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On Sasakian manifolds with vanishing contact Bochner curvature tensor II

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§ 1. Introduction

This paper is a continuation of our previous one [7] with the same title, in which we proved the following theorems.

THEOREM A. *Let M be a $(2n+1)$-dimensional Sasakian manifold with constant scalar curvature R whose contact Bochner curvature tensor vanishes. If the square of the length of the η-Einstein tensor is less than*

$$\frac{(n-1)(n+2)^2(R+2n)^2}{2n(n+1)^2(n-2)^2}, \qquad n\geqq 3,$$

then M is a space of ϕ-holomorphic sectinal curvature.

THEOREM B. *Let M be a 5-dimensional Sasakian manifold with constant scalar curvature whose contact Bochner curvatur tensor vanishes. If the scalar curvature is not -4, then M is a space of constant ϕ-holomorphic sectional curvature.*

The above two results are analogous to the following theorems proved by S. I. Goldberg, M. Okumura and Y. Kubo.

THEOREM C (S. I. Goldberg and M. Okumura [4]). *Let M be an n-dimensional compact conformally flat Riemannian manifold with constant scalar curvature R. If the length of the Ricci tensor is less than $R/\sqrt{n-1}$, $n\geqq 3$, then M is a space of constant curvature.*

THEOREM D (Y. Kubo [8]). *Let M be a real n-dimensional Kaehlerian manifold with constant scalar curvature R whose Bochner curvatur tensor vanishes. If the length of the Ricci tensor is not greater than $R/\sqrt{n-2}$, $n\geqq 4$, then M is a space of constant holomorphic sectinal curvature.*

REMARK. We improved Theorem D, in [6], as the inequality with respect to the Ricci tensor is the best possible.

Also, S. I. Goldberg, M. Okumura and Y. Kubo have proved the following theorems under the different conditions from Theorem C and D.

THEOREM E (S. I. Goldberg and M. Okumura [4]). *Let M be an n-dimensional compact conformally flat Riemannian manifold. If the length of the Ricci tensor is constant and less than $R/\sqrt{n-1}$, $n \geqq 3$, then M is a space of constant curvature.*

THEOREM F (Y. Kubo [8]). *Let M be a real n-dimensional Kaehlerian manifold with vanishing Bochner curvature tensor. If the length of the Ricci tensor is constant and less than $R/\sqrt{n-2}$, $n \geqq 4$, then M is a space of constant holomorphic sectional curvature.*

The purposes of this paper are to obtain the theorem, analogous to the above two theorems, for a Sasakian manifold with vanishing contact Bochner curvature tensor and to give an example of Sasakian manifold which shows that the inequality in Theorem A is the best possible.

§ 2. Preliminaries

First, we would like to recall definitions and som fundamental properties of Sasakian manifolds. Let M be a $(2n+1)$-dimensional Sasakian manifold with the Riemannian metric g_{ij} and an almost contact structure (ϕ, ξ, η) satisfying

$$\phi_a{}^i\phi_j{}^a = -\delta^i{}_j + \eta_j\xi^i, \quad \phi_a{}^i\xi^a = 0, \quad \eta_a\phi_i{}^a = 0, \quad \eta_a\xi^a = 1,$$

$$(2.1) \qquad N_{ij}{}^k + (\partial_i\eta_j - \partial_j\eta_i)\xi^k = 0, \quad \phi_{ij} = \frac{1}{2}(\partial_i\eta_j - \partial_j\eta_i),$$

$$g_{ab}\phi_i{}^a\phi_j{}^b = g_{ij} - \eta_i\eta_j \text{ and } \xi^i = \eta_a g^{ai},$$

where $N_{ij}{}^k$ is the Nijenhuis tensor with respect to ϕ_{ij} and $\partial_i = \partial/\partial x^i$ is the partial differential operator with respect to the local coordinate (x^i). In the view of the last equation, we shall write η^i instead of ξ^i in the sequel.

In the following, let R_{hijk}, R_{ij} and R denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature. If the Ricci tensor R_{ij} of a Sasakian manifold M satisfies

$$(2.2) \qquad R_{ij} = \left(\frac{R}{2n} - 1\right) g_{ij} - \left(\frac{R}{2n} - 2n - 1\right) \eta_i\eta_j,$$

then we say that M is an η-Einstein manifold. We define the η-Einstein tensor S_{ij} by

$$(2.3) \qquad S_{ij} := R_{ij} - \left(\frac{R}{2n} - 1\right) g_{ij} + \left(\frac{R}{2n} - 2n - 1\right) \eta_i\eta_j.$$

Note that if the η-Einstein tensor of a Sasakian manifold M vanishes, then

M is an η-Einstein manifold.

Let M be a Sasakian manifold of dimension $2n+1$. As an analogue of the Bochner curvature tensor of a Kaehlerian manifold, we define the contact Bochner curvature tensor B_{hijk} of M by

$$
\begin{aligned}
B_{hijk} := R_{hijk} &- \frac{1}{2(n+2)} (R_{ij}g_{hk} - R_{ik}g_{hj} + g_{ij}R_{hk} - g_{ik}R_{hj} \\
&- R_{ij}\eta_h\eta_k + R_{ik}\eta_h\eta_j - \eta_i\eta_j R_{hk} + \eta_i\eta_k R_{hj} + H_{ij}\phi_{hk} \\
&- H_{ik}\phi_{hj} + \phi_{ij}H_{hk} - \phi_{ik}H_{hj} - 2H_{hi}\phi_{jk} - 2\phi_{hi}H_{jk}) \\
&+ \frac{R-6n-8}{4(n+1)(n+2)} (g_{ij}g_{hk} - g_{ik}g_{hj}) \\
&+ \frac{R+4n^2+6n}{4(n+1)(n+2)} (\phi_{ij}\phi_{hk} - \phi_{ik}\phi_{hj} - 2\phi_{hi}\phi_{jk}) \\
&- \frac{R+2n}{4(n+1)(n+2)} (g_{ij}\eta_h\eta_k - g_{ik}\eta_h\eta_j + \eta_i\eta_j g_{hk} - \eta_i\eta_k g_{hj}) ,
\end{aligned} \tag{2. 4}
$$

where $H_{ij} := R_i{}^a\phi_{aj}$.

The following lemmas are well-known.

LEMMA A. *In a Sasakian manifold with vanishing contact Bochner curvature tensor, there exists the following identity*:

$$
\begin{aligned}
\nabla_k R_{ij} - \nabla_j R_{ik} = \frac{1}{4(n+1)} &\{(\nabla_k R)(g_{ij} - \eta_i\eta_j) - (\nabla_j R)(g_{ik} - \eta_i\eta_k) \\
&+ (\nabla^a R)(\phi_{ak}\phi_{ij} - \phi_{aj}\phi_{ik} - 2\phi_{ai}\phi_{jk})\} \\
&- (\eta_k H_{ij} - \eta_j H_{ik} - 2\eta_i H_{jk}) + 2n(\eta_k\phi_{ij} - \eta_j\phi_{ik} - 2\eta_i\phi_{jk}) ,
\end{aligned} \tag{2. 5}
$$

where ∇_k denotes the operator of covariant differentiation.

LEMMA B. *Let M be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M is an η-Einstein manifold, then M is a space of constant ϕ-holomorphic sectional curvature.*

M. Okumura proved the following lemmas.

LEMMA C [10]. *Let a_i, $i=1, 2, \cdots, m$ $(m \geqq 2)$, be real numbers satisfying*

$$
\sum_{i=1}^{m} a_i = 0 \quad and \quad \sum_{i=1}^{m} a_i^2 = k^2 \qquad (k \geqq 0) . \tag{2. 6}
$$

Then we have

$$
-\frac{m-2}{\sqrt{m(m-1)}} k^3 \leqq \sum_{i=1}^{m} a_i^3 \leqq \frac{m-2}{\sqrt{m(m-1)}} k^3 . \tag{2. 7}
$$

LEMMA D [9]. *If a given set of* $m+1$ $(m\geqq 2)$ *real numbers* $a_1, a_2, \cdots, a_m$ *and* k *satisfies the inequality*

$$(2.8)\qquad \sum_{i=1}^{m} a_i^2+k<\frac{1}{m-1}\left(\sum_{i=1}^{m} a_i\right)^2,$$

then, for any pair of distinct i *and* j, *we have*

$$(2.9)\qquad k<2a_ia_j\,.$$

§ 3. Main result

Let M be a $(2n+1)$-dimensional Sasakian manifold. From the definition of the η-Einstein tensor S_{ij}, we have $S_a{}^i\phi_j{}^a=\phi_a{}^iS_j{}^a$. From the brief computations, we see that

$$(3.1)\qquad \text{trace } S=S^a{}_a=0\,,$$

$$(3.2)\qquad \text{trace } S^2=S_{ab}S^{ab}=R_{ab}R^{ab}-\frac{1}{2n}R^2+2R-4n^2-2n\geqq 0$$

and

$$(3.3)\qquad \begin{aligned}\text{trace } S^3=S^a{}_bS^b{}_cS^c{}_a&=R^a{}_bR^b{}_cR^c{}_a-3\left(\frac{R}{2n}-1\right)R_{ab}R^{ab}+\frac{1}{2n^2}R^3\\&\quad-\frac{3}{n}R^2+6(n+1)R-4n(n+1)(2n+1)\\&=R^a{}_bR^b{}_cR^c{}_a-3\left(\frac{R}{2n}-1\right)S_{ab}S^{ab}-\frac{1}{4n^2}R^3+\frac{3}{2n}R^2-3R\\&\quad-2n(2n-1)(2n+1)\,.\end{aligned}$$

Note that if trace S^2 is zero, then M is an η-Einstein manifold.

The following lemma is fundamental and may be found in [7].

LEMMA E. *Put* $f^2:=S_{ab}S^{ab}$ $(f\geqq 0)$ *and let* c_i, $i=1, 2, \cdots, 2n+1$, *be the characteristic roots of* $S_i{}^j$. *Then we have*

$$(3.4)\qquad -\frac{n-2}{\sqrt{2n(n-1)}}f^3\leqq \sum_{i=1}^{2n+1} c_i^3\leqq \frac{n-2}{\sqrt{2n(n-1)}}f^3.$$

PROOF. From $S^i{}_a\eta^a=0$ and the comutativity of $S_i{}^j$ and $\phi_i{}^j$, we see that the characteristic roots of $S_i{}^j$ are $c_1, \cdots, c_n, c_1, \cdots, c_n$ and 0. Combining this fact with Lemma C, we obtain this lemma.

By a straightforward computation, the Laplacian of the square length of the Ricci tensor becomes

$$\frac{1}{2}\Delta(R_{ab}R^{ab}) = (\nabla_c R_{ab})(\nabla^c R^{ab}) + \frac{1}{2} R^{ab}\nabla_a\nabla_b R + R^a{}_b R^b{}_c R^c{}_a - R_{abcd}R^{ad}R^{bc} + R^{ab}\nabla^c(\nabla_c R_{ab} - \nabla_b R_{ac}). \tag{3.5}$$

Using Lemma A and making use of (3.2) and (3.3), we get

$$\frac{1}{2}\Delta(R_{ab}R^{ab}) = (\nabla_c R_{ab})(\nabla^c R^{ab}) + \left\{\frac{n-1}{2(n+1)} R^{ab} + \frac{R-2n}{4(n+1)} g^{ab}\right\}\nabla_a\nabla_b R + \frac{n}{n+2}\,\mathrm{trace}\, S^3 + \frac{R-2(n+2)}{2(n+1)}\,\mathrm{trace}\, S^2 - \frac{1}{n}\left\{\mathrm{R} - 2n(2n+1)\right\}^2. \tag{3.6}$$

On the other hand, by a straightfo₁ward calculation, we see that

$$\begin{aligned}&\left\{\nabla_c R_{ab} + \phi_c{}^d(R_{da}\eta_b + R_{db}\eta_a) - 2n(\phi_{ca}\eta_b + \phi_{cb}\eta_a)\right\}\\ &\cdot\left\{\nabla^c R^{ab} + \phi^{ce}(R_e{}^a\eta^b + R_e{}^b\eta^a) - 2n(\phi^{ca}\eta^b + \phi^{cb}\eta^a)\right\}\\ &= (\nabla_c R_{ab})(\nabla^c R^{ab}) - 2R_{ab}R^{ab} + 8nR - 16n^3 - 8n^2\\ &= (\nabla_c R_{ab})(\nabla^c R^{ab}) - 2\,\mathrm{trace}\, S^2 - \frac{1}{n}\left\{R - 2n(2n+1)\right\}^2 \geqq 0.\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), we have

$$\begin{aligned}\frac{1}{2}\Delta(R_{ab}R^{ab}) &= \left\{\nabla_c R_{ab} + \phi_c{}^d(R_{da}\eta_b + R_{db}\eta_a) - 2n(\phi_{ca}\eta_b + \phi_{cb}\eta_a)\right\}\\ &\quad\cdot\left\{\nabla^c R^{ab} + \phi^{ce}(R_e{}^a\eta^b + R_e{}^b\eta^a) - 2n(\phi^{ca}\eta^b + \phi^{cb}\eta^a)\right\}\\ &\quad+\left\{\frac{n-1}{2(n+1)} R^{ab} + \frac{R-2n}{4(n+1)} g^{ab}\right\}\nabla_a\nabla_b R\\ &\quad+\frac{n}{n+2}\,\mathrm{trace}\, S^3 + \frac{R+2n}{2(n+1)}\,\mathrm{trace}\, S^2.\end{aligned} \tag{3.8}$$

From $R^i{}_a\eta^a = 2n\eta^i$ and the commutativity of $R_i{}^j$ and $\phi_i{}^j$, we see that the characteristic roots of $R_i{}^j$ are $b_1, \cdots, b_n$, $b_1, \cdots, b_n$ and $2n$. Here we assume that the length of the η-Einstein tensor is less than $\dfrac{R-2n}{\sqrt{2n(n-1)}}$. Then we have the inequality

$$\sum_{i=1}^{n} b_i^2 < \frac{1}{n-1}\left(\sum_{i=1}^{n} b_i\right)^2. \tag{3.9}$$

Since the n-real numbers $b_1, \cdots, b_n$ satisfy the inequality (3.9), applying Lemma D, we find $2b_ib_j > 0$ $(i \neq j)$. From this fact and $R > 2n$, $R_i{}^j$ is positive definite.

Next, assume that the length of the Ricci tensor is constant. Then the left hand of (3. 8) vanishes.

From the assumption with respect to the length of the η-Einstein tensor and Lemma E, we have

$$\begin{aligned}&\frac{n}{n+2}\text{ trace } S^3+\frac{R+2n}{2(n+1)}\text{ trace } S^2\\ &\geqq f^2\left\{-\frac{n(n-2)}{\sqrt{2n(n-1)}\,(n+2)}f+\frac{R+2n}{2(n+1)}\right\}\geqq 0\,.\end{aligned}\tag{3. 10}$$

Thus (3. 8) yields the inequality

$$\left\{\frac{n-1}{2(n+1)}R^{ab}+\frac{R-2n}{4(n+1)}g^{ab}\right\}\nabla_a\nabla_b R\leqq 0\,.\tag{3. 11}$$

If M is compact, E. Hopf's well-known theorem says that R is constant, because $R_i{}^j$ is positive definite. Hence the second term of the right hand of (3. 8) vanishes. Since the first and third terms of the right hand of (3. 8) are not negative, it is clear that $f^2=\text{trace } S^2=0$, that is, M is an η-Einstein mainfold.

Consequently we have the following

THEOREM. *Let M be a $(2n+1)$-dimensional compact Sasakian manifold with vanishing contact Bochner curvature tensor. If the length of the Ricci tensor is constant and the length of the η-Einstein tensor is less than $\frac{R-2n}{\sqrt{2n(n-1)}}$, $n\geqq 2$, then M is a space of constant ϕ-holomorphic sectional curvature.*

§ 4. Example

We shall now give the example of a Sasakian manifold which shows that the inequality in Theorem A is the best possible.

Let $M_1(\tilde{g}_{\beta\gamma}, \tilde{\phi}_\beta{}^\gamma, \tilde{\eta}^\gamma)$ be a $(2n-1)$-dimensional Sasakian manifold of constant ϕ-holomorphic sectional curvature $c-3\,(c>0)$. Let $M_2(g_{\mu\nu}, J_\mu{}^\nu)$ be a real 2-dimensional Kaehlerian manifold of constant holomorphic sectional curvature $-c$ with 1-form ω_μ satisfying $J_{\mu\nu}=(1/2)\,(\partial_\mu\omega_\nu-\partial_\nu\omega_\mu)$. There exist such manifolds. For example, we put $M_2:=\left\{(x, y)\in \boldsymbol{R}^2 | x<0\right\}$, $(g'_{\mu\nu})=\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}$, $(J_\mu{}^\nu)=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$ and $(\omega_\mu)=(0, v)$, where $a=e^{\sqrt{c}\,x}/(1-e^{\sqrt{c}\,x})^2$ and $v=\frac{2}{\sqrt{c}\,(1-e^{\sqrt{c}\,x})}$. We put $M:=M_1\times M_2$ and induce a new Sasakain structure $(g_{ij}, \phi_i{}^j, \eta^j)$ on M as follows:

(4.1) $$(g_{ij}) := \left(\begin{array}{c|c} \tilde{g}_{\beta\gamma} & \tilde{\eta}_{\beta}\omega_{\nu} \\ \hline \omega_{\mu}\tilde{\eta}_{\gamma} & g'_{\mu\nu}+\omega_{\mu}\omega_{\nu} \end{array}\right),$$

(4.2) $$(\phi_i{}^j) := \left(\begin{array}{c|c} \tilde{\phi}_{\beta}{}^{\gamma} & 0 \\ \hline -J_{\mu}{}^{\lambda}\omega_{\lambda}\tilde{\eta}^{\gamma} & J_{\mu}{}^{\nu} \end{array}\right),$$

(4.3) $$(\eta^i) := \left(\begin{array}{c} \tilde{\eta}^{\beta} \\ \hline 0 \end{array}\right).$$

Then we have

(4.4) $$(\eta_i) = (\tilde{\eta}_{\beta}|\,\omega_{\mu}),$$

(4.5) $$(g^{ij}) = \left(\begin{array}{c|c} \tilde{g}^{\beta\gamma}+\tilde{\eta}^{\beta}\tilde{\eta}^{\gamma} & -\tilde{\eta}^{\beta}\omega^{\nu} \\ \hline -\omega^{\mu}\tilde{\eta}^{\gamma} & g'^{\mu\nu} \end{array}\right),$$

and

(4.6) $$(\phi_{ij}) = \left(\begin{array}{c|c} \tilde{\phi}_{\beta\gamma} & 0 \\ \hline 0 & J_{\mu\nu} \end{array}\right).$$

For ϕ-basis $\{e_0, e_1, \cdots, e_{2n}\}$ at every point of M such that $e_0=\eta$, $e_1, \cdots, e_{n-1}$, $e_{n+1}=\phi e_1, \cdots, e_{2n-1}=\phi e_{n-1}$ are tangent to M_1 and e_n, $e_{2n}=\phi e_n$ tangent to M_2, we obtain

(4.7)
$$K(e_i, e_{n+i}) = c-3 \qquad (i=1, \cdots, n-1),$$
$$K(e_i, e_j) = \frac{c}{4} \qquad (i, j=1, \cdots, n-1\,; i\neq j),$$
$$K(e_n, e_{2n}) = -c-3$$
and
$$K(e_i, e_n) = 0 \qquad (i=1, \cdots, n-1),$$

where $K(X, Y)$ means the sectional curvature with respect to the plane spaned by X and Y.

This shows that M is an example of $(2n+1)$-dimensional Sasakian manifold with vanishing contact Bochner curvature tensor which is not of constant ϕ-holomorphic sectional curvature. Furthermore we have

(4.8) $$R = (n-2)(n+1)c-2n$$

and

(4.9) $$R_{ab}R^{ab} = (1/2)(n^3-n^2+4)\,c^2-4(n-2)(n+1)\,c+4n^2+8n$$

on this Sasakain manifold M.

Therefore, if $n \geqq 3$, then we have

$$S_{ab}S^{ab} = R_{ab}R^{ab} - (1/2n)\, R^2 + 2R - 4n^2 - 2n$$
$$= \frac{(n-1)(n+2)^2(R+2n)^2}{2n(n-2)^2(n+1)^2}\,.$$

This shows that the inequality in Theorem A is the best possible.

If $n=2$, we have $R=-4$. This shows that M is an example of a 5-dimensional Sasakian manifold with vanishing contact Bochner curvature tensor and constant scalar curvature -4 which is not of constant ϕ-holomorphic sectional curvature.

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