{(-c)}=\tilde{F}_{(-c)}$ and $F_{(c)}=\tilde{F}_{(c)}$.

(3) *The case where both c_1 and c_2 are non-positive*: $F(M)=\bar{F}(M)=F_{(0)}=\tilde{F}_{(0)}$.

(4) *The case where* $c_1>\mathrm{Max}\,(0, c_2)$: $F_{(\nu)}=0$ *for* $\nu>0$ *or* $\nu<-c_1$, $\tilde{F}_{(\nu)}=0$ *for* $\nu\neq 0$, *and* $F_{(0)}=\tilde{F}_{(0)}$. *The case where* $c_2>\mathrm{Max}\,(0, c_1)$: $F_{(\nu)}=0$ *for* $\nu<0$ *or* $\nu>c_2$, $\tilde{F}_{(\nu)}=0$ *for* $\nu\neq 0$, *and* $F_{(0)}=\tilde{F}_{(0)}$.

PROOF. We see from Propositions 2.6 and 2.7 that the operator N leaves invariant the finite dimensional subspace $F(M)$ of $C^{\infty}(M)$ and from Proposition 2.2 that N is a self-adjoint operator with respect to the inner product (,) on $F(M)$. Hence we have

$$F(M)=\sum_{\nu}F_{(\nu)} \qquad \text{(direct sum)}\,.$$

Similarly we see that the operator N leaves invariant the subspace $F(M)\cap\bar{F}(M)$ of $C^{\infty}(M)$ and hence we obtain

$$F(M)\cap\bar{F}(M)=\sum_{\nu}\tilde{F}_{(\nu)} \qquad \text{(direct sum)}\,.$$

By Proposition 2.4, we see that the correspondence $f\rightarrow\bar{f}$ gives an isomorphism of $\tilde{F}_{(\nu)}$ onto $\tilde{F}_{(-\nu)}$, and hence

$$\dim\tilde{F}_{(\nu)}=\dim\tilde{F}_{(-\nu)}\,,$$

proving (1).

Hereafter we assume that σ_i are constant. We first assert that $\overline{W}_1=0$. Indeed, for any $X\in S_x^1$, we have

$$\begin{aligned} g(X, \overline{W}_1) &= g(X, \sum_{a,b}\nabla_b R^*_{a\bar{b}}\bar{e}_a)=g(X, \sum_{a,b}\nabla_a R^*_{b\bar{b}}\bar{e}_a) \\ &= r(r+1)X\sigma_1=0\,, \end{aligned}$$

and hence obtain $\overline{W}_1=0$. In the same way, we can show that $\overline{W}_2=0$.

Let $f\in F_{(\nu)}$. By Proposition 2.6, we have

$$A_i f=0\,, \qquad i=1, 2, 3\,.$$

It is easy to see from Proposition 2.8 that if $\nu=0$, then

$$\bar{A}_i f = 0\,, \qquad i = 1, 2, 3\,.$$

Hence we have $F_{(0)} = \tilde{F}_{(0)} \subset \bar{F}(M)$.

Suppose that there is a positive number ν such that $F_{(\nu)} \neq 0$. We claim that $\nu \leqq c_2$ and $c_1 \leqq c_2$. For this purpose, let f be a non-trivial function contained in $F_{(\nu)}$. By Proposition 2. 8, we have

$$r(r+1)\,\nu(\nu - c_1)\,(f, f) + 2(r+1)\,\nu(\Box_1 f, f) = (\bar{A}_1 f, f) \geqq 0\,, \tag{3. 1}$$

$$s(s+1)\,\nu(\nu + c_2)\,(f, f) - 2(s+1)\,\nu(\Box_2 f, f) = (\bar{A}_2 f, f) \geqq 0\,, \tag{3. 2}$$

$$-s\nu(\Box_1 f, f) + r\nu(\Box_2 f, f) - rs\nu^2(f, f) = (\bar{A}_3 f, f) \geqq 0\,. \tag{3. 3}$$

Hence we obtain

$$\frac{1}{r}(\Box_1 f, f) \geqq \frac{1}{2}(c_1 - \nu)\,(f, f)\,,$$

$$\frac{1}{s}(\Box_2 f, f) \leqq \frac{1}{2}(c_2 + \nu)\,(f, f)\,,$$

$$-\frac{1}{r}(\Box_1 f, f) + \frac{1}{s}(\Box_2 f, f) \geqq \nu(f, f)\,.$$

From the second and the third inequalities, we have

$$\frac{1}{r}(\Box_1 f, f) \leqq \frac{1}{2}(c_2 - \nu)\,(f, f)\,.$$

Since $\Box_1$ is a positive semi-definite operator, we have $\nu \leqq c_2$. By using the first and the fourth inequalities, we have $c_1 \leqq c_2$. These prove our assertions. Similarly we can prove that if there is a negative number ν such that $F_{(\nu)} \neq 0$, then $\nu \geqq -c_1$ and $c_2 \geqq c_1$. From these facts, we obtain (3) and (4).

Let us prove (2). Assume that $c_1 = c_2$ $(=c) > 0$. Let ν be a positive number such that $F_{(\nu)} \neq 0$. Take any $f \in F_{(\nu)}$. It is easily verified that equalities hold in (3. 1), (3. 2) and (3. 3). Hence it follows from Proposition 2. 6 that $f \in \tilde{F}_{(\nu)}$. By Proposition 2. 8, we have $\Box_1 f = \frac{r}{2}(c - \nu) f$ and $\Box_2 f = \frac{s}{2}(c + \nu) f$. Since $A_3 f = 0$, we have

$$\Box_1 \Box_2 f = \frac{rs}{4}(c - \nu)\,(c + \nu) f = 0\,.$$

If $f \neq 0$, then we have $\nu = c$. We have thus shown that $F_{(\nu)} = 0$ for $\nu \neq c$, and $F_{(\nu)} = \tilde{F}_{(\nu)}$. In the same manner, we can prove that, for a negative number ν, $F_{(\nu)} = 0$ except $\nu = -c$, and $F_{(-c)} = \tilde{F}_{(-c)}$. We have thus completed the proof of Proposition 3. 1. q. e. d.

For each $\nu \in \boldsymbol{R}$, we define a subspace $\mathfrak{g}_{(\nu)}$ of $\mathfrak{g}(M)$ and a subspace

$\tilde{\mathfrak{g}}_{(\nu)}$ of $\boldsymbol{Ca}(M)$ respectively by

$$\mathfrak{g}_{(\nu)} = \left\{u \in \mathfrak{g}(M) \middle| \sqrt{-1}\,[\xi, u] = \nu u\right\},$$
$$\tilde{\mathfrak{g}}_{(\nu)} = \left\{u \in \boldsymbol{Ca}(M) \middle| \sqrt{-1}\,[\xi, u] = \nu u\right\}.$$

It is easy to see that the assignment $u \to f_u$ gives an isomorphism of $\mathfrak{g}_{(\nu)}$ onto $F_{(\nu)}$ and an isomorphism of $\tilde{\mathfrak{g}}_{(\nu)}$ onto $\tilde{F}_{(\nu)}$.

THEOREM 3.2. *Let M be a compact non-degenerate PC manifold of index r. Assume that $r \geqq 1$ and M satisfies conditions* (C.1) *and* (C.2).

(1) $\mathfrak{g}(M) = \sum \mathfrak{g}_{(\nu)}$, $\boldsymbol{Ca}(M) = \sum \tilde{\mathfrak{g}}_{(\nu)}$ *(direct sum) and* $\dim \tilde{\mathfrak{g}}_{(\nu)} = \dim \tilde{\mathfrak{g}}_{(-\nu)}$. *Moreover* $\mathfrak{g}(M)$ *and* $\boldsymbol{Ca}(M)$ *become graded Lie algebras with respect to these decompositions.*

(2) *If an infinitesimal automorphism X of M is contained in* $\mathfrak{a}(M) \cap \mathfrak{g}_{(0)}$, *then X satisfies* $[X, \Gamma(S^i)] \subset \Gamma(S^i)$, $i=1, 2$.

Assume further that the scalar curvatures σ_1 and σ_2 are equal to real constants c_1 and c_2.

(3) *The case where* $c_1 = c_2\ (=c) > 0$: $\mathfrak{g}_{(\nu)} = 0$ *for* $\nu \neq 0, -c, c$, $\mathfrak{g}_{(0)} = \tilde{\mathfrak{g}}_{(0)}$, $\mathfrak{g}_{(c)} = \tilde{\mathfrak{g}}_{(c)}$, $\mathfrak{g}_{(-c)} = \tilde{\mathfrak{g}}_{(-c)}$ *and* $\mathfrak{g}(M) = \boldsymbol{Ca}(M) = \mathfrak{g}_{(0)} + \mathfrak{g}_{(c)} + \mathfrak{g}_{(-c)}$.

(4) *The case where both c_1 and c_2 are non-positive*: $\mathfrak{g}_{(\nu)} = \tilde{\mathfrak{g}}_{(\nu)} = 0$ for $\nu \neq 0$, $\mathfrak{g}_{(0)} = \tilde{\mathfrak{g}}_{(0)}$ *and* $\mathfrak{g}(M) = \boldsymbol{Ca}(M) = \mathfrak{g}_{(0)}$.

(5) *The case where* $c_1 > \mathrm{Max}\,(0, c_2)$: $\mathfrak{g}_{(\nu)} = 0$ *for* $\nu > 0$ *or* $\nu < -c_1$, $\tilde{\mathfrak{g}}_{(\nu)} = 0$ *for* $\nu \neq 0$, $\mathfrak{g}_{(0)} = \tilde{\mathfrak{g}}_{(0)}$ *and* $\boldsymbol{Ca}(M) = \mathfrak{g}_{(0)}$. *The case where* $c_2 > \mathrm{Max}\,(0, c_1)$: $\mathfrak{g}_{(\nu)} = 0$ *for* $\nu < 0$ *or* $\nu > c_2$, $\tilde{\mathfrak{g}}_{(\nu)} = 0$ *for* $\nu \neq 0$, $\mathfrak{g}_{(0)} = \tilde{\mathfrak{g}}_{(0)}$ *and* $\boldsymbol{Ca}(M) = \mathfrak{g}_{(0)}$.

PROOF. We prove only (2). The other assertions follow immediately from Proposition 3.1. Let X be an infinitesimal automorphism of M which is contained in $\mathfrak{a}(M) \cap \mathfrak{g}_{(0)}$. Then we see that the function f_X corresponding to X satisfies $\Box_1 \Box_2 f_X = N f_X = 0$. By Proposition 2.4, we have $\Box_2 \Box_1 f_X = \Box_1 \Box_2 f_X = 0$. Take any cross section Y (resp. W) of S^1 (resp. of S^2). Then we have

$$\nabla_Y \nabla_{\bar{W}} f_X = \nabla_W \nabla_{\bar{Y}} f_X = 0\,,$$

which implies that

$$(d\theta)\,(\nabla_Y X^S, \bar{W}) = (d\theta)\,(\nabla_W X^S, \bar{Y}) = 0\,,$$

where X^S is the S-component of X. Therefore we have $\nabla_Y X^S \in \Gamma(S^1)$ and $\nabla_W X^S \in \Gamma(S^2)$. From the fact that X is an infinitesimal automorphism and the fact that P and S are parallel with respect to the canonical affine connection ∇, we obtain

$$[X, Y] = [X, Y]^S = [f_X\xi, Y]^S + [X^S, Y]^S + [\bar{X}^S, Y]^S$$
$$= f_X \nabla_\xi Y + \nabla_{X^S} Y - \nabla_Y X^S .$$

Since S^1 is parallel with respect to ∇, we have $[X, Y] \in \Gamma(S^1)$. Similarly we obtain $[X, W] \in \Gamma(S^2)$. Therefore we have $[X, \Gamma(S^i)] \subset \Gamma(S^i)$. q. e. d.

3. 2. Condition (C. 3) and the space $F(M)$. In the rest of this section, we further assume the following condition:

(C. 3) The Ricci tensor R^* satisfies

$$R^*(X, \bar{Y}) = (r+1)\, c_1 g(X, \bar{Y}) \qquad \text{for any} \quad X, Y \in S_x^1 ,$$
$$R^*(X, \bar{Y}) = (s+1)\, c_2 g(X, \bar{Y}) \qquad \text{for any} \quad X, Y \in S_x^2 ,$$

where c_1 and c_2 are real constants.

First of all, we remark the following equalities:

$$A_1 = \Box_1^2 - \Box_1 N - (r+1)\, c_1 \Box_1 ,$$
$$A_2 = \Box_2^2 + \Box_2 N - (s+1)\, c_2 \Box_2 ,$$
$$A_3 = \Box_1 \Box_2 = \Box_2 \Box_1 ,$$
$$\bar{A}_1 = A_1 + (r+1)\,(rN^2 - rc_1 N + 2\Box_1 N) ,$$
$$\bar{A}_2 = A_2 + (s+1)\,(sN^2 + sc_2 N - 2\Box_2 N) ,$$
$$\bar{A}_3 = A_3 - s\Box_1 N + r\Box_2 N - rsN^2 .$$

We also remark that the operators A_i, $\Box_i$ and N commute one another. For each triple $(\lambda_1, \lambda_2, \nu)$ of real numbers, we define a subspace $F_{(\lambda_1, \lambda_2, \nu)}$ of $C^\infty(M)$ *by*

$$F_{(\lambda_1, \lambda_2, \nu)} = \left\{ f \in C^\infty(M) \,\middle|\, \Box_1 f = \lambda_1 f,\ \Box_2 f = \lambda_2 f,\ Nf = \nu f \right\}$$

Moreover we define subspaces F^i, $i=0, \cdots, 4$ of $C^\infty(M)$ as follows: First we put $F^0 = F_{(0,0,0)}$. If $c_1 \neq 0$, then we put $F^1 = F_{((r+1)c_1, 0, 0)}$ and $F^3 = F_{(rc_1, 0, -c_1)}$, and if $c_1 = 0$, then we put $F^1 = F^3 = 0$. Similarly if $c_2 \neq 0$, then we put $F^2 = F_{(0, (s+1)c_2, 0)}$ and $F^4 = F_{(0, sc_2, c_2)}$, and if $c_2 = 0$, then we put $F^2 = F^4 = 0$.

PROPOSITION 3. 3. *Let M be a compact non-degenerate PC manifold of index r. Assume that $r \geqq 1$ and M satisfies conditions* (C. 1), (C. 2) *and* (C. 3).

(1) $F(M) = \sum_{i=1}^{4} F^i$ *(direct sum).*

(2) F^0 *consists of all constant functions.*

(3) *If $c_1 \leqq 0$ (resp. If $c_2 \leqq 0$), then $F^1 = 0$ (resp. $F^2 = 0$).*

(4) *If $c_1 < c_2$ or $c_1 \leqq 0$ (resp. If $c_1 > c_2$ or $c_2 \leqq 0$), then $F^3 = 0$ (resp. $F^4 = 0$).*

PROOF. We first show that $F^i \subset F(M)$, $i = 0, \cdots, 4$. Let $f \in F^i$. By (3. 4), (3. 5) and (3. 6), we have

$$A_i f=0\,, \qquad i=1,2,3\,.$$

Hence it follows from Proposition 2.6 that $f\in F(M)$, which proves our assertion. Conversely we will show that $F(M)\subset\sum_i F^i$. First of all, we remark that $\dim F(M)<+\infty$ and that the operators $\square_1$, $\square_2$ and N are self-adjoint and leave $F(M)$ invariant and commute one another. Therefore we obtain

$$F(M)=\sum F^*_{(\lambda_1,\lambda_2,\nu)}\,,$$

where $F^*_{(\lambda_1,\lambda_2,\nu)}$ is the subspace of $F(M)$ defined by

$$F^*_{(\lambda_1,\lambda_2,\nu)}=F(M)\cap F_{(\lambda_1,\lambda_2,\nu)}\,.$$

Let $(\lambda_1, \lambda_2, \nu)$ be a triple such that $F^*_{(\lambda_1,\lambda_2,\nu)}\neq 0$. We must show that $F^*_{(\lambda_1,\lambda_2,\nu)}$ is contained in some F^i. Since $A_i f=0$, we have

(3.4) $\lambda_1^2-\nu\lambda_1-(r+1)\,c_1\lambda_1=0\,,$

(3.5) $\lambda_2^2+\nu\lambda_2-(s+1)\,c_2\lambda_2=0\,,$

(3.6) $\lambda_1\lambda_2=0\,.$

Since $\square_1$, $\square_2$, $\overline{\square}_1$ and $\overline{\square}_2$ are positive semi-definite operators, we obtain

(3.7) $\lambda_1\geqq 0\,,$

(3.8) $\lambda_2\geqq 0\,,$

(3.9) $\lambda_1+r\nu\geqq 0\,,$

(3.10) $\lambda_2-s\nu\geqq 0\,.$

Moreover since $\bar{A}_i$ are positive semi-definite operators, it follows that

(3.11) $(r+1)\,(r\nu^2-rc_1\nu+2\lambda_1\nu)\geqq 0\,,$

(3.12) $(s+1)\,(s\nu^2+sc_2\nu-2\lambda_2\nu)\geqq 0\,,$

(3.13) $-s\lambda_1\nu+r\lambda_2\nu-rs\nu^2\geqq 0\,.$

By (3.6), (3.7) and (3.8), it suffices to consider the following three cases.
1] The case where $\lambda_1=\lambda_2=0$: By (3.9) and (3.10), we have $\nu=0$.
2] The case where $\lambda_1>0$ and $\lambda_2=0$: By (3.4), we obtain

$$\lambda_1=\nu+(r+1)\,c_1\,.$$

Substituting $\lambda_1=\nu+(r+1)\,c_1$ into (3.9), we have

$$\nu+c_1\geqq 0\,.$$

By (3.10), we have $\nu\leqq 0$. Hence we obtain

$$-c_1 \leqq \nu \leqq 0 .$$

On the other hand, it follows from (3. 11) that

$$\nu(\nu+c_1) \geqq 0 .$$

From these facts, we see that $\nu=0$ or $\nu=-c_1$. If $\nu=0$, then we have $(\lambda_1, \lambda_2, \nu)=((r+1)c_1, 0, 0)$ and $c_1 \geqq 0$. If $\nu=-c_1$, then we have $(\lambda_1, \lambda_2, \nu)=(rc_1, 0, -c_1)$ and $c_1 \geqq 0$. Moreover by (3. 12), we have $c_1 \geqq c_2$.

3] The case where $\lambda_1=0$ and $\lambda_2>0$: In the same manner as in 2], we see that the triple $(\lambda_1, \lambda_2, \nu)$ coincides with $(0, (s+1)c_2, 0)$ or $(0, sc_2, c_2)$. Furthermore we see that if $(\lambda_1, \lambda_2, \nu)=(0, (s+1)c_2, 0)$, then $c_2 \geqq 0$ and if $(\lambda_1, \lambda_2, \nu)=(0, sc_2, c_2)$ then $c_2 \geqq 0$ and $c_2 \geqq c_1$.

We have thus shown that $F(M) \subset \sum F^i$ and hence have completed the proof of (1). (3) and (4) follow immediately from the discussions in 1], 2], and 3]. Let f be a function contained in F^0, then we have $\Box_1 f=\Box_2 f=Nf=0$. By Proposition 2. 4, we have $\overline{\Box}_1 f=\overline{\Box}_2 f=0$. Hence it follows that $\nabla f=0$. Thus f is constant, proving (2). q. e. d.

Let $f \in F_{(\lambda_1, \lambda_2, \nu)}$. Then we see that $\bar{f} \in F_{(\lambda_1+r\nu, \lambda_2-s\nu, -\nu)}$. Therefore putting $\tilde{F}^i=F^i \cap \bar{F}(M)$, we obtain

COROLLARY 3. 4. (1) *$F(M) \cap \bar{F}(M)=\sum \tilde{F}^i$ (direct sum).*

(2) *$F^0=\tilde{F}^0$, $F^1=\tilde{F}^1$ and $F^2=\tilde{F}^2$.*

(3) *If $c_1=c_2$, then $F^3=\tilde{F}^3$ and $F^4=\tilde{F}^4$.*

(4) *If $c_1 \neq c_2$, then $\tilde{F}^3=\tilde{F}^4=0$.*

3. 3. Condition (C. 3) and the structure theorems on the Lie algebras $\mathfrak{g}(M)$ and $C\mathfrak{a}(M)$. Let us denote by $\mathfrak{g}^i$ the subspace of $\mathfrak{g}(M)$ which corresponds to F^i through the isomorphism $u \rightarrow f_u$ of $\mathfrak{g}(M)$ onto $F(M)$.

THEOREM 3. 5. *Let M be a compact non-degenerate PC manifold of index r. Assume that $r \geqq 1$ and M satisfies conditions (C. 1), (C. 2) and (C. 3).*

(1) *$\mathfrak{g}(M)=\sum_{i=1}^{4} \mathfrak{g}^i$ (vector space direct sum).*

(2) *$\mathfrak{g}^0=\{C\xi\}$, $\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2=\{u \in \mathfrak{g}(M) \mid [\xi, u]=0\}$ $(=\mathfrak{g}_{(0)})$. If $c_1 \neq 0$, then $\mathfrak{g}^3=\{u \in \mathfrak{g}(M) \mid \sqrt{-1}[\xi, u]=-c_1 u\}$ $(=\mathfrak{g}_{(-c_1)})$. If $c_2 \neq 0$, then $\mathfrak{g}^4=\{u \in \mathfrak{g}(M) \mid \sqrt{-1}[\xi, u]=c_2 u\}$ $(=\mathfrak{g}_{(c_2)})$.*

(3) *If $u \in \mathfrak{g}^1$ or $u \in \mathfrak{g}^3$ (resp. If $u \in \mathfrak{g}^2$ or $u \in \mathfrak{g}^4$), then the cross section U^S of S corresponding to u satisfies $U^S_x \in S^1_x$ (resp. $U^S_x \in S^2_x$) at any point x of M.*

(4) *$\mathfrak{g}^i$, $i=0, 1, 2$, are subalgebras of $\mathfrak{g}(M)$, and $\mathfrak{g}_{(0)}=\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2$ (direct sum of Lie algebras).*

(5) $[\mathfrak{g}^i, \mathfrak{g}^3]\subset\mathfrak{g}^3$, $i=0, 1, 2$, *and* $[\mathfrak{g}^i, \mathfrak{g}^4]\subset\mathfrak{g}^4$, $i=0, 1, 2$.

(6) *If* $c_1\leqq 0$ (*resp. If* $c_2\leqq 0$), *then* $\mathfrak{g}^1=0$ (*resp.* $\mathfrak{g}^2=0$).

(7) *If* $c_1<c_2$ *or* $c_1\leqq 0$ (*resp. If* $c_2<c_1$ *or* $c_2\leqq 0$), *then* $\mathfrak{g}^3=0$ (*resp.* $\mathfrak{g}^4=0$).

(8) *If* $c_1=c_2>0$, *then* $[\mathfrak{g}^3, \mathfrak{g}^4]\subset\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2$.

PROOF. (1) follows immediately from (1) of Proposition 3.3.

From (2) of Proposition 3.3, we have $\mathfrak{g}^0=\{C\xi\}$. It follows from Proposition 3.3 that $F^0+F^1+F^2=F_{(0)}$, if $c_1\neq 0$, then $F^3=F_{(-c_1)}$, and if $c_2\neq 0$, then $F^4=F_{(c_2)}$. This implies that $\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2=\mathfrak{g}_{(0)}$, if $c_1\neq 0$, then $\mathfrak{g}^3=\mathfrak{g}_{(-c_1)}$, and if $c_2\neq 0$ then $\mathfrak{g}^4=\mathfrak{g}_{(c_2)}$, proving (2).

Now let us prove (3). Let $u\in\mathfrak{g}^1$. Then the corresponding function f_u satisfies $\Box_2 f_u=0$, which implies that $\nabla_{\bar{Y}} f_u=0$ for any $\bar{Y}\in\bar{S}^2_x$. We have

$$(d\theta)(U^S_x, \bar{Y}) = -\nabla_{\bar{Y}} f_u = 0\,,$$

and hence $U^S_x\in S^1_s$. The other assertions of (3) are quite similar.

Let u, $v\in\mathfrak{g}_{(0)}$. Let us denote by U and V the cross sections of $T(=S+P)$ which correspond to u and v. We first remark that $\xi f_u=\xi f_v=0$ and $[\xi, U^S]=[\xi, V^S]=0$ and $[U, V]=(U^S f_v-V^S f_u)\,\xi+[U^S, V^S]$. Therefore we obtain $f_{[u,v]}=U^S f_v-V^S f_u$ and $Nf_{[u,v]}=0$.

Assume that u, $v\in\mathfrak{g}^1$. By a direct calculation, we have

$$\begin{aligned}\Box_1(U^S f_v) &= U^S(\Box_1 f_v)+\sqrt{-1}\sum_{a,b}\nabla_a\nabla_{\bar{a}} f_u \nabla_b\nabla_{\bar{a}} f_v+U^S(Nf_v)+\sqrt{-1}\,\Box_1 f_u Nf_v\\ &=(r+1)\,c_1 U^S f_v+\sqrt{-1}\sum_{a,b}\nabla_a\nabla_{\bar{b}} f_u \nabla_b\nabla_{\bar{a}} f_v\,.\end{aligned}$$

In the same manner, we obtain

$$\Box_1(V^S f_u)=(r+1)\,c_1 V^S f_u+\sqrt{-1}\sum_{a,b}\nabla_a\nabla_{\bar{b}} f_u \nabla_b\nabla_{\bar{a}} f_v\,,$$

Therefore we obtain

$$\Box_1 f_{[u,v]}=(r+1)\,c_1 f_{[u,v]}\,.$$

Similarly we obtain

$$\Box_2 f_{[u,v]}=0\,.$$

From these facts we obtain $f_{[u,v]}\in F^1$, implying that $[\mathfrak{g}^1, \mathfrak{g}^1]\subset\mathfrak{g}^1$. In the same manner, we have $[\mathfrak{g}^2, \mathfrak{g}^2]\subset\mathfrak{g}^2$.

Let $u\in\mathfrak{g}^1$ and $v\in\mathfrak{g}^2$ By (3), we have $U^S_x\in S^1_x$ and $V^S_x\in S^2_x$. Since $\overline{\Box}_2 f_u=\overline{\Box}_1 f_v=0$, we have

$$f_{[u,v]}=U^S f_v-V^S f_u=0\,,$$

implying that $[u, v]=0$. Therefore we have $[\mathfrak{g}^1, \mathfrak{g}^2]=0$. We have thus proved (4).

(5) follows directly from (2). (6) and (7) are also immediate from (3) and (4) of Proposition 3.3. Finally (8) follows from (2). q. e. d.

As a consequence of Corollary 3.4, we have

THEOREM 3.6. *Let M be a compact non-degenerate PC manifold of index r. Assume that $r\geqq 1$ and M satisfies conditions (C.1), (C.2) and (C.3).*

(1) *If $c_1=c_2$, then $\boldsymbol{C}\mathfrak{a}(M)=\sum_{i=1}^{4}\mathfrak{g}^i$. In particular, if $c_1=c_2\leqq 0$, then $\boldsymbol{C}\mathfrak{a}(M)=\mathfrak{g}(M)=\mathfrak{g}^0$.*

(2) *If $c_1\neq c_2$, then $\boldsymbol{C}\mathfrak{a}(M)=\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2$.*

§4. Applications and examples (the non-degenerate case)

4.1. Some general facts on holomorphic line bundles. Let $\tilde{M}$ be an $(n-1)$-dimensional complex manifold, and F a holomorphic line bundle over $\tilde{M}$ with a hermitian metric h. Let P be the principal $\boldsymbol{C}^*=GL(1,\boldsymbol{C})$-bundle associated with F, and π the projection of P onto $\tilde{M}$. For each $a\in\boldsymbol{C}^*$, let R_a denote the right translation, that is, $R_a x=xa$, $x\in P$.

Let $\{U_\alpha\}$ be an open covering of $\tilde{M}$ (with sufficiently small U_α's), and, for each α, let e^α be a local frame of F defined on U_α. Let us consider the corresponding holomorphic trivializations $x\rightarrow(\pi(x), z^\alpha(x))$ of $\pi^{-1}(U_\alpha)$ onto $U_\alpha\times\boldsymbol{C}^*$, and the corresponding system of transition functions, $\{\tau_{\alpha\beta}\}$. Then we have

$$z^\alpha(xa)=z^\alpha(x)\,a,\quad x\in\pi^{-1}(U_\alpha),\ a\in\boldsymbol{C}^*,$$
$$z^\alpha(x)=\tau_{\alpha\beta}\big(\pi(x)\big)\,z^\beta(x),\quad x\in\pi^{-1}(U_\alpha\cap U_\beta)\,.$$

For each α we define a function h^α on U_α by $h^\alpha(y)=h(e_y^\alpha, e_y^\alpha)$, $y\in U_\alpha$, and put $\omega_\alpha=\partial\log h^\alpha$. As is well known, the 1-forms $\pi^*\omega^\alpha+\dfrac{1}{z_\alpha}dz^\alpha$ on $\pi^{-1}(U_\alpha)$ define a global 1-form ω on P, which is a connection form in the principal bundle P and represents the canonical connection of the hermitian holomorphic line bundle F (cf. [7]). Let us now consider the curvature form $\Omega=d\omega$ of ω. Then we know that there is a unique 2-form Φ on $\tilde{M}$ such that $\pi^*\Phi=\dfrac{\sqrt{-1}}{2\pi}\Omega$, which is usually called the first Chern form of F. Note that Φ is a real form of type (1.1).

Let M be the $U(1)$-reduction of P defined by h, which is a real hypersurface of P, and let S be the induced PC structure on M. It is easy to see that $M\cap\pi^{-1}(U_\alpha)$ is defined by the equation $\pi^*h^\alpha|z^\alpha|^2=1$. From this fact we easily obtain

PROPOSITION 4.1 (cf. [7]). *For each* $x \in M$, S_x *consists of all* $X \in T^{1,0}(P)_x$ *such that* $\omega(X)=0$.

Let ξ_P be the vector field on P induced from the 1-parameter group of right translations $R_{e^{\sqrt{-1}t}}$, $t \in \boldsymbol{R}$. Clearly ξ_P is tangent to M, and hence the restriction ξ of ξ_P to M becomes an infinitesimal automorphism of the PC manifold M. We define a 1-form θ on M by

$$\theta = -\sqrt{-1}\, i^* \omega ,$$

i being the injection $M \to P$. Then we have $\theta(\xi)=1$ and $\theta(S)=0$ (Proposition 4.1). Especially we see from this fact that M satisfies condition (C. 1) with respect to ξ. Let L be the Levi form on M corresponding to the real 1-form θ. Then we have

PROPOSITION 4.2. $L(X, Y) = -\Omega(X, \bar{Y})$, $X, Y \in S_x$.

Let us now consider the Lie algebra $\mathfrak{g}(M)$ of all holomorphic vector fields on M and its subspaces $\mathfrak{g}_{(\nu)}$ (see § 1 and § 3). From the definition of ξ we easily obtain

PROPOSITION 4.3. (1) *If* ν *is not an integer, then* $\mathfrak{g}_{(\nu)}=0$.

(2) *If* ν *is equal to an integer* m, *then*

$$\mathfrak{g}_{(m)} = \left\{ u \in \mathfrak{g}(M) \,\middle|\, R_{a*} u = a^m u,\ u \in U(1) \right\}.$$

We denote by $\mathfrak{g}(P)$ the Lie algebra of all holomorphic vector fields on P, and, for any integer m, define a subspace $\mathfrak{g}(P)_{(m)}$ by

$$\mathfrak{g}(P)_{(m)} = \left\{ X \in \mathfrak{g}(P) \,\middle|\, R_{a*} X = a^m X, a \in \boldsymbol{C}^* \right\}.$$

Then we have

PROPOSITION 4.4. *The assignment* $X \to X|M$ *gives an isomorphism of* $\mathfrak{g}(P)_{(m)}$ *onto* $\mathfrak{g}_{(m)}$.

By virture of this fact the study of $\mathfrak{g}_{(m)}$ is reduced to that of $\mathfrak{g}(P)_{(m)}$.

We denote by $C^\infty(P)_{(m)}$ the space of all functions f on P such that $R_a^* f = a^{-m} f$, $a \in \boldsymbol{C}^*$. We will construct a linear mapping $f \to \tilde{f}$ of $C^\infty(P)_{(m)}$ to $\Gamma(F^m)$, where F^m denotes the m-th tensor product of F if $m \geqq 0$, and the $(-m)$-th tensor product of the dual bundle F^* of F if $m<0$. Take any $f \in C^\infty(P)_{(m)}$. For each α we define a function f^α on $\pi^{-1}(U_\alpha)$ by $f^\alpha = (z^\alpha)^m f$. Then we have $R_a^* f^\alpha = f^\alpha$, $a \in \boldsymbol{C}^*$, and hence there is a unique function $\tilde{f}^\alpha$ on U_α such that $f^\alpha = \pi^* \tilde{f}^\alpha$. We have $\tilde{f}^\alpha = (\tau_{\alpha\beta})^m \tilde{f}^\beta$. Therefore the local cross sections $\tilde{f}^\alpha \otimes (e^\alpha)^m$ of F^m give rise to a global cross section $\tilde{f}$ of F^m, where $(e^\alpha)^m$ denotes the local frame of F^m naturally induced from e^α. This completes our construction. It is easy to see that the assignment $f \to \tilde{f}$ gives an isomorphism of $C^\infty(P)_{(m)}$ onto $\Gamma(F^m)$ and that f is holomorphic if and only if $\tilde{f}$ is holomorphic.

We now denote by $\Gamma(T^{1,0}(P))_{(m)}$ the space of all cross sections X of $T^{1,0}(P)$ such that $R_{a*}X=a^m X$, $a\in \boldsymbol{C}^*$. We construct a linear mapping $X\to\tilde{X}$ of $\Gamma(T^{1,0}(P))_{(m)}$ to $\Gamma(T^{1,0}(\tilde{M})\otimes F^m)$ as follows: For each α we define a vector field X^α on $\pi^{-1}(U_\alpha)$ by $X^\alpha=(z^\alpha)^m X$. Then we see as above that there is a unique vector field $\tilde{X}^\alpha$ on U_α such that $\tilde{X}^\alpha=\pi_*(X^\alpha)$ and that the local cross sections $\tilde{X}^\alpha\otimes(e^\alpha)^m$ of $T^{1,0}(\tilde{M})\otimes F^m$ give rise to a global cross section $\tilde{X}$ of $T^{1,0}(\tilde{M})\otimes F^m$ completing our construction. It is easy to see that if X is holomorphic, so is $\tilde{X}$.

For any $X\in\Gamma(T^{1,0}(P))_{(m)}$ we put $\rho_X=\omega(X)$, which is an element of $C^\infty(P)_{(m)}$. Then we notice that the assignment $X\to(\tilde{X}, \tilde{\rho}_X)$ gives an isomorphism of $\Gamma(T^{1,0}(P))_{(m)}$ onto $\Gamma(T^{1,0}(\tilde{M})\otimes F^m)\times\Gamma(F^m)$.

As we have just seen, the mapping $X\to\tilde{X}$ induces a linear mapping of $\mathfrak{g}(P)_{(m)}$ to $\Gamma_{hol}(T^{1,0}(\tilde{M})\otimes F^m)$, which we denote by κ_m. As before let Φ be the Chern form of F. Let us assume that Φ_x is non-degenerate at any point x of $\tilde{M}$. Then we have a linear isomorphism $\lrcorner\Phi$ of $T^{1,0}(\tilde{M})\otimes F^m$ onto $(T^{0,1}(\tilde{M})^*\otimes F^m)$, which is naturally induced from the isomorphism $X\to X\lrcorner\Phi$ of $T^{1,0}(\tilde{M})$ onto $(T^{0,1}(\tilde{M}))^*$. Let $\bar\partial$: $\Gamma(F^m)\to\Gamma(T^{0,1}(\tilde{M}))^*\otimes F^m)$ be the Cauchy-Riemann operator, and let $\xi_P^{(1,0)}$ be the $(1,0)$-part of the real vector field ξ_P. Then the next theorem determines the image and the kernel of the linear mapping κ_m.

THEOREM 4.5. *Assume that Φ_x is non-degenerate at any point x of $\tilde{M}$.*
(1) Im κ_m *consists of all $Y\in\Gamma(T^{1,0}(\tilde{M})\otimes F^m)$ such that $Y\lrcorner\Phi$ is $\bar\partial$-exact.*
(2) Ker κ_m *consists of all holomorphic vector fields of the form $\rho\xi_P^{(1,0)}$, where ρ is a holomorphic function in $C^\infty(P)_{(m)}$. Hence* Ker κ_m *is isomorphic to $\Gamma_{hol}(F^m)$.*

PROOF. Let $X\in\mathfrak{g}(P)_{(m)}$. If we put $\psi_X=X\lrcorner\Omega$, we see that ψ_X is a 1-form of type $(0,1)$ on P, and satisfies: $\psi_X(\xi_P)=0$ and $R_a^*\psi_X=a^{-m}\psi_X$, $a\in\boldsymbol{C}^*$. For each α we define a 1-form ψ_X^α on $\pi^{-1}(U_\alpha)$ by $\psi_X^\alpha=(z^\alpha)^m\psi_X$. Then it follows that there is a unique 1-form $\tilde\psi_X^\alpha$ of type $(0,1)$ on U_α such that $\pi^*\tilde\psi_X^\alpha=\psi_X^\alpha$. Since $\pi^*\Phi=\dfrac{\sqrt{-1}}{2\pi}\Omega$, we easily obtain

$$\tilde{X}\lrcorner\Phi=\frac{\sqrt{-1}}{2\pi}\tilde\psi_X^\alpha\otimes(e^\alpha)^m \qquad \text{on } U_\alpha\,.$$

Since X is holomorphic, we have $[X,\bar{Y}]\in\Gamma(T^{0,1}(P))$ for any $\bar{Y}\in\Gamma(T^{0,1}(P))$, and hence

$$\psi_X(\bar{Y}) = \Omega(X, \bar{Y}) = X\omega(\bar{Y}) - \bar{Y}\omega(X) - \omega\big([X, \bar{Y}]\big)$$
$$= -\bar{Y}\rho_X,$$

meaning that $\psi_X = -\bar{\partial}\rho_X$. Therefore we have $\tilde{\psi}_X^\alpha = -\bar{\partial}\tilde{\rho}_X^\alpha$, and hence

$$\tilde{X} \lrcorner \Phi = -\frac{\sqrt{-1}}{2\pi}\bar{\partial}\tilde{\rho}_X.$$

Conversely let Y be a holomorphic cross section of $T^{1,0}(\tilde{M}) \otimes F^m$ such that $Y \lrcorner \Phi$ is $\bar{\partial}$-exact. Take a cross section $\tilde{\rho}$ of F^m such that

$$Y \lrcorner \Phi = -\frac{\sqrt{-1}}{2\pi}\bar{\partial}\tilde{\rho}.$$

As we have remarked before, we can find a unique $X \in (T^{1,0}(P))_{(m)}$ such that $\tilde{X} = Y$ and $\tilde{\rho}_X = \tilde{\rho}$. We show that X is holomorphic, which will complete the proof of (1).

Fix α and consider a coordinate system $w^1, \cdots, w^{n-1}$ on U_α. Then the vector field $\tilde{X}^\alpha$ may be expressed as follows:

$$\tilde{X}^\alpha = \sum_i \tilde{\xi}_i \frac{\partial}{\partial w^i}$$

Furthermore the n functions w^i, z^α form a coordinate system on $\pi^{-1}(U_\alpha)$, and the vector field X^α may be expressed as follows:

$$X^\alpha = \sum_i \xi_i \frac{\partial}{\partial w^i} + \eta \frac{\partial}{\partial z^\alpha}.$$

For our purpose it suffices to prove that X^α is holomorphic, *i.e.*, the functions ξ_i and η are all holomorphic. Since $\pi_* X^\alpha = \tilde{X}^\alpha$, we have $\xi^i = \pi^* \tilde{\xi}^i$. Since $\tilde{X}^\alpha$ is holomorphic, we see that ξ^i are holomorphic.

If we put $\eta^\alpha = \eta / z^\alpha$, we have $R_a^* \eta^\alpha = \eta^\alpha$, $a \in \boldsymbol{C}^*$. Hence there is a unique function $\tilde{\eta}^\alpha$ on U_α such that $\eta^\alpha = \pi^* \tilde{\eta}^\alpha$. Since $\omega = \pi^* \omega^\alpha + \dfrac{1}{z^\alpha} dz^\alpha$, we have

$$\omega(X^\alpha) = \pi^*\big(\omega^\alpha(\tilde{X}^\alpha)\big) + \pi^* \tilde{\eta}^\alpha.$$

On the other hand we have

$$\omega(X^\alpha) = (z^\alpha)^m \omega(X) = (z^\alpha)^m \rho_X = \rho_X^\alpha = \pi^* \tilde{\rho}_X^\alpha.$$

Hence it follows that $\tilde{\eta}^\alpha = \tilde{\rho}_X^\alpha - \omega^\alpha(\tilde{X}^\alpha)$. Since $\tilde{X}^\alpha$ is holomorphic, we have

$$\bar{Y}\omega^\alpha(\tilde{X}^\alpha) = -d\omega^\alpha(\tilde{X}^\alpha, \bar{Y}) + \tilde{X}^\alpha \omega^\alpha(\bar{Y}) - \omega^\alpha\big([\tilde{X}^\alpha, \bar{Y}]\big)$$
$$= -d\omega^\alpha(\tilde{X}^\alpha, \bar{Y}) = 2\pi\sqrt{-1}\,\Phi(\tilde{X}^\alpha, \bar{Y}) = \bar{Y}\tilde{\rho}_X^\alpha.$$

where $\bar{Y} \in \Gamma(T^{0,1}(\tilde{M}))$. This means that $\tilde{\eta}^\alpha$ and hence η are holomorphic.

We have thus shown that X is holomorphic, and hence have completed the proof of (1).

It remains to prove (2). Let $X \in \mathfrak{g}(P)_{(m)}$. Then we see that $X \in \operatorname{Ker} \kappa_m$ if and only if $\pi_* X^\alpha = \tilde{X}^\alpha = 0$. Clearly this last condition means that X is of the form $\rho \xi_P^{(1,0)}$, where ρ is a holomorphic function on P and $\rho \in C^\infty(P)_{(m)}$. These prove (2). q. e. d.

4. 2. Non-degenerate PC manifolds associated with pairs of positive line bundles. For each $i=1$ or 2, let $\tilde{M}_i$ be a compact complex manifold of dimension r_i, and let F_i be a holomorphic line bundle over $\tilde{M}_i$ with a hermitian metric h_i. We assume that the first Chern form Φ_i of the hermitian holomorphic line bundle F_i is positive (cf. [16]).

Let $\tilde{g}_i$ be the Kählerian metric on $\tilde{M}_i$ associated with Φ_i, and let $(\tilde{M}, \tilde{g})$ be the product of the two Kählerian manifolds $(\tilde{M}_1, \tilde{g}_1)$ and $(\tilde{M}_2, \tilde{g}_2)$. For each point $y=(y^1, y^2) \in \tilde{M} = \tilde{M}_1 \times \tilde{M}_2$, we define a 1-dimensional vector space F_y by

$$F_y = (F_1)_{y^1} \otimes (F_2)^*_{y^2} ,$$

$(F_2)^*_{y^2}$ being the dual space of $(F_2)_{y^2}$, and put $F = \bigcup_y F_y$. Then F is a holomorphic line bundle over $\tilde{M}$. Let $(h_2)^*$ be the hermitian metric of the dual bundle $(F_2)^*$ of F_2 which is naturally induced from h_2. Then we define a hermitian metric h of F by

$$h_y = (h_1)_{y^1} \otimes (h_2)^*_{y^2} ,$$

or

$$h(u \otimes v, u \otimes v) = h_1(u, u)\, h_2^*(v, v) ,$$

where $u \in (F_1)_{y^1}$ and $v \in (F_2)^*_{y^2}$.

In this paragraph we consider the PC manifold M which is associated with the hermitian holomorphic line bundle F over $\tilde{M}$, thus obtained. Recall that M satisfies condition (C .1).

Proposition 4. 6. *The PC manifold M is non-degenerate of index r_1, and satisfies condition (C. 2).*

Proof. Let x be any point of M. For $i=1$ or 2, we define a subspace S_x^i of S_x as follows: We first remark that the differential π'_* of the projection $\pi' : M \to \tilde{M}$ maps S_x isomorphically onto $T^{1,0}(\tilde{M})_y$, where $y = \pi'(x)$, and that $T^{1,0}(\tilde{M})_y$ is naturally decomposed as follows:

$$T^{1,0}(\tilde{M})_y = T^{1,0}(\tilde{M}_1)_{y^1} + T^{1,0}(\tilde{M}_2)_{y^2} ,$$

where $y=(y^1, y^2)$. Now S_x^i is defined to be the subspace of S_x which corresponds to the subspace $T^{1,0}(\tilde{M}_i)_{y^i}$ of $T^{1,0}(\tilde{M})_y$ through the isomorphism

$\pi'_*|S_x$. It is clear that $S^i=\bigcup_x S^i_x$ is a subbundle of S and that $S=S^1+S^2$ (direct sum). Let us apply Proposition 4.2 to the three line bundles F^1, F^2 and F. Then it is not difficult to see that the Levi form L_x of M at x is negative definite (resp. positive definite) on S^1_x (resp. on S^2_x) and that S^1_x and S^2_x are mutually orthogonal with respect to L_x, implying that M is non-degenerate of index r_1. Furthermore it is not difficult to see that both S^1 and S^2 are parallel with respect to the canonical affine connection ∇ of (M, ξ). Thus we have seen that M satisfies condition (C. 2). q. e. d.

Hereafter we assume that both $\tilde{M}_1$ and $\tilde{M}_2$ are Einstein manifolds;

$$\tilde{R}^*_i=(r_i+1)\,c_i\,\tilde{g}_i \qquad (i=1,2),$$

where $\tilde{R}^*_i$ are the Ricci tensors of the Kählerian manifolds $\tilde{M}_i$, and c_i are real constants. A direct calculation shows that the Ricci tensor of the canonical affine connection ∇ satisfies the equalities

$$R^*(X,\bar{Y})=(r_i+1)\,c_i g(X,\bar{Y}), \qquad X,\ Y\in S^i_x\ (i=1,2),$$

implying that M satisfies condition (C. 3). Therefore we know from Theorem 3.5 that the Lie algebra $\mathfrak{g}(M)$ is decomposed as follows:

$$\mathfrak{g}(M)=\sum_{i=0}^{4}\mathfrak{g}^i.$$

THEOREM 4.7. (1) *If* $c_1>0$ (*resp. If* $c_2>0$), *then* $\mathfrak{g}^1$ (*resp.* $\mathfrak{g}^2$) *is isomorphic to* $\Gamma_{hol}(T^{1,0}(\tilde{M}_1))$ (*resp. to* $\Gamma_{hol}(T^{1,0}(\tilde{M}_2))$). *If* $c_1\leqq 0$ (*resp. If* $c_2\leqq 0$), *then* $\mathfrak{g}^1=0$ (*resp.* $\mathfrak{g}^2=0$).

(2) *If* $\mathfrak{g}^3\neq 0$ (*resp. If* $\mathfrak{g}^4\neq 0$), *then* $\tilde{M}_1$ (*resp.* $\tilde{M}_2$) *is biholomorphic to the* r_1*-dimensional complex projective space* $P^{r_1}(\boldsymbol{C})$ (*resp. to the* r_2*-dimensional complex projective space* $P^{r_2}(\boldsymbol{C})$), *and correspondingly* F_1 (*resp.* F_2) *is isomorphic to the hyperplane bundle* H_1 *over* $P^{r_1}(\boldsymbol{C})$ (*resp. to the hyperplane bundle* H_2 *over* $P^{r_2}(\boldsymbol{C})$).

PROOF. Assume that $c_1>0$. Let $u\in\mathfrak{g}^1$. Since $\mathfrak{g}^1\subset\mathfrak{g}_{(0)}$, we have a unique element X of $\mathfrak{g}(P)_{(0)}$ such that $X|M=u$. We claim that $(\kappa_0(X))_y\in T^{1,0}(\tilde{M})_{y^1}$ for any $y=(y^1,y^2)\in\tilde{M}$. Indeed the cross section U^S of S corresponding to u satisfies $U^S_x\in S^1_x$, which implies that $\kappa_0(X)_y\in T^{1,0}(\tilde{M})_y$. Let us remark that $\Gamma_{hol}(T^{1,0}(\tilde{M}))$ is naturally decomposed as follows:

$$\Gamma_{hol}\left(T^{1,0}(\tilde{M})\right)=\Gamma_{hol}\left(T^{1,0}(\tilde{M}_1)\right)+\Gamma_{hol}\left(T^{1,0}(\tilde{M}_2)\right)$$

(direct sum). Hence we have $\kappa_0(X)\in\Gamma_{hol}(T^{1,0}(\tilde{M}_1))$.

Let us prove that the assignment $u\to\kappa_0(X)$ gives an isomorphism of $\mathfrak{g}^1$ onto $\Gamma_{hol}(T^{1,0}(\tilde{M}_1))$. Suppose that $\kappa_0(X)=0$. Let f_u be the element of F^1

which corresponds to u. It can be easily seen that $U^S=0$ and hence $\Box_1 f_u=0$. Since $f_u\in F^1$, we have $\Box_1 f_u=(r_1+1)c_1 f_u$, and hence $f_u=0$. This implies that $u=0$. We have thus shown that the linear mapping $u\to\kappa_0(X)$ of $\mathfrak{g}^1$ to $\Gamma_{hol}(T^{1,0}(\widetilde{M}))$ is injective. Take any $Y\in\Gamma_{hol}(T^{1,0}(\widetilde{M}_1))$. We remark that $H^{0,1}(\widetilde{M})$ is naturally decomposed as follows:

$$H^{0,1}(\widetilde{M})=H^{0,1}(\widetilde{M}_1)+H^{0,1}(\widetilde{M}_2)\qquad \text{(direct sum)}.$$

It is easy to see that the cohomology class $[Y\lrcorner\Phi]$ determined by the closed 1-form $Y\lrcorner\Phi$ is contained in $H^{0,1}(\widetilde{M}_1)$. Since $c_1>0$, we see that the canonical line bundle $k(\widetilde{M}_1)$ of $\widetilde{M}_1$ is a negative line bundle. By Kodaira's vanishing theorem, we have $H^{0,1}(\widetilde{M}_1)=0$ and hence $Y\lrcorner\Phi$ is $\bar{\partial}$-exact. It follows from Theorem 4.5 that $Y\in\operatorname{Im}\kappa_0$, which implies that $\kappa_0(\mathfrak{g}^1)=\Gamma_{hol}(T^{1,0}(\widetilde{M}_1))$. In the same manner, we can prove that if $c_2>0$, then $\mathfrak{g}^2$ is isomorphic to $\Gamma_{hol}(T^{1,0}(\widetilde{M}_2))$. Furthermore it follows from Theorem 3.5 that if $c_1\leqq 0$ (resp. if $c_2\leqq 0$), then $\mathfrak{g}^1=0$ (resp. $\mathfrak{g}^2=0$). We have thus proved (1).

To prove (2), we first remark that $\chi_1=(r_1+1)\,c_1\Phi_1$, where χ_1 (resp. Φ_1) denotes the Ricci form of the Kählerian manifold $\widetilde{M}_1$ (resp. the first Chern form of the hermitian holomorphic line bundle F_1). Assume that $\mathfrak{g}^3\neq 0$. From Theorem 3.5 and Proposition 4.3 and the fact that $\mathfrak{g}^3\subset\mathfrak{g}_{(-c_1)}$, it follows that c_1 is a positive integer. Hence we have the following inequality

$$\chi_1\geqq(r_1+1)\,\Phi_1$$

According to the result of Kobayashi-Ochiai [8], we see that $\widetilde{M}_1$ is biholomorphic to the complex projective space $P^{r_1}(\boldsymbol{C})$ and F_1 is isomorphic to the hyperplane bundle H_1 over $P^{r_1}(\boldsymbol{C})$. In the same manner, we see that if $\mathfrak{g}^4\neq 0$, then M_2 is biholomorphic to $P^{r_2}(\boldsymbol{C})$ and F_2 is isomorphic to the hyperplane bundle H_2 over $P^{r_2}(\boldsymbol{C})$. q. e. d.

4.3. Special cases of 4.2. For $i=1$ or 2, let $\widetilde{M}_i$ be the r_i-dimensional complex projective space $P^{r_i}(\boldsymbol{C})$, and let F_i be the k_i-th tensor product $(H_i)^{k_i}$ of the hyperplane bundle over $P^{r_i}(\boldsymbol{C})$, where $k_i\geqq 1$. Let $z_0^i,\cdots,z_{r_i}^i$ be homogeneous coordinates of $P^{r_i}(\boldsymbol{C})$, and for each $0\leqq\alpha\leqq r_i$, let U_α^i be the open subset of $P^{r_i}(\boldsymbol{C})$ defined by $z_\alpha^i\neq 0$. We define a function h_α^i on U_α^i by

$$h_\alpha^i=\Bigl(\sum_{\gamma=0}^{r_i}|z_\gamma^i|^2/|z_\alpha^i|^2\Bigr)^{k_i}.$$

Then we see that the functions h_α^i, $\alpha=0,\cdots,r_i$, satisfy the relations

$$h_\alpha^i=h_\beta^i\Bigl(|z_\beta^i|/|z_\alpha^i|\Bigr)^{2k_i}\qquad \text{on } U_\alpha^i\cup U_\beta^i,$$

and hence define a hermitian metric of the line bundle F_i.

In this paragraph we apply the arguments in 4.2 to the hermitian holomorphic line bundles F_i over $\widetilde{M}_i$, thus obtained. First of all, we remark that the Kählerian metric $\tilde{g}_i$ are of constant holomorphic sectional curvature $2/k_i$, from which follows that

$$\tilde{R}_i^* = \frac{r_i+1}{k_i}\tilde{g}_i .$$

We will calculate the dimensions of the subspaces $\mathfrak{g}^i$ of $\mathfrak{g}(M)$. By (2) of Theorem 3.5 we have $\dim\mathfrak{g}^0=1$. By (1) of Theorem 4.7 we have $\dim\mathfrak{g}^1=\dim\Gamma_{hol}(T^{1,0}(P^{r_1}(\boldsymbol{C})))$. It is well known that $\Gamma_{hol}(T^{1,0}(P^{r_1}(\boldsymbol{C}))$ is isomorphic to $\mathfrak{gl}(r_1+1, \boldsymbol{C})/center$. Hence we have $\dim\mathfrak{g}^1=\dim\Gamma_{hol}(T^{1,0}(P^{r_1}(\boldsymbol{C})))=r_1(r_1+2)$, and similarly $\dim\mathfrak{g}^2=r_2(r_2+2)$. Let us now calculate $\dim\mathfrak{g}^3$. We first recall that $\mathfrak{g}^3=\mathfrak{g}_{(-\frac{1}{k_i})}\cong\mathfrak{g}(P)_{(-\frac{1}{k_i})}$ ((2) of Theorem 3.5 and Proposition 4.4). From (1) of Proposition 4.3 it follows that if $k_1\geqq 2$, then $\mathfrak{g}^3=0$. Thus we consider the case where $k_1=1$. We assert that the linear mapping κ_{-1} gives an isomorphism of $\mathfrak{g}(P)_{(-1)}$ onto $\Gamma_{hol}(T^{1,0}(\widetilde{M})\otimes F^{-1})$. Indeed by the Künneth formula we have

$$H^0(\widetilde{M}, F^{-1})\cong H^0\left(P^{r_1}(\boldsymbol{C}), H_1^{-1}\right)\otimes H^0\left(P^{r_2}(\boldsymbol{C}), H_2^{k_2}\right),$$

$$H^1(\widetilde{M}, F^{-1})\cong H^0\left(P^{r_1}(\boldsymbol{C}), H_1^{-1}\right)\otimes H^1\left(P^{r_2}(\boldsymbol{C}), H_2^{k_2}\right)$$
$$+H^1\left(P^{r_1}(\boldsymbol{C}), H_1^{-1}\right)\otimes H^0\left(P^{r_2}(\boldsymbol{C}), H_2^{k_2}\right)$$

Since $H^0(P^{r_1}(\boldsymbol{C}), H_1^{-1})=H^1(P^{r_1}(\boldsymbol{C}), H_1^{-1})=0$ (cf. [1]), it follows that $H^0(\widetilde{M}, F^{-1})=H^1(\widetilde{M}, F^{-1})=0$. Hence our assertion follows immediately from Theorem 4.5. Therefore using the Künneth formula again, we obtain

$$\begin{aligned}\mathfrak{g}^3 &\cong \Gamma_{hol}\left(T^{1,0}(\widetilde{M})\otimes F^{-1}\right)\\ &\cong H^0\left(P^{r_1}(\boldsymbol{C}), T^{1,0}\left(P^{r_1}(\boldsymbol{C})\right)\otimes H_1^{-1}\right)\otimes H^0\left(P^{r_2}(\boldsymbol{C}), H_2^{k_2}\right)\\ &\quad +H^0\left(P^{r_1}(\boldsymbol{C}), H_1^{-1}\right)\otimes H^0\left(P^{r_2}(\boldsymbol{C}), T^{1,0}\left(P^{r_2}(\boldsymbol{C})\right)\otimes H_2^{k_2}\right)\\ &\cong H^0\left(P^{r_1}(\boldsymbol{C}), T^{1,0}\left(P^{r_1}(\boldsymbol{C})\right)\otimes H_1^{-1}\right)\otimes H^0\left(P^{r_2}(\boldsymbol{C}), H_2^{k_2}\right),\end{aligned}$$

whence $\dim\mathfrak{g}^3=(r_1+1)\binom{r_2+k_2}{k_2}$ (cf. [1]). In the same manner as above, we can show that if $k_2\geqq 2$, then $\mathfrak{g}^4=0$ and if $k_2=1$, then $\dim\mathfrak{g}^4=(r_2+1)\binom{r_1+k_1}{k_1}$.

We have thus proved the following

PROPOSITION 4.8. (1) $\dim\mathfrak{g}^0=1$, $\dim\mathfrak{g}^1=r_1(r_1+2)$, *and* $\dim\mathfrak{g}^2=r_2(r_2+2)$.
(2) *If* $k_1, k_2\geqq 2$, *then* $\mathfrak{g}^3=\mathfrak{g}^4=0$.

(3) *If* $k_1=1$ *and* $k_2\geqq 2$, *then* $\mathfrak{g}^4=0$ *and* $\dim\mathfrak{g}^3=(r_1+1)\binom{r_2+k_2}{k_2}$. *If* $k_1\geqq 2$ *and* $k_2=1$, *then* $\mathfrak{g}^3=0$ *and* $\dim\mathfrak{g}^4=(r_2+1)\binom{r_1+k_1}{k_1}$.

(4) *If* $k_1=k_2=1$, *then* $\dim\mathfrak{g}^3=\dim\mathfrak{g}^4=(r_1+1)(r_2+1)$.

This proposition combined with Theorem 3.5 gives the following

COROLLARY 4.9. (1) *If* k_1, $k_2\geqq 2$, *then* $\mathfrak{g}(M)=C\mathfrak{a}(M)=\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2$.

(2) *If* $k_1=1$ *and* $k_2\geqq 2$, *then* $\mathfrak{g}(M)=\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2+\mathfrak{g}^3$ *and* $C\mathfrak{a}(M)=\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2$. *If* $k_1\geqq 2$ *and* $k_2=1$, *then* $\mathfrak{g}(M)=\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2+\mathfrak{g}^4$ *and* $C\mathfrak{a}(M)=\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2$.

(3) *If* $k_1=k_2=1$, *then* $\mathfrak{g}(M)=C\mathfrak{a}(M)=\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2+\mathfrak{g}^3+\mathfrak{g}^4$.

REMARK. Put $n=r_1+r_2+1$ and consider the n-dimensional complex projective space $P^n(\boldsymbol{C})$. Let $z_0,\cdots,z_n$ be homogeneous coordinates of $P^n(\boldsymbol{C})$, and let Q_{r_1} be the hermitian quadric of $P^n(\boldsymbol{C})$ defined by

$$\sum_{\alpha=0}^{r_1}|z_\alpha|^2-\sum_{\beta=r_1+1}^{n}|z_\beta|^2=0\,,$$

which is a non-degenerate PC manifold of index r_1. Then it can be shown that if $k_1=k_2=1$, then M is naturally isomorphic to Q_{r_1} as PC manifolds.

4.4. Non-degenerate PC manifolds associated with complex tori. Let $\tilde{M}$ be an $(n-1)$-dimensional complex torus. In his paper [11], Matsushima constructed a class of hermitian holomorphic line bundles F over $\tilde{M}$, whose Chern forms are non-degenerate and indefinite everywhere. We assert without proof that the PC manifolds M associated with F are non-degenerate of positive index, and satisfy conditions (C. 1), (C. 2) and (C. 3), where (C. 3) is satisfied with constants $c_1=c_2=0$. Therefore we know from Theorem 3.5 that $\mathfrak{g}(M)=C\mathfrak{a}(M)=\{C\xi\}$.

§ 5. The structures of the Lie algebras $\mathfrak{g}(M)$ and $C\mathfrak{a}(M)$. (the strongly pseudo-convex case)

5.1. A general structure theorem on the Lie algebras $\mathfrak{g}(M)$ and $C\mathfrak{a}(M)$. Let M be a compact strongly pseudo-convex manifold of dimension $2n-1$ satisfying condition (C. 1). First of all we remark that M automatically satisfies condition (C. 2): $S^1=0$ and $S^2=S$. Thus we have the operators $\square_2$, $\overline{W}_2$, A_2 and the scalar curvature σ_2, which will be simply denoted by $\square$, $\overline{W}$, A and σ respectively. Note that the operator A is defined by

$$A=\square^2+\square N+\sum R^{*}_{\alpha\bar{\beta}}\nabla_\beta\nabla_{\bar{\alpha}}+\overline{W}\,.$$

Let us recall the results of § 2.

PROPOSITION 5.1. *Let $f\in C^{\infty}(M)$. Then the following conditions are mutually equivalent*:

(1) $f\in F(M)$.

(2) $Af=0$.

(3) $(Af, f)=0$.

PROPOSITION 5.2. (1) $\square$, *N and A are self-adjoint operators. Moreover $\square$ and A are positive semi-definite.*

(2) $\square N=N\square, \qquad AN=NA$.

(3) $\bar{\square}-\square=-(n-1)N, \qquad \bar{N}=-N$.

(4) $\bar{A}-A=n(n-1)N^{2}+n(n-1)\sigma N-2n\square N+W-\bar{W}$.

Although the finite dimensionality of the Lie algebra $\mathfrak{g}(M)$ is not valid any more for a strongly pseudo-convex manifold M (cf. [2]), we know the following

THEOREM B ([4], [12] and [13]). *Let M be a $(2n-1)$-dimensional non-degenerate PC manifold, then*

$$\dim_{R}\mathfrak{a}(M)\leqq n^{2}+2n .$$

Therefore we know from Proposition 1.5 that $F(M)\cap\bar{F}(M)$ is finite dimensional. As in § 3, for each $\nu\in\boldsymbol{R}$, let us define a subspace $F_{(\nu)}$ of $F(M)$ and a subspace $\tilde{F}_{(\nu)}$ of $F(M)\cap\bar{F}(M)$ respectively by

$$F_{(\nu)}=\left\{f\in F(M)\middle| Nf=\nu f\right\},$$

$$\tilde{F}_{(\nu)}=\left\{f\in F(M)\cap\bar{F}(M)\middle| Nf=\nu f\right\}.$$

PROPOSITION 5.3. *Let M be a compact strongly pseudo-convex manifold satisfying condition* (C. 1).

(1) *Each $F_{(\nu)}$ is finite dimensional.*

(2) $F(M)\cap\bar{F}(M)=\sum_{\nu}\tilde{F}_{(\nu)}$ *(direct sum), and* $\dim\tilde{F}_{(\nu)}=\dim\tilde{F}_{(-\nu)}$.

Assume further that the scalar curvature σ is equal to a real constant c.

(3) $F_{(0)}=\tilde{F}_{(0)}$.

(4) *The case where $c>0$: $F_{(\nu)}=0$ for $\nu>c$, and $\tilde{F}_{(\nu)}=0$ for $|\nu|>c$.*

(5) *The case where $c\leqq 0$: $F_{(\nu)}=0$ for $\nu>0$, and $\tilde{F}_{(\nu)}=0$ for $\nu\neq 0$.*

PROOF. First of all, we see that any $f\in F_{(\nu)}$ satisfies

$$(A+N^{4})f=\nu^{4}f .$$

Since the operator $A+N^{4}$ is a self-adjoint strongly elliptic differential operator, it follows that $F_{(\nu)}$ is finite dimensional, implying (1). We see from Proposition 5.2 that N is a self-adjoint operator and leaves invariant the finite dimensional subspace $F(M)\cap\bar{F}(M)$ of $C^\infty(M)$. Hence we have

$$F(M)\cap\bar{F}(M)=\sum_\nu \tilde{F}_{(\nu)} \qquad \text{(direct sum)}.$$

It follows from Proposition 5.2 that the assignment $f\to\bar{f}$ gives an isomorphism of $\tilde{F}_{(\nu)}$ onto $\tilde{F}_{(-\nu)}$. Hence we obtain $\dim\tilde{F}_{(\nu)}=\dim\tilde{F}_{(-\nu)}$, proving (2).

In the following we assume that σ is equal to a constant c. As in the proof of Proposition 3.1, we have $\overline{W}=0$. Take any $f\in F_{(0)}$. Then it follows from Proposition 5.1 that $Af=Nf=0$. By (4) of Proposition 5.2, we have $\bar{A}f=0$. Thus we obtain $f\in F_{(0)}$, implying (3).

To prove (4) and (5), it suffices to prove that $F_{(\nu)}=0$ for $\nu>\mathrm{Max}(c,0)$. Let us assume that $\nu>\mathrm{Max}(c,0)$ and take any $f\in F_{(\nu)}$. Then we have $Af=0$ and $Nf=\nu f$. Moreover we have

$$\bar{A}f=n(n-1)\,\nu(\nu+c)f-2n\nu\Box f.$$

By Proposition 5.2, we have

$$0\leqq(\bar{A}f,f)=n(n-1)\,\nu(\nu+c)\,(f,f)-2n\nu(\Box f,f).$$

Hence we have

$$(\Box f,f)\leqq\frac{(n-1)\,(c+\nu)}{2}(f,f).$$

By Proposition 5.2, we have

$$(\overline{\Box}f,f)\leqq\frac{(n-1)\,(c-\nu)}{2}(f,f)$$

It follows from (1) of Proposition 5.1 that

$$0\leqq\frac{(n-1)\,(c-\nu)}{2}(f,f).$$

Hence we obtain $f=0$. This proves our assertion. q. e. d.

As in §3 we define a subspace $\mathfrak{g}_{(\nu)}$ of $\mathfrak{g}(M)$ and a subspace $\tilde{\mathfrak{g}}_{(\nu)}$ of $C\mathfrak{a}(M)$ respectively by

$$g_{(\nu)}=\left\{u\in\mathfrak{g}(M)\middle|\sqrt{-1}\,[\xi,u]=\nu u\right\},$$

$$\tilde{g}_{(\nu)}=\left\{\mathfrak{u}\in C\mathfrak{a}(M)\middle|\sqrt{-1}\,[\xi,u]=\nu u\right\}.$$

Then by Proposition 5.3, we have the following

THEOREM 5.4. *Let M be a compact strongly pseudo-convex manifold satisfying condition* (C. 1)

(1) $\mathfrak{g}_{(\nu)}$ *are finite dimensional, and* $[\mathfrak{g}_{(\mu)},\mathfrak{g}_{(\nu)}]\subset\mathfrak{g}_{(\mu+\nu)}$.

(2) $\mathfrak{Ca}(M)=\sum\tilde{\mathfrak{g}}_{(\nu)}$ *(direct sum), and* $\dim\tilde{\mathfrak{g}}_{(\nu)}=\dim\tilde{\mathfrak{g}}_{(-\nu)}$

Assume further that the scalar curvature σ is equal to a real constant c.

(3) $\mathfrak{g}_{(0)}=\tilde{\mathfrak{g}}_{(0)}$.

(4) *The case where $c>0$: $\mathfrak{g}_{(\nu)}=0$ for $\nu>c$, and $\tilde{\mathfrak{g}}_{(\nu)}=0$ for $|\nu|>c$.*

(5) *The case where $c\leqq 0$: $\mathfrak{g}_{(\nu)}=0$ for $\nu>0$, and $\tilde{\mathfrak{g}}_{(\nu)}=0$ for $\nu\neq 0$.*

5.2. Condition (C. 3) and structure theorems on the Lie algebras $\mathfrak{g}(M)$ and $\mathfrak{Ca}(M)$. As in 3.2, we assume the following condition:

(C. 3) The Ricci tensor R^* satisfies

$$R^*(X,\bar{Y})=nc\,g(X,\bar{Y})\qquad\text{for any}\quad X,\ Y\in S_x,$$

where c is a real constant.

For each $\nu\in\boldsymbol{R}$, we define subspaces $F^1_{(\nu)}$ and $F^2_{(\nu)}$ of $C^\infty(M)$ respectively by

$$F^1_{(\nu)}=\left\{f\in C^\infty(M)\middle|\Box f=0,\ Nf=\nu f\right\},$$

$$F^2_{(\nu)}=\left\{f\in C^\infty(M)\middle|\Box f=(nc-\nu)f,\ Nf=\nu f\right\}.$$

Then we obtain the following

PROPOSITION 5.5. *Let M be a compact strongly pseudo-convex manifold satisfying conditions (C. 1) and (C. 3).*

(1) *$F_{(\nu)}=F^1_{(\nu)}+F^2_{(\nu)}$ (direct sum) for $\nu\neq nc$, and $F_{(nc)}=F^1_{(nc)}=F^2_{(nc)}$.*

(2) *$F^1_{(\nu)}$ consists of all holomorphic functions satisfying $Nf=\nu f$. In particular, $F^1_{(0)}$ consists of all constant functions.*

(3) *The case where $c>0$: $F^1_{(\nu)}=0$ for $\nu>0$ or $0>\nu>-c$, and $F^2_{(\nu)}=0$ for $\nu>c$ or $c>\nu>0$.*

(4) *The case where $c\leqq 0$: $F^1_{(\nu)}=0$ for $\nu>0$, and $F^2_{(\nu)}=0$ for $\nu>nc$.*

(5) *The case where $c>0$: $F(M)\cap\bar{F}(M)=F^1_{(0)}+F^2_{(0)}+F^1_{(-c)}+F^2_{(c)}$ (direct sum), and $\dim F^1_{(-c)}=\dim F^2_{(c)}$. The case where $c\leqq 0$: $F(M)\cap\bar{F}(M)=F^1_{(0)}$*

PROOF. First of all, we remark that condition (C. 3) impies that

$$A=\Box^2+N\Box-nc\Box.$$

Hence we have $F^1_{(\nu)}\subset F_{(\nu)}$ and $F^2_{(\nu)}\subset F_{(\nu)}$.

Conversely let us prove that $F_{(\nu)}\subset F^1_{(\nu)}+F^2_{(\nu)}$. It follows from Proposition 5.2 that $\Box$ is a self-adjoint operator leaving invariant the finite dimensional subspace $F_{(\nu)}$ of $C^\infty(M)$. Hence we have

$$F_{(\nu)}=\sum_\lambda F_{(\lambda,\nu)}\qquad\text{(direct sum)}$$

where we put $F_{(\lambda,\nu)}=\{f\in F_{(\nu)}|\Box f=\lambda f\}$. Take any $f\in F_{(\lambda,\nu)}$, then we have

$$Af=\lambda(\lambda+\nu-nc)f=0.$$

Hence we see that if $F_{(\lambda,\nu)}\neq 0$, then $\lambda=0$ or $\lambda=nc-\nu$. We have thus proved our assertion. From this, (1) follows immediately.

Let $f\in F^1_{(\nu)}$. From the fact that $\square f=0$, it follows that $\bar{Y}f=0$ for any $Y\in S_x$, $x\in M$. Therefore f is a holomorphic function. In particular, if $f\in F^1_{(0)}$, then we have $\square f=Nf=0$ and hence $\bar{\square} f=0$. These imply that $\nabla f=0$, and hence f is constant. We have thus proved (2).

Since $\bar{\square}$ and $\bar{A}$ are positive semi-definite operators (Proposition 5. 2), we see that if $F^1_{(\nu)}\neq 0$, then $-(n-1)\nu\geqq 0$ and $n(n-1)\nu(\nu+c)\geqq 0$. Similarly we see that if $F^2_{(\nu)}\neq 0$, then $nc-\nu\geqq 0$, $n(c-\nu)\geqq 0$ and $n(n+1)\nu(\nu-c)\geqq 0$. Now (3) and (4) follow from these facts.

By Proposition 5. 3, we have

$$F(M)\cap\bar{F}(M)=\sum_{\nu}\tilde{F}_{(\nu)} \qquad \text{(direct sum)}\,.$$

The operator $\square$ leaves invariant the subspace $\tilde{F}_{(\nu)}$. Therefore by (1) we have

$$\tilde{F}_{(\nu)}=\tilde{F}^1_{(\nu)}+\tilde{F}^2_{(\nu)} \qquad \text{(direct sum) for } \nu\neq nc\,,$$

$$\tilde{F}_{(nc)}=\tilde{F}^1_{(nc)}=\tilde{F}^2_{(nc)}\,,$$

where we set $\tilde{F}^1_{(\nu)}=\tilde{F}_{(\nu)}\cap F^1_{(\nu)}$ and $\tilde{F}^2_{(\nu)}=\tilde{F}_{(\nu)}\cap F^2_{(\nu)}$. It is easily verified that $F^1_{(0)}=\tilde{F}^1_{(0)}$, $F^2_{(0)}=\tilde{F}^2_{(0)}$, $F^1_{(-c)}=\tilde{F}^1_{(-c)}$ and $F^2_{(c)}=\tilde{F}^2_{(c)}$. Moreover the assignment $f\to\bar{f}$ gives an isomorphism of $F^1_{(-c)}$ onto $F^2_{(c)}$. On the other hand we see from Proposition 5. 2 that if $\nu\neq 0$, $-c$, then $F^1_{(\nu)}\cap\bar{F}(M)=0$, and if $\nu\neq 0$, c, then $F^2_{(\nu)}\cap\bar{F}(M)=0$. Hence we have $\tilde{F}^1_{(\nu)}=0$ for $\nu\neq 0$, $-c$, and $\tilde{F}^2_{(\nu)}=0$ for $\nu\neq 0$, c. We have thus proved (5). q. e. d.

Let $\mathfrak{g}^i_{(\nu)}$ be the subspace of $\mathfrak{g}_{(\nu)}$ which corresponds to the subspace $F^i_{(\nu)}$ of $F_{(\nu)}$ through the isomorphism $u\to f_u$ of $\mathfrak{g}_{(\nu)}$ onto $F_{(\nu)}$.

THEOREM 5. 6. *Let M be a compact strongly pseudo-convex manifold satisfying conditions* (C. 1) *and* (C. 3).

(1) $\mathfrak{g}_{(\nu)}=\mathfrak{g}^1_{(\nu)}+\mathfrak{g}^2_{(\nu)}$ *(vector space direct sum) for $\nu\neq nc$, and* $\mathfrak{g}_{(nc)}=\mathfrak{g}^1_{(nc)}=\mathfrak{g}^2_{(nc)}$.

(2) $\mathfrak{g}^1_{(\nu)}=\{f\xi\mid f$ *is a holomorphic function satisfying* $Nf=\nu f\}$. *In particular,* $\mathfrak{g}_{(0)}=\{C\xi\}$.

(3) $[\mathfrak{g}^1_{(\mu)},\mathfrak{g}^1_{(\nu)}]\subset\mathfrak{g}^1_{(\mu+\nu)}$, $[\mathfrak{g}^1_{(\nu)},\mathfrak{g}^1_{(\nu)}]=0$, $[\mathfrak{g}^2_{(\mu)},\mathfrak{g}^2_{(\nu)}]\subset\mathfrak{g}^2_{(\mu+\nu)}$, $[\mathfrak{g}^2_{(0)},\mathfrak{g}^1_{(\nu)}]\subset\mathfrak{g}^1_{(\nu)}$ *and* $[\mathfrak{g}^1_{(0)},\mathfrak{g}^2_{(\nu)}]\subset\mathfrak{g}^2_{(\nu)}$.

(4) *The case where* $c>0$: $\mathfrak{g}^1_{(\nu)}=0$ *for* $\nu>0$ *or* $0>\nu>-c$, *and* $\mathfrak{g}^2_{(\nu)}=0$ *for* $\nu>c$ *or* $c>\nu>0$.

(5) *The case where* $c\leqq 0$: $\mathfrak{g}^1_{(\nu)}=0$ *for* $\nu>0$, *and* $\mathfrak{g}^2_{(\nu)}=0$ *for* $\nu>nc$.

PROOF. We will show the proof of (3). The other assertions follow directly from Proposition 5.5. Let $u \in \mathfrak{g}_{(\mu)}$ and $v \in \mathfrak{g}_{(\nu)}$. Let us denote by U and V the cross sections of T, which correspond to u and v. First of all, we have

$$[U, V] = \left\{\sqrt{-1}(\mu-\nu)f_u f_v + U^S f_v - V^S f_u\right\}\xi$$
$$+[U^S, V^S] + \sqrt{-1}\,\mu f_v U^S - \sqrt{-1}\,\nu f_u V^S .$$

Hence it follows that

$$(5.1) \qquad f_{[u,v]} = \sqrt{-1}(\mu-\nu)f_u f_v + U^S f_v - V^S f_u .$$

By a simple calculation we obtain

$$\square(f_u f_v) = \square f_u f_v + f_v \square f_u + \sqrt{-1}(V^S f_u + U^S f_v),$$
$$\square(U^S f_v) = U^S(\square f_v) - \nu(U^S f_v) + \sqrt{-1}\,\nu(\square f_u) f_v$$
$$-\sqrt{-1}\sum_{\alpha,\beta} \nabla_\alpha \nabla_{\bar\beta} f_u \nabla_\beta \nabla_{\bar\alpha} f_v ,$$
$$\square(V^S f_u) = V^S(\square f_u) - \mu(V^S f_u) + \sqrt{-1}\,\mu(\square f_v) f_u$$
$$-\sqrt{-1}\sum_{\alpha,\beta} \nabla_\alpha \nabla_{\bar\beta} f_u \nabla_\beta \nabla_{\bar\alpha} f_v ,$$

and hence

$$(5.2) \qquad \square f_{[u,v]} = \sqrt{-1}\,\mu(\square f_u) f_v - \sqrt{-1}\,\nu f_u(\square f_v) + \nu(V^S f_u)$$
$$-\mu(U^S f_v) + U^S(\square f_v) - V^S(\square f_u) .$$

Let $u \in \mathfrak{g}^1_{(\mu)}$ and $v \in \mathfrak{g}^1_{(\nu)}$. Since f_u and f_v are holomorphic, we have $U^S = V^S = 0$. By (5.2), we have $\square f_{[u,v]} = 0$. Therefore we have $f_{[u,v]} \in F^1_{(\mu+\nu)}$, which implies $[\mathfrak{g}^1_{(\mu)}, \mathfrak{g}^1_{(\nu)}] \subset \mathfrak{g}^1_{(\mu+\nu)}$. In particular if $\mu = \nu$, then we easily see from (5.1) that $f_{[u,v]} = 0$. Hence we have $[\mathfrak{g}^1_{(\nu)}, \mathfrak{g}^1_{(\nu)}] = 0$.

Let $u \in \mathfrak{g}^2_{(\mu)}$ and $v \in \mathfrak{g}^2_{(\nu)}$. Then it follows that $\square f_u = (nc-\mu) f_u$ and $\square f_v = (nc-\nu) f_v$. By (5.2), we have

$$\square f_{[u,v]} = (n-\mu-\nu) f_{[u,v]}$$

Therefore we see that $f_{[u,v]} \in F^2_{(\mu+\nu)}$, and hence $[\mathfrak{g}^2_{(\mu)}, \mathfrak{g}^2_{(\nu)}] \subset \mathfrak{g}^2_{(\mu+\nu)}$.

Let $u \in \mathfrak{g}^2_{(0)}$ and $v \in \mathfrak{g}^1_{(\nu)}$. We have $\square f_v = 0$ and $V^S = 0$. Therefore it follows from (5.2) that $\square f_{[u,v]} = 0$. Consequently we have $[\mathfrak{g}^2_{(0)}, \mathfrak{g}^1_{(\nu)}] \subset \mathfrak{g}^1_{(\nu)}$. Finally from (2) together with the definition of $\mathfrak{g}^2_{(\nu)}$, it is clear that $[\mathfrak{g}^1_{(0)}, \mathfrak{g}^2_{(\nu)}] \subset \mathfrak{g}^2_{(\nu)}$ q. e. d.

As a consequence of (5) of Proposition 5.5, we have

THEOREM 5.7. *Let M be a compact strongly pseudo-convex manifold satisfying conditions* (C. 1) *and* (C. 3).

(1) *The case where* $c>0$: $\boldsymbol{C}\mathfrak{a}(M)=\mathfrak{g}^1_{(0)}+\mathfrak{g}^2_{(0)}+\mathfrak{g}^1_{(-c)}+\mathfrak{g}^2_{(c)}$, *(vector space direct sum), and* $\dim \mathfrak{g}^1_{(-c)}=\dim \mathfrak{g}^2_{(c)}$.

(2) *The case where* $c\leqq 0$: $\boldsymbol{C}\mathfrak{a}(M)=\mathfrak{g}^1_{(0)}$.

§ 6. Applications and examples (the strongly pseudo-convex case)

6. 1. Finite dimensional subalgebras of the Lie algebra $\mathfrak{g}(M)$. In this paragraph, by using Theorem 5. 6 and Theorem 5. 7, we will prove the following

THEOREM 6. 1. *Let M be a compact strongly pseudo-convex manifold satisfying conditions (C. 1) and (C. 3) with a positive constant c. If $\mathfrak{g}^1_{(-c)}\neq 0$, then $\boldsymbol{C}\mathfrak{a}(M)$ is a maximal finite dimensional subalgebra of $\mathfrak{g}(M)$.*

PROOF. Let $\mathfrak{b}$ be a finite dimensional subalgebra of $\mathfrak{g}(M)$ which contains $\boldsymbol{C}\mathfrak{a}(M)$. Since $\xi\in\boldsymbol{C}\mathfrak{a}(M)\subset\mathfrak{b}$, we have $[\xi,\mathfrak{b}]\subset\mathfrak{b}$. For each $\nu\in\boldsymbol{R}$, we define a subspace $\mathfrak{b}_{(\nu)}$ of $\mathfrak{b}$ by

$$\mathfrak{b}_{(\nu)}=\left\{u\in\mathfrak{b}\,\middle|\,\sqrt{-1}\,[\xi,u]=\nu u\right\}=\mathfrak{b}\cap\mathfrak{g}_{(\nu)}\,.$$

Then we have $\mathfrak{b}=\sum_\nu \mathfrak{b}_{(\nu)}$ (vector space direct sum) and $[\mathfrak{b}_{(\mu)},\mathfrak{b}_{(\nu)}]\subset\mathfrak{b}_{(\mu+\nu)}$.

By Theorem 5. 7 we have $\mathfrak{g}^1_{(-c)}\subset\mathfrak{b}_{(-c)}$, and hence $[\mathfrak{b}_{(\nu)},\mathfrak{g}^1_{(-c)}]\subset\mathfrak{b}_{(\nu-c)}$. For any $u\in\mathfrak{b}_{(\nu)}$, we define a linear mapping A_u of $\mathfrak{g}^1_{(-c)}$ to $\mathfrak{b}_{(\nu-c)}$ by

$$A_u(v)=[u,v]\,,\qquad v\in\mathfrak{g}^1_{(-c)}\,.$$

We recall that if $u\in\mathfrak{b}_{(\nu)}$ and $v\in\mathfrak{g}^1_{(-c)}$, then

$$[U,V]=\left\{\sqrt{-1}\,(\nu+c)f_uf_v+U^Sf_v\right\}\xi+\sqrt{-1}\,f_v\,U^S\,, \tag{6.1}$$

where we have used the fact that $V^S=0$ (see the proof of Theorem 5. 6).

LEMMA 6. 2. *Assume that $\mathfrak{b}_{(\nu)}\neq 0$ for some ν, and let u be a non-zero element of $\mathfrak{b}_{(\nu)}$. Assume further that* 1) $\nu\neq 0$, $-c$ *or* 2) $\nu=-c$ *and* $u\notin\mathfrak{g}^1_{(-c)}$. *Then the linear mapping A_u of $\mathfrak{g}^1_{(-c)}$ to $\mathfrak{b}_{(\nu-c)}$ is injective.*

PROOF. We first remark that $\{x\in M\mid f_u(x)\neq 0\}$ is an open dense subset of M, because the function f_u satisfies the strongly elliptic differential equation $(\square^2+N^4)f_u=\nu^4 f_u$. Similarly we remark that if $v\in\mathfrak{g}^1_{(-c)}$ and $v\neq 0$, then $\{x\in M\mid f_v(x)\neq 0\}$ is an open dense subset of M. Clearly it suffices to consider the following two cases.

A] The case where $\nu\neq 0$ and $u\notin\mathfrak{g}^1_{(\nu)}$. Let $v\in\mathfrak{g}^1_{(-c)}$ be such that $A_u(v)=0$. From (6. 1), it follows that $\nu f_v U^S=0$. If $v\neq 0$, then we have $U^S=0$. This contradicts the fact that $u\notin\mathfrak{g}^1_{(\nu)}$.

B] The case where $\nu\neq -c$ and $u\in\mathfrak{g}^1_{(\nu)}$. Let $v\in\mathfrak{g}^1_{(-c)}$ be such that $A_u(v)=0$. From (6. 1) and the fact that $U^S=0$, it follows that $(\nu+c)f_u f_v=0$. Hence we have $f_v=0$, implying that $v=0$. q. e. d.

We are now in position to complete the proof of Theorem 6. 1. Suppose that $\mathfrak{b}\neq \boldsymbol{C}\mathfrak{a}(M)$. Let ν_0 be the real number defined by

$$\nu_0=\mathrm{Min}\left\{\nu\in\boldsymbol{R}\,\middle|\,\mathfrak{b}_{(\nu)}\neq 0\right\}$$

Since $\mathfrak{b}\supset\mathfrak{g}^1_{(-c)}$, we have $\nu_0\leqq -c$. Consequently it suffices to consider the following three cases.

1) The case where $\nu_0<-c$. By Lemma 6. 2, we have $\mathfrak{b}_{(\nu_0-c)}\neq 0$, which contradicts the definition of ν_0.

2) The case where $\nu_0=-c$ and there is real number ν_1 such that $\mathfrak{b}_{(\nu_1)}\neq 0$ and $-c<\nu_1<0$. By Lemma 6. 2, we have $\mathfrak{b}_{(\nu_1-c)}\neq 0$, which contradicts the definition of ν_0.

3) The case where $\nu_0=-c$ and $\mathfrak{b}_{(\nu)}=0$ for $-c<\nu<0$. We assert that $\mathfrak{b}_{(-c)}\supsetneqq\mathfrak{g}^1_{(-c)}$. Indeed, by Theorem 5. 7, we have $\boldsymbol{C}\mathfrak{a}(M)=\mathfrak{g}^1_{(0)}+\mathfrak{g}^2_{(0)}+\mathfrak{g}^1_{(-c)}+\mathfrak{g}^2_{(c)}$. Furthermore by Theorem 5. 6, we have $\mathfrak{b}_{(\nu)}=0$ $(\nu>0,\ \nu\neq c)$, $\mathfrak{b}_{(c)}=\mathfrak{g}^2_{(c)}$, and $\mathfrak{b}_{(0)}=\mathfrak{g}_{(0)}$. From these facts follows easily our assertion. Now let u be an element of $\mathfrak{b}_{(-c)}$ such that $u\notin\mathfrak{g}^1_{(-c)}$. Then it follows from Lemma 6. 2 that the mapping A_u of $\mathfrak{g}^1_{(-c)}$ to $\mathfrak{b}_{(-2c)}$ is injective. Hence we have $\mathfrak{b}_{(-2c)}\neq 0$, which contradicts the definition of ν_0. q. e. d.

REMARK 1. Assume that $\mathfrak{g}^1_{(-c)}=0$. By Theorem 5. 7, we have $\boldsymbol{C}\mathfrak{a}(M)=\mathfrak{g}^1_{(0)}+\mathfrak{g}^2_{(0)}$. From (2) of Theorem 5. 6 we see that, for each $\nu\in\boldsymbol{R}$, $\mathfrak{g}^1_{(0)}+\mathfrak{g}^2_{(0)}+\mathfrak{g}^1_{(-\nu)}$ is a finite dimensional subalgebra of $\mathfrak{g}(M)$. Let us further assume that the vector field ξ is induced from a $U(1)$-action: $(x, a)\in M\times U(1)\rightarrow xa\in M$. Then we can show that there are a $U(1)$-invariant open subset M^* of M and a hermitian holomorphic line bundle F_0 over an $(n-1)$-dimensional complex manifold $\widetilde{M}_0$ such that the Chern form of F_0 is negative and such that the PC manifold M^* is equivariantly isomorphic to the PC manifold M_0 associated with F_0 (the $U(1)$-reduction of the principal $\boldsymbol{C}^*$-bundle P_0 associated with F_0). Furthermore by using this fact and reasoning in the same manner as in [15], we can show that the space $F^1_{(-\nu)}$, which is equal to the space of all holomorphic functions f on M such that $Nf=-\nu f$, is not reduced to the zero space for any sufficiently large integer ν. Accordingly we know that, for any sufficiently large integer ν, $\mathfrak{g}_{(-\nu)}\neq 0$ and hence $\mathfrak{g}^1_{(0)}+\mathfrak{g}^2_{(0)}+\mathfrak{g}^1_{(-\nu)}$ is a finite dimensional subalgebra of $\mathfrak{g}(M)$ containing $\boldsymbol{C}\mathfrak{a}(M)$ as a proper subalgebra.

REMARK 2. Let M be a compact strongly pseudo-convex manifold.

Under the assumption of Theorem 6.1, we can show that M is isomorphic to the unit sphere S^{2n-1} of $\boldsymbol{C}^n$ as PC manifolds. In the forthcoming paper, we will prove this fact more generally.

6.2. The Lie algebra $\mathfrak{a}(M)$ and infinitesimal isometries. As in 4.1, let $\tilde{M}$ be a complex manifold of dimension $n-1$ and F a holomorphic line bundle over $\tilde{M}$ with a hermitian metric h. We assume that $\tilde{M}$ is compact and the first Chern form Φ of the hermitian holomorphic line bundle F is positive. Let $\tilde{g}$ be the Kählerian metric associated with Φ. Let us denote by $\mathfrak{k}(\tilde{M})$ the Lie algebra of all infinitesimal isometries of the Kählerian manifold $(\tilde{M}, \tilde{g})$, which may be considered as a subspace of $\Gamma_{hol}(T^{1,0}(\tilde{M}))$. Let M be the PC manifold associated with the hermitian holomorphic line bundle F (see § 4). In that section, we defined the spaces $\mathfrak{g}(P)_{(m)}$ and the linear mappings κ_m of $\mathfrak{g}(P)_{(m)}$ to $\Gamma_{hol}(T^{1,0}(\tilde{M})\otimes F^m)$. We define a subspace $\mathfrak{a}(P)_{(0)}$ of $\mathfrak{g}(P)_{(0)}$ by

$$\mathfrak{a}(P)_{(0)}=\left\{X\in\mathfrak{g}(P)_{(0)}\middle|X|M\in\mathfrak{a}(M)\right\}.$$

PROPOSITION 6.3. (1) $\operatorname{Ker}\kappa_0\cap\mathfrak{a}(P)_{(0)}=\{\boldsymbol{R}\xi_P^{(1,0)}\}$.
(2) $\kappa_0(\mathfrak{a}(P)_{(0)})=\mathfrak{k}(\tilde{M})\cap\operatorname{Im}\kappa_0$.

PROOF. It is obvious that $\operatorname{Ker}\kappa_0\cap\mathfrak{a}(P)_{(0)}\supset\{\boldsymbol{R}\xi_P^{(1,0)}\}$. Conversely let $X\in\operatorname{Ker}\kappa_0\cap\mathfrak{a}(P)_{(0)}$. By Theorem 4.5, we see that $X=\pi^*\tilde{\rho}\xi_P^{(1,0)}$, where $\tilde{\rho}$ is a holomorphic function on $\tilde{M}$. Since $X|M\in\mathfrak{a}(M)$, $\tilde{\rho}$ is a real valued function, and hence $\tilde{\rho}$ is constant. Therefore we have $\operatorname{Ker}\kappa_0\cap\mathfrak{a}(P)_{(0)}\subset\{\boldsymbol{R}\xi_P^{(1,0)}\}$, proving (1).

Let $X\in\mathfrak{a}(P)_{(0)}$ and let η the real part of X. Then we see that η is tangent to M and the restriction $\eta|M$ to M is an infinitesimal automorphism of M. Since η is invariant by the $U(1)$-action, it follows from Proposition 4.1 that $\mathcal{L}_\eta\omega=0$. This implies that $\mathcal{L}_\eta\Omega=\mathcal{L}_\eta d\omega=d\mathcal{L}_\zeta\omega=0$. Therefore we see that the real part $\tilde{\eta}$ of $\tilde{X}=\kappa_0(X)$ satisfies $\mathcal{L}_{\tilde{\eta}}\Phi=0$, which implies that $\tilde{\eta}$ is an infinitesimal isometry of M. Hence we have $\kappa_0(\mathfrak{a}(P)_{(0)})\subset\mathfrak{k}(\tilde{M})\cap\operatorname{Im}\kappa_0$.

Conversely let $Y\in\mathfrak{k}(\tilde{M})\cap\operatorname{Im}\kappa_0$. Then the real part ζ of Y satisfies

$$0=2\mathcal{L}_\zeta\Phi=2d(\zeta\lrcorner\Phi)=d(Y\lrcorner\Phi)+d\overline{(Y\lrcorner\Phi)}\,.$$

By Theorem 4.6, we have a function ρ on $\tilde{M}$ such that $Y\lrcorner\Phi=-\dfrac{\sqrt{-1}}{2\pi}\bar{\partial}\rho$. Then we have

$$\partial\bar{\partial}(\rho+\bar{\rho})=\overline{d\bar{\partial}\rho}-d\overline{\bar{\partial}\rho}=2\pi\sqrt{-1}\left\{d(Y\lrcorner\Phi)+d\overline{(Y\lrcorner\Phi)}\right\}=0\,,$$

implying that $\rho+\bar{\rho}$ is a pluriharmonic function on $\tilde{M}$. Hence we see that the real part of ρ is constant. Let us denote by q the imaginary part of

ρ, then we have $\bar{\partial}\rho=\sqrt{-1}\,\bar{\partial}q$. Let X be the vector field on P such that X is π-related to Y and $\omega(X)=\sqrt{-1}\,\pi^*q$. As in the proof of Theorem 4.5, we can show that X is holomorphic. Since $\omega(X)$ is a pure imaginary valued function, we see that the real part of X is tangent to M, and hence $X|M\in\mathfrak{a}(M)$. q. e. d.

6.3. Holomorphic line bundles and Einstein Kählerian manifolds. In this paragraph we further assume that $\tilde{M}$ is an Einstein Kählerian manifold:

$$\tilde{R}^* = nc\,\tilde{g}$$

where $\tilde{R}^*$ is the Ricci tensor of $\tilde{M}$ and c is a real constant. By Proposition 4.2, we see that M is a compact strongly pseudo-convex manifold. Moreover it is easy to see that the Levi form associated with $-\xi$ is positive definite and the pair $(M, -\xi)$ satisfies conditions (C. 1) and (C. 3).

Let us apply the results in § 5 to the pair $(M, -\xi)$. First we remark that $\mathfrak{g}_{(\nu)}=\{u\in\mathfrak{g}(M)|\sqrt{-1}\,[\xi, u]=-\nu u\}$. It follows from Theorem 5.6 that $\mathfrak{g}_{(\nu)}$ is decomposed as follows: $\mathfrak{g}_{(\nu)}=\mathfrak{g}^1_{(\nu)}+\mathfrak{g}^2_{(\nu)}$ (direct sum) for $\nu\neq nc$. As in Proposition 4.3, we know that if ν is not an integer, then $\mathfrak{g}_{(\nu)}=0$ and that for each integer m, the assignment $X\rightarrow X|M$ gives an isomorphism of $\mathfrak{g}(P)_{(m)}$ onto $\mathfrak{g}_{(-m)}$. For $i=1$ or 2, let us denote by $\mathfrak{g}^i(P)_{(m)}$ the subspace of $\mathfrak{g}(P)_{(m)}$ which corresponds to $\mathfrak{g}^i_{(-m)}$.

PROPOSITION 6.4. (1) $\mathfrak{g}^1(P)_{(m)}=\operatorname{Ker}\kappa_m$ *and* $\mathfrak{g}^1(P)_{(m)}$ *is isomorphic to* $\Gamma_{hol}(F^m)$.

(2) κ_m *induces an isomorphism of* $\mathfrak{g}^2(P)_{(m)}$ *onto* $\Gamma_{hol}(T^{1,0}(\tilde{M})\otimes F^m)$ *for* $m\neq -nc$.

PROOF. Let $X\in\mathfrak{g}^1(P)_{(m)}$. Since $X|M\in\mathfrak{g}^1_{(-m)}$, we can express $X|M$ as follows: $X|M=f\xi$, where f is a holomorphic function on M satisfying $R_a^*f=a^{-m}f$, $a\in U(1)$. Since $R_{a*}X=a^m X$, $a\in \boldsymbol{C}^*$, it follows that $X=\rho\xi_P^{(1,0)}$, where ρ is a holomorphic function in $C^\infty(P)_{(m)}$ satisfying $\rho|M=f$. By (2) of Theorem 4.5, we have $X\in\operatorname{Ker}\kappa_m$. Conversely it is clear from (2) of Theorem 4.5 that $\operatorname{Ker}\kappa_m\subset\mathfrak{g}^1(P)_{(m)}$. We have thus proved that $\mathfrak{g}^1(P)_{(m)}=\operatorname{Ker}\kappa_m$. The second assertion follows from Theorem 4.5, proving (1).

To prove (2), let us consider the line bundle $F^m\otimes k(\tilde{M})^{-1}$, where $k(\tilde{M})$ denotes the canonical line bundle of $\tilde{M}$. If $m>-nc$, then $F^m\otimes k(\tilde{M})^{-1}$ is a positive line bundle over $\tilde{M}$. By Kodaira's vanishing theorem, we obtain $H^1(F^m)=0$. It follows from Theorem 4.5 that $\kappa_m(g(P)_{(m)})=\Gamma_{hol}(T^{1,0}(\tilde{M})\otimes F^m)$. If $m<-nc$, then $F^{-m}\otimes k(\tilde{M})$ is a positive line bundle over $\tilde{M}$. We have

$$H^0\Big(T^{1,0}(\tilde{M})\otimes F^m\Big) = H^{n-1}\Big(\Omega^{n-1}\Big(T^{1,0}(\tilde{M})^*\otimes F^{-m}\Big)\Big)$$

$$= H^{n-1}\Big(\Omega^1\Big(F^{-m}\otimes k(\tilde{M})\Big)\Big) = 0\,.$$

By Theorem 5.6, we see that $\mathfrak{g}^2(P)_{(m)}=0$, and hence $\mathfrak{g}^2(P)_{(m)}$ is isomorphic to $\Gamma_{hol}(T^{1,0}(\widetilde{M})\otimes F^m)$. q. e. d.

Let χ be the Ricci form of $\widetilde{M}$. Then we have $\chi=nc\Phi$. Assume that there exists a positive integer m such that $\mathfrak{g}^2(P)_{(-m)}\neq 0$. Applying Theorem 5.6, we see that $\mathfrak{g}^2(P)_{(-c)}\neq 0$ and $c>0$. As in Proposition 4.3, we see that c is a positive integer. Hence we obtain the inequality $\chi\geqq n\Phi$. According to the result of Kobayashi-Ochiai [8], we see that $\widetilde{M}$ is biholomorphic to the $(n-1)$-dimensional complex projective space $P^{n-1}(\boldsymbol{C})$ and F is isomorphic to the hyperplane bundle over $P^{n-1}(\boldsymbol{C})$. This fact combined with (2) of Proposition 6.4 implies

THEOREM 6.5. *Let $\widetilde{M}$ be a compact complex manifold of dimension $n-1$ and F a holomorphic line bundle over $\widetilde{M}$. Assume that there is a hermitian metric h of F satisfying the following conditions:*

[a] The first Chern from Φ of the hermitian holomorphic line bundle F is a positive form.

[b] The Kählerian metric $\tilde{g}$ associated with Φ is an Einstein metric.

If the vector bundle $T^{1,0}(\widetilde{M})\otimes F^{-1}$ admits a non-trivial holomorphic cross section, then 1] *$\widetilde{M}$ is biholomorphic to the $(n-1)$-dimensional complex projective space $P^{n-1}(\boldsymbol{C})$ and F is isomorphic to the hyperplane bundle over $P^{n-1}(\boldsymbol{C})$, or* 2] *$n=2$, $\widetilde{M}$ is biholomorphic to the 1-dimensional complex projective space $P^1(\boldsymbol{C})$ and F is isomorphic to the tangent bundle of $P^1(\boldsymbol{C})$.*

PROOF. From condition [b], we have $\chi=nc\Phi$, c being a real constant. It follows from Proposition 6.4 that if $nc\neq 1$, then $\Gamma_{hol}(T^{1,0}(\widetilde{M})\otimes F^{-1})$ is isomorphic to the space $\mathfrak{g}^2(P)_{(-1)}$. Hence we have $\mathfrak{g}^2(P)_{(-1)}\neq 0$. As we have just seen above, this implies that $\widetilde{M}$ is biholomorphic to the complex projective space $P^{n-1}(\boldsymbol{C})$ and F is isomorphic to the hyperplane bundle over $P^{n-1}(\boldsymbol{C})$.

Assume that $nc=1$. Since the Ricci tensor χ of $\widetilde{M}$ is positive, we have $H^{0,1}(\widetilde{M})=0$. It follows that every line bundle is completely determined by its Chern class. Since $\Phi=\chi$, we obtain $F^{-1}=K(\widetilde{M})$, where $K(\widetilde{M})$ is the canonical line bundle of $\widetilde{M}$. Therefore we have

$$\begin{aligned}\Gamma_{hol}\left(T^{1,0}(\widetilde{M})\otimes F^{-1}\right) &= H^0\left(\widetilde{M}, T^{1,0}(\widetilde{M})\otimes K(\widetilde{M})\right)\\ &= H^{n-1}\left(\widetilde{M}, \Omega^{n-1}\left(T^{1,0}(\widetilde{M})^*\otimes K(\widetilde{M})^*\right)\right)\\ &= H^{1,n-1}(\widetilde{M})\,.\end{aligned}$$

Since $\widetilde{M}$ is a Kählerian manifold, we have

$$\dim H^{1,n-1}(\widetilde{M})=\dim H^{n-1,1}(\widetilde{M})=\dim H^1\left(\widetilde{M}, K(\widetilde{M})\right).$$

Therefore we obtain $\dim H^1(\widetilde{M}, K(\widetilde{M})) = \dim \Gamma_{hol}(T^{1,0}(\widetilde{M}) \otimes F^{-1})$.

On the other hand, from Kodaira-Nakano's vanishing theorem, it follows that if $n-1>1$, then

$$H^1\left(\widetilde{M}, K(\widetilde{M})\right) = 0 .$$

Hence, if $n \geqq 3$, then $\Gamma_{hol}(T^{1,0}(\widetilde{M}) \otimes F^{-1}) = 0$, which is a contradiction. If $n=2$, then $\widetilde{M}$ is biholomorphic to the complex projective space $P^1(\boldsymbol{C})$, because $K(\widetilde{M})$ is a negative line bundle. q. e. d.

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Department of Mathematics
Hokkaido University