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Some remarks on separable extensions

Dedicated to Professor Goro Azumaya
on his 60th birthday

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Many equivalent conditions for a ring extension to be H-separable have been studied by Sugano and Nakamoto in [2] and [4] etc.. In this connection we give some equivalent conditions for separable extensions in Theorem 1. In § 2, we consider the automorphism group of an H-separable extension, and in Theorem 3 we give an exact sequence which turns into the result of Rosenberg and Zelinsky [3] Theorem 7 in case of Λ is an Azumaya algebra.

§ 1. Separable extension

Throughout this paper we assume that all rings have the identity 1 and subrings contain this element and modules are unitary. For any two-sided module M over a ring A, M^A means the set $\{m\in M | am=ma$ for all $a\in A\}$. Thus for a ring Λ and a subring Γ of Λ, denote by Δ, $\Delta=\Lambda^\Gamma=\{d\in\Lambda | d\gamma=\gamma d,\ \gamma\in\Gamma\}$, the commutator of Γ in Λ, and by C, $C=\Lambda^\Lambda$, the center of Λ.

THEOREM 1. *Let Λ be a ring with the center C, Γ a subring of Λ. Then the following conditions are equivalent.*

(1) *Λ is separable over Γ.*

(2) *$(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ is $(\Lambda\otimes_\Gamma\Lambda)^\Gamma$-finitely generated projective and $\Lambda\cong\mathrm{Hom}_{(\Lambda\otimes_\Gamma\Lambda)^\Gamma}((\Lambda\otimes_\Gamma\Lambda)^\Lambda,\ \Lambda\otimes_\Gamma\Lambda)$ as two-sided Λ-modules.*

(3) *For any two-sided Λ-module M, $M^\Lambda\cong(\Lambda\otimes_\Gamma\Lambda)^\Lambda\otimes_{(\Lambda\otimes_\Gamma\Lambda)^\Gamma}M^\Gamma$ as C-modules.*

(4) *$C\cong(\Lambda\otimes_\Gamma\Lambda)^\Lambda\otimes_{(\Lambda\otimes_\Gamma\Lambda)^\Gamma}\Delta$ as C-modules.*

(5) *$(\Lambda\otimes_\Gamma\Lambda)^\Lambda\cdot\Delta=C$.*

(6) *There exists an element $\sum x_i\otimes y_i$ in $(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ such that $\sum x_i y_i=1$.*

PROOF. (1)⇒(2). By the definition Λ is separable over Γ means that Λ is a two-sided Λ-direct summand of $\Lambda\otimes_\Gamma\Lambda$, ${}_\Lambda\Lambda_\Lambda<\oplus\Lambda\otimes_\Gamma\Lambda$. Then by Theorem 1.2 [1], (1) implies (2). Note that, in this case, $(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ is a direct summand right ideal of $(\Lambda\otimes_\Gamma\Lambda)^\Gamma\cong\mathrm{Hom}_{\Lambda,\Lambda}(\Lambda\otimes_\Gamma\Lambda,\ \Lambda\otimes_\Gamma\Lambda)$. (2)⇒(3).

Let M be a two-sided Λ-module. Then we have following isomorphisms $(\Lambda\otimes_\Gamma\Lambda)^\Lambda\otimes_{(\Lambda\otimes_\Gamma\Lambda)^\Gamma}M^\Gamma\cong(\Lambda\otimes_\Gamma\Lambda)^\Lambda\otimes_{(\Lambda\otimes_\Gamma\Lambda)^\Gamma}\mathrm{Hom}_{\Lambda,\Lambda}(\Lambda\otimes_\Gamma\Lambda, M)\cong\mathrm{Hom}_{\Lambda,\Lambda}(\mathrm{Hom}_{(\Lambda\otimes_\Gamma\Lambda)^\Gamma}((\Lambda\otimes_\Gamma\Lambda)^\Lambda, \Lambda\otimes_\Gamma\Lambda), M)\cong\mathrm{Hom}_{\Lambda,\Lambda}(\Lambda, M)\cong M^\Lambda$. (3)$\Rightarrow$(4). Take Λ as M in (3). (4)$\Rightarrow$(5). The isomorphism in (4) is given by $(\Lambda\otimes_\Gamma\Lambda)^\Lambda\otimes_{(\Lambda\otimes_\Gamma\Lambda)^\Gamma}\Delta\ni(x\otimes y)\otimes d\to(x\otimes y)\cdot d=xdy\in C$. So $(\Lambda\otimes_\Gamma\Lambda)^\Lambda\cdot\Delta=C$. (5)$\Rightarrow$(6). From the assumption, there are elements $\alpha_i=\sum_j x_{ij}\otimes y_{ij}\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ and $d_i\in\Delta$ such that $1=\sum_i\alpha_i d_i=\sum_{ij}x_{ij}d_iy_{ij}$. Clearly $\sum_{ij}x_{ij}\otimes d_iy_{ij}$ is in $(\Lambda\otimes_\Gamma\Lambda)^\Lambda$. (6)$\Rightarrow$(1). This is clear.

Let $\mathcal{M}^\Gamma$ (resp. $\mathcal{M}^\Lambda$) be the category of two-sided Δ (resp. C)-modules M^Γ (resp. M^Λ) obtained from two-sided Λ-modules M. If Λ is an H-separable extension of Γ, then we have $M^\Gamma\cong\Delta\otimes_C M^\Lambda$ for any two-sided Λ-module M. Since H-separable means separable we have, by Theorem 1, following theorem which turns into the Morita equivalence in case of Λ is an Azumaya algebra.

THEOREM 2. *$\Delta\otimes_C$- and $(\Lambda\otimes_\Gamma\Lambda)^\Lambda\otimes_{(\Lambda\otimes_\Gamma\Lambda)^\Gamma}$- give an equivalence between $\mathcal{M}^\Lambda$ and $\mathcal{M}^\Gamma$.*

§ 2. Automorphism group

In this section we shall assume that Λ is an H-separable extension of Γ. Let σ be a ring automorphism of Λ leaving invariant the elements of Γ. Then Λ is converted to new two-sided Λ-module ${}_\sigma\Lambda$ by defining $\lambda\cdot x=\sigma(\lambda)x$ for λ and x in Λ. Put $J_\sigma=({}_\sigma\Lambda)^\Lambda$. Then, since $({}_\sigma\Lambda)^\Gamma=\Delta$ and Λ is an H-separable extension of Γ, $\Delta\cong\Delta\otimes_C J_\sigma\cong J_\sigma\otimes_C\Delta$. As C is a direct summand of Δ and Δ is C-finitely generated projective, J_σ is a C-finitely generated projective module of rank 1.

LEMMA 1. *σ is inner if and only if J_σ is C-free of rank* 1.

PROOF. If $\sigma\lambda=u_\sigma\lambda u_\sigma^{-1}$, $\lambda\in\Lambda$, for some unit u_σ (in Δ), then $u_\sigma\in J_\sigma$. For any $x\in J_\sigma$ and $\lambda\in\Lambda$, $\sigma(\lambda)xu_\sigma^{-1}=x\lambda u_\sigma^{-1}=xu_\sigma^{-1}\sigma(\lambda)$, so $xu_\sigma^{-1}\in C$. Thus $J_\sigma=Cu_\sigma$ is C-free. Conversely if $J_\sigma=Cu$ is C-free, then, since $\Delta=J_\sigma\Delta=\sigma(\Delta)J_\sigma=\Delta J_\sigma$, it follows that $\Lambda J_\sigma=\Lambda\Delta J_\sigma=\Lambda\Delta=\Lambda$ and $J_\sigma\Lambda=J_\sigma\Delta\Lambda=\Delta\Lambda=\Lambda$. So we have $\Lambda u=\Lambda=u\Lambda$, u is a unit and $\sigma(\lambda)=u\lambda u^{-1}$ for $\lambda\in\Lambda$.

Let $O(\Lambda/\Gamma)=\mathrm{Aut}(\Lambda/\Gamma)/\mathrm{Inn}(\Lambda/\Gamma)$ be the Γ-automorphism group of Λ modulo inner, and $I(C)$ the set of isomorphism classes of finitely generated projective C-modules of rank 1. $I(C)$ is an abelian group by the multiplication $\otimes_C$ with the identity $[C]$, the class of C. Now we shall prove that the map $\alpha:\sigma\to[J_\sigma]$ is a group homomorphism of $\mathrm{Aut}(\Lambda/\Gamma)$ to $I(C)$. Since Λ is H-separable over Γ, there exist elements $\sum_j x_{ij}\otimes y_{ij}\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ and $d_i\in\Delta$ such that $\sum_{ij}x_{ij}\otimes y_{ij}d_i=1\otimes1$. (cf. [2], p. 296). Bythe map f: $d\otimes x\to$

dx, $J_\sigma \otimes_C \Lambda$ is isomorphic to ${}_\sigma\Lambda$. The inverse map is given by $g: x \to \sum_{ij} \sigma(x_{ij}) y_{ij} \otimes d_i x = \sum_{ij} \sigma(x_{ij}) y_{ij} \otimes \sigma^{-1}(x) d_i$, since, for $d \in J_\sigma$ and $x \in \Lambda$, $g \circ f(d \otimes x) = \sum_{ij} \sigma(x_{ij}) y_{ij} \otimes d_i dx = \sum_{ij\alpha\beta} \sigma(x_{ij}) y_{ij} \otimes x_{\alpha\beta} d_i dy_{\alpha\beta} d_\alpha x = \sum \sigma(x_{ij}) y_{ij} x_{\alpha\beta} d_i dy_{\alpha\beta} \otimes d_\alpha x = \sum \sigma(x_{\alpha\beta}) \sigma(x_{ij}) y_{ij} d_i dy_{\alpha\beta} \otimes d_\alpha x = \sum_{\alpha\beta} \sigma(x_{\alpha\beta}) dy_{\alpha\beta} \otimes d_\alpha x = \sum dx_{\alpha\beta} y_{\alpha\beta} \otimes d_\alpha x = d \otimes \sum x_{\alpha\beta} y_{\alpha\beta} d_\alpha x = d \otimes x$. Thus $J_\sigma \otimes_C \Lambda \cong {}_\sigma\Lambda$ and $J_\sigma = ({}_\sigma\Lambda)^\Lambda \cong (J_\sigma \otimes_C \Lambda)^\Lambda$. Similarly we can prove that, for Γ-automorphisms σ and τ of Λ, $J_\sigma \otimes_C J_\tau \otimes_C \Lambda \cong {}_{\sigma\tau}\Lambda$ by the map $d_1 \otimes d_2 \otimes x \to d_1 d_2 x$. Therefore we have $J_{\sigma\tau} = ({}_{\sigma\tau}\Lambda)^\Lambda \cong (J_\sigma \otimes_C J_\tau \otimes_C \Lambda)^\Lambda \cong J_\sigma \otimes_C J_\tau$ and $\alpha: \sigma \to [J_\sigma]$ is a homomorphism of $\mathrm{Aut}(\Lambda/\Gamma)$ to $I(C)$. By Lemma 1, $\mathrm{Ker}\,\alpha = \mathrm{Inn}(\Lambda/\Gamma)$.

Let P be a two-sided Λ-module satisfying the following conditions:

(1) ${}_\Lambda P_\Lambda < \oplus \Lambda \oplus \cdots \oplus \Lambda$

(2) $P^\Gamma_{\mathfrak{m}} \cong \Delta_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of C.

Since $P^\Lambda \cong \mathrm{Hom}_{\Lambda,\Lambda}(\Lambda, P) < \oplus \sum \mathrm{Hom}_{\Lambda,\Lambda}(\Lambda, \Lambda) = \sum C$, P^Λ is a finitely generated projective C-module. Furthermore, $P^\Gamma \cong \mathrm{Hom}_{\Lambda,\Lambda}(\Lambda \otimes_\Gamma \Lambda, P) < \oplus \sum \mathrm{Hom}_{\Lambda,\Lambda}(\Lambda \otimes_\Gamma \Lambda, \Lambda) \cong \sum \Delta$, P^Γ is Δ-(and so C-) finitely generated and projective. As $P^\Gamma \cong \Delta \otimes_C P^\Lambda$, P^Λ is a C-finitely generated projective module of rank 1, by the condition (2). We have also from (1) $P \cong \Lambda \otimes_C \mathrm{Hom}_{\Lambda,\Lambda}(\Lambda, P) \cong \Lambda \otimes_C P^\Lambda$ by Theorem 1. 2 [1].

Let $I(\Lambda)$ be the set of left Γ- and right Λ-isomorphism classes of two-sided Λ-modules P with the properties (1) and (2) above. $I(\Lambda)$ becomes a group by the multiplication $\otimes_\Lambda$. As $P \cong \Lambda \otimes_C P^\Lambda$ with rank 1 projective C-module P^Λ and $P_1 \otimes_\Lambda P_2 \cong \Lambda \otimes_C P_1^\Lambda \otimes_C P_2^\Lambda$ for two-sided Λ-modules P_1 and P_2 satisfying (1) and (2), $I(\Lambda)$ is a homomorphic image of $I(C)$. Let $[J] \in I(C)$ be in the kernel of $\beta: I(C) \to I(\Lambda)$, $\beta([J]) = [J \otimes_C \Lambda]$. Then $J \otimes_C \Lambda = P$ is isomorphic to Λ as a (Γ, Λ)-module. Put $P = u\Lambda$. Then for any $\gamma \in \Gamma$, we have $\gamma u = u\gamma$, and for any $\lambda \in \Lambda$, we have $\lambda u = u\lambda'$ for some $\lambda' \in \Lambda$. It can be easily seen that the map $\sigma: \sigma(\lambda) = \lambda'$ is a Γ-endomorphism of Λ. As P is a submodule of $\Lambda \oplus \cdots \oplus \Lambda$, $cu = uc$ for all $c \in C$. Therefore σ fixes the elements of C. By Theorem 2 in [2], σ is a Γ-automorphism of Λ and $P \cong {}_\sigma\Lambda$. Thus we have proved the following theorem.

THEOREM 3. *If Λ is an H-separable extension of Γ, then*

$$1 \longrightarrow O(\Lambda/\Gamma) \longrightarrow I(C) \longrightarrow I(\Lambda) \longrightarrow 1$$

is exact.

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