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Notes on complete noncompact Riemannian manifolds with convex exhaustion functions

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§ 0. Introduction

Let M be a connected, complete and noncompact Riemannian manifold without boundary, and let K_{σ} be the sectional curvature of M determined by a plane section σ. Every geodesic on M is parametrized by arc length. Let C_{p} (resp. Q_{p}) be the tangent cut locus (resp. the tangent first conjugate locus) with respect to a point $p\in M$, and let $C(p)=\exp_{p}C_{p}$, where $\exp_{p}$: $M_{p}\rightarrow M$ is the exponential map. The injectivity radius function of the exponential map is a continuous function $i: M\rightarrow \boldsymbol{R}\cup\{\infty\}$ determined by $i(p)=\inf\{d(p, q)\,;\, q\in C(p)\}$, where d is the distance function of M induced from the Riemannian metric of M. And the injectivity radius $i(M)$ of M is defined as the infimum of $i(p)$, $p\in M$.

Toponogov ([11]) and Maeda ([8], [9]) have shown the following theorem which relates the injectivity radius with the curvature of M;

THEOREM A ([8], [9], [11]) *If the sectional curvature K_{σ} of M satisfies $0<K_{\sigma}\leqq\lambda$ for all σ, then we have $i(p)\geqq\pi/\sqrt{\lambda}$ for all p of M.*

Recently, Sharafutdinov ([10]) has extended the above result as follows;

THEOREM B ([10]) *If M is homeomorphic to a Euclidean space and if $0\leqq K_{\sigma}\leqq\lambda$ for all σ, then we have $i(p)\geqq\pi/\sqrt{\lambda}$ for all p of M.*

The proof of this estimate given in [8] and [9] is based on the fact that there is a continuous filtration of compact totally convex sets $\{C_{t}\}_{t\geqq 0}$ if $K_{\sigma}>0$ holds for all σ (see [2]).

Now if $K_{\sigma}\geqq 0$ holds for all σ, then every Busemann function f_{γ} with respect to a ray γ : $[0, \infty)\rightarrow M$ is convex ([1]). Moreover if $K_{\sigma}\geqq 0$ for all σ, then $F=\sup\{f_{\gamma}: \gamma(0)=p\}$ is a convex exhaustion function and $\{F^{-1}((-\infty, t])\}_{t\geqq 0}$ gives a filtration by compact totally convex sets, where sup is taken over all rays emanating from a fixed point p of M. And it has been proved by Greene and Wu ([3]) that if $K_{\sigma}>0$ then the above F can be replaced by a strongly convex exhaustion function g. Namely, g satisfies; for every compact set A in M there is a $\delta>0$ such that the second difference quotient along every geodesic at any point on A is bounded below by δ.

First, we shall give generalizations of Theorem A and B from a convex functional view point.

THEOREM 1. *Let M admits a convex exhaustion function φ; $M\to \boldsymbol{R}$ and $K_\sigma \leqq \lambda$ for any σ. Then either $i(M)\geqq \pi/\sqrt{\lambda}$ or $i(M)$ is attained at a point belonging to the minimum set of φ.*

COROLLARY. *If M admits a strictly convex exhaustion function and $K_\sigma \leqq \lambda$ for any σ, then we have $i(M)\geqq \pi/\sqrt{\lambda}$.*

Concerning the problem on the position of cut locus and conjugate locus mentioned by Weinstein ([12]), Gromoll and Meyer have proved in [6] that if the sectional curvature of M satisfies $K_\sigma>0$ for any σ, then there exists a point p of M such that $C_p\cap Q_p\neq\emptyset$. If a strongly convex exhaustion function on M is replaced by a (weaker) strictly convex exhaustion function (for definition see below), then we have a generalization of the above result as follows;

THEOREM 2. *Assume that M admits a strictly convex exhaustion function. If there is a point q of M at which $C_q\neq\emptyset$, then there exists a point p of M such that $C_p\cap Q_p\neq\emptyset$.*

As is seen later, in the above Theorems 1, 2 and Corollary, the assumptions are optimal in the following sense; If in Theorem 1 the exhaustion condition is removed, and if in Corollary one of the three conditions, strict convexity, exhaustion and $K_\sigma\leqq\lambda$ is removed, then there is a counter example which violates the conclusion. If in Theorem 2, one of the three conditions, strict convexity, exhaustion and $C_q\neq\emptyset$ is removed, then there is a counter example which violates the conclusion.

Definitions and some auxiliary results are given in the section 1, the proof of Theorem 1 and remarks are given in the section 2 and the proof of Theorem 2 and some other related results are given in the section 3.

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§ 1. Preliminaries

Hereafter let M be an n-dimensional complete noncompact Riemannian manifold without boundary. First of all, we shall define the concept of convexity and exhaustion for a real valued function on M. A function $\varphi: M\to\boldsymbol{R}$ is said to be convex if for every geodesic $\gamma: \boldsymbol{R}\to M$ and every t_1, $t_2\in\boldsymbol{R}$ and $\lambda\in[0, 1]$, $\varphi\circ\gamma$ satisfies

$$\varphi\circ\gamma\Big((1-\lambda)t_1+\lambda t_2\Big)\leqq(1-\lambda)\varphi\circ\gamma(t_1)+\lambda\varphi\circ\gamma(t_2)\,.$$

It the above inequality is strict for any $\lambda\in(0, 1)$, then φ is called to be strictly convex. A convex function is locally Lipschitz continuous and hence it is differentiable at almost all points in M. And a strongly convex function is a strictly convex function but the converse is not generally true. A function $\varphi: M\to\boldsymbol{R}$ is said to be exhaustion if $\varphi^{-1}((-\infty, a])$ is compact for any $a\in\boldsymbol{R}$. We denote the sublevel set $\varphi^{-1}((-\infty, a])$ by $M^a(\varphi)$ or simply M^a. Furthermore, a subset A in M is by definition totally convex if every geodesic segment from p to q is contained in A for any points p and q of A. Any sublevel set M^a of a convex function is closed totally convex. Hence convex exhaustion functions take their minima and thus if φ is a strictly convex exhaustion function, then the minimum set is a single point.

Hereafter let $\varphi: M\to\boldsymbol{R}$ be a convex function and let m be the infimum of $\varphi(p)$, $p\in M$, unless otherwise mentioned. The following notions are used by Sharafutdinov ([10]) to prove Theorem C and D. For $v\in M_p$, we denote by $\varphi'(p, v)$ the value of the right derivative of $\varphi(t)=\varphi(\exp_p tv)$ at $t=0$. Then $\varphi'(p, \cdot): M_p\to\boldsymbol{R}$ is Lipschitz continuous, convex and positive homogeneous, *i.e.*, $\varphi'(p, tv)=t\varphi'(p, v)$ for $t\geqq 0$. Hence it turns out that $\varphi'(p, \cdot)$ restricted to the set S_pM of unit tangent vectors to M at p attains its negative minimum at a unique vector in S_pM if $\varphi(p)>m$. So we can define the vector $X(p)=\varphi'(p, v)v$, where v is the unit tangent vector such that $\varphi'(p, v)=\min\{\varphi'(p, w)\,;\ w\in S_pM\}$.

Next let $c: [a, b]\to M$ be a continuous curve and v be a tangent vector at $c(t)$, $t\in(a, b]$. Then v is said to be the left tangent vector to the curve c at t if for every smooth function $f: M\to\boldsymbol{R}$, the left derivative of $f\circ c$ at t is equal to vf. And we write as $v=\dot{c}_-(t)$. Under these notations, we define the following;

DEFINITION. Let the point p of M and the number T be such that $\varphi(p)=t_0>T\geqq m$. The continuous curve $c_p: [T, t_0]\to M$ is called to be an integral curve for the field $\boldsymbol{X}$ emanating from p if it satisfies the following three conditions;

(1) $c_p(t_0)=p$,

(2) c_p is locally Lipschitz continuous on $(T, t_0]$,

(3) For any $t\in(T, t_0]$, $\varphi(c_p(t))>m$ and

$$\dot{c}_{p-}(t)=\boldsymbol{X}\Big(c_p(t)\Big)/|\boldsymbol{X}\Big(c_p(t)\Big)|^2\,.$$

If moreover $T=m$, then c_p is called a maximal integral curve for the field $\boldsymbol{X}$ emanating from p.

The following facts were proved by Sharafutdinov in [10] and see [10] for details.

THEOREM C ([10]). *Let φ be a convex exhaustion function on M and let M^a be an arbitrarily given sublevel set of φ. Then for every point p of M^a such that $\varphi(p)>m$, there exists a unique maximal integral curve for the field X emanating from p.*

THEOREM D ([10]). *Let φ be a convex exhaustion function. For every t with $m\leqq t\leqq a$, define a* map $\psi_t: M^a\to M^t$ *by $\psi_t(p)=p$ if $\varphi(p)\leqq t$ and $\psi_t(p)=c_p(t)$ if $\varphi(p)=t_0>t$, where $c_p: [m, t_0]\to M^a$ is a maximal integral curve for X emanating from p. Then we have $L(\gamma)\geqq L(\psi_t\circ\gamma)$ for any rectifiable curve $\gamma: [0, 1]\to M^a$, where $L(\gamma)$ denotes the length of the curve γ.*

§ 2. Proof of Theorem 1 and Remarks

PROOF OF THEOREM 1. Assume that $i(M)<\pi/\sqrt{\lambda}$ and let $\{p_n\}$ be a sequence of points in M such that $i(p_n)\to i(M)$ as $n\to\infty$. Fix an $a_n\in \boldsymbol{R}$ so that $\varphi(p_n)\leqq a_n$. Since φ is exhaustion, M^{a_n} is compact and the injectivity radius function restricted to M^{a_n} takes minimum at some point q_n of M^{a_n}. Let q_n' in $C(q_n)$ be a point such that $d(q_n, q_n')=d(q_n, C(q_n))$. Then from the theorem of Morse-Schoenberg and Lemma 2 [5; p 226], there exists a geodesic loop $\gamma: =\gamma^{(n)}: [0, 2i(q_n)]\to M$ such that $\gamma(0)=\gamma(2i(q_n))=q_n$. Since M^{a_n} is totally convex, $\gamma([0, 2i(q_n)])$ is contained in M^{a_n}. If $\dot\gamma(0)\neq\dot\gamma(2i(q_n))$ then $d(q_n', C(q_n'))<d(q_n', q_n)$. Hence $i(q_n')<i(q_n)$. This contradicts the choice of q_n. So γ is a closed geodesic.

Let $\varphi\circ\gamma([0, 2i(q_n)]=b_n$. If $b_n=m$ holds for all n, then $i(M)$ is attained at a point on M^m. Assume that $b_n>m$, and reparametrize γ by $\gamma: [0, 1]\to M$. For any $t\in[m, b_n]$, define a curve $\tilde\gamma_t: =\tilde\gamma_t^{(n)}: [0, 1]\to M$ by putting $\tilde\gamma_t=\psi_t\circ\gamma$, where $\psi_t: M^{b_n}\to M^t$ is the map defined in Theorem D. Then by Theorem D, $\tilde\gamma_t$ is a closed curve with $L(\tilde\gamma_t)\leqq L(\gamma)<2\pi/\sqrt{\lambda}$. Let $t_0: =t^{(n)}$ be the infimum of the number $t\in[m, b_n]$ such that $L(\psi_t\circ\gamma)=L(\gamma)$ and such that $\psi_t\circ\gamma$ is a closed geodesic. If $t_0>m$, then take a sequence $t_k\in(m, t_0)$, $k=1, 2, \cdots$, tending to t_0. If $L(\tilde\gamma_{t_k})=L(\tilde\gamma_{t_0})$, $\tilde\gamma_{t_k}$ is not a closed geodesic by the definition of t_0. Therefore we can get a closed curve $\gamma_{t_k}: =\gamma_{t_k}^{(n)}: [0, 1]\to M^{t_k}$ such that $L(\gamma_{t_k})<L(\tilde\gamma_{t_k})$ and such that $\sup\{d(\gamma_{t_k}(s), \tilde\gamma_{t_k}(s)): 0\leqq s\leqq 1\}<1/k$, $k=1, 2, \cdots$, by exchanging a subarc of $\tilde\gamma_{t_k}$ for a minimal geodesic segment. Then the sequence $\{\gamma_{t_k}\}$ converges uniformly to $\tilde\gamma_{t_0}$. Namely there exists a family of closed curves γ_{t_k}, converging towards $\tilde\gamma_{t_0}$, each of which is shorter than $\tilde\gamma_{t_0}$ with $L(\tilde\gamma_{t_0})<2\pi/\sqrt{\lambda}$. Now we are led to the contradiction by the same argument as Klingenberg's ([7]). Hence $t_0=m$, that is, we may suppose that $\gamma=\gamma^{(n)}\ni q_n$ belongs to the minimum set M^m of φ for any n. Then by the compactness of M^m, $i(M)$ is attained at a point of M^m.

REMARK 1. In order to derive the conclusion of Theorem 1 and Corollary, we can not omit the exhaustion condition on the assumption of φ. We can construct a smooth non-exhaustion strongly convex function on a surface of revolution in $\boldsymbol{R}^3$ such that Gaussian curvature has a positive maximum λ and $i(p_n)$ tends to zero for a certain sequence of points $\{p_n\}$ in the surface. Let $h:\boldsymbol{R}\to\boldsymbol{R}$ be a smooth non-decreasing function such that $h(t)=0$ if $t\leqq a$, $0<h(t)<1$ if $a<t<b$ and $h(t)=1$ if $t\geqq b$, where a and b are constants such that $f:\boldsymbol{R}\to\boldsymbol{R}$, $f(t)=e^t+h(t)$, satisfies $f''(t_0)<0$ for some $t_0\in(a, b)$. For this function f, let M be the surface of revolution in $\boldsymbol{R}^3$ with parametrization

$$\chi(u, v)=\left(f(v)\cos u,\ f(v)\sin u,\ v\right),$$

where $0<u<2\pi$ and $-\infty<v<+\infty$. Note that Gaussian curvature G is bounded above by a positive constant since $G(p)>0$ at p with $v(p)=t_0$ and $G(q)<0$ at every q with $v(q)\notin(a, b)$. And note also that $i(p)\to 0$ as $v(p)\to -\infty$. Hence we have only to show the existence of a non-exhaustion strongly convex function on M. Consider the function $\varphi: M\to\boldsymbol{R}$ with $\varphi(u, v)=f^n(v)$, where n is a positive integer. We assert that φ is strongly convex for sufficiently large n. Using differential equation of the geodesic $\gamma:\boldsymbol{R}\to M$, we get

$$\begin{aligned}\frac{d^2}{ds^2}\varphi\circ\gamma(s)&=nf^n(f')^2(\dot{u})^2/\left\{(f')^2+1\right\}\\&\quad+nf^{n-2}\left\{(n-1)(f')^4+(n-1)(f')^2+ff''\right\}(\dot{v})^2/\left\{(f')^2+1\right\}.\end{aligned}$$

Since $\frac{d^2}{ds^2}\varphi\circ\gamma(s)$ is positive at $v<a$ and $v>b$, we have only to consider the case; $a\leqq v\leqq b$. Then there are positive constants C_0, C_1 and C_2 such that $f>C_0$, $f'>C_1$ and $|f''|<C_2$. So we have

$$(n-1)(f')^4+(n-1)(f')^2+ff''>(n-1)C_1^4+(n-1)C_1^2-C_0C_2.$$

Consequently φ is smooth strongly convex for sufficiently large n and not an exhaustion function by definition.

REMARK 2. In order to derive the conclusion of Corollary, we can not omit the strict convexity condition, also. Let M be a flat cylinder $S^1\times\boldsymbol{R}$ with induced metric from $\boldsymbol{R}^3$, where S^1 is a unit circle. Then the function $\varphi: S^1\times\boldsymbol{R}\to\boldsymbol{R}$, $\varphi(x, t):=|t|$, is not-strictly convex but exhaustion. However the conclusion does not hold.

REMARK 3. The manifold which we consider in Corollary is homeomorphic to a Euclidean space by consequence of Theorem F in Greene and Shiohama ([4]) because the minimum set of every structly convex exhaustion function is a single point. And this is a natural requirement to derive the conclusion of Corollary. See Remark in [9] for details.

§ 2. Proof of Theorem 2 and Remarks

PROOF OF THEOREM 2. Fix an $a \in \boldsymbol{R}$ so that $\varphi(q)<a$. Since M^a is compact, the injectivity radius function restricted to M^a takes the minimum at a point p of M^a. Then $i(p)$ is finite since $i(q)$ is finite by assumption. If $C_p \cap Q_p=\emptyset$, then there is a geodesic loop γ; $[0, 2i(p)] \rightarrow M$ such that $\gamma(0)=\gamma(2i(p))=p$. And γ is shown to be a closed geodesic by the same discussion as that of the proof of Theorem 1. Therefore this contradicts the strict convexity of φ.

REMARK 4. In order to derive the conclusion of Theorem 2, we can not omit any one of the three conditions, exhaustion, strict convexity and the existence of a point q such that $C_q \neq \emptyset$. Simple counter examples are given as follows; Let M be the surface of revolution in $\boldsymbol{R}^3$ with parametrization

$$\chi(u, v)=(e^v \cos u,\ e^v \sin u,\ v),$$

where $0<u<2\pi$ and $-\infty<v<+\infty$. Then let $\varphi_1: M \rightarrow \boldsymbol{R}$ be the function defined by $\varphi(u, v)=e^v$. φ_1 is strongly convex but not exhaustion. Secondly, let $\varphi_2: S^1 \times \boldsymbol{R} \rightarrow \boldsymbol{R}$ be the function defined by $\varphi_2(x, t)=|t|$, where $S^1 \times \boldsymbol{R}$ is a flat cylinder with induced metric from $\boldsymbol{R}^3$. And $\boldsymbol{R}^n$ furnishes the trivial counter example for which $C_q \neq \emptyset$ can not be omitted.

Moreover, we can get an information about the point at which φ takes its minimum.

PROPOSITION. *Let M admits a strictly convex exhaustion function φ: $M \rightarrow \boldsymbol{R}$ and let p be the point at which φ attains the minimum. Then we have either $C_p=\emptyset$ or $C_p \cap Q_p \neq \emptyset$.*

PROOF. Suppose that $C_p \neq \emptyset$. If $C_p \cap Q_p=\emptyset$, then there is a geodesic loop γ: $[0, 2i(p)] \rightarrow M$ with $\gamma(0)=\gamma(2i(p))=p$. Since $M^{\min \varphi}=\{p\}$ is a totally convex set, $\gamma([0, 2i(p)])$ is contained in $\{p\}$. Hence $i(p)=0$. This is a contradiction.

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