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An example of a certain Kaehlerian manifold

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Kubo [5] proved that a real $n(\geqq 4)$-dimensional Kaehlerian manifold with constant scalar curvature and vanishing Bochner curvature tensor is a space of constant holomorphic sectional curvature if a certain inequality for the Ricci tensor and the scalar curvature holds. In connection with this, Hasegawa and Nakane [3] remarked that a real 4-dimensional Kaehlerian manifold with non-zero constant scalar curvature and vanishing Bochner curvature tensor is of constant holomorphic sectional curvature. Then, it is natural to ask whether a real 4-dimensional Kaehlerian manifold with zero scalar curvature and vanishing Bochner curvature tensor is locally flat. The answer is negative. The purpose of the present paper is to give a counter example to the above question.

Correspondingly, we also give an example of a 5-dimensional Sasakian manifold with constant scalar curvature -4 and vanishing contact Bochner curvature tensor which is not of constant ϕ-holomorphic sectional curvature -3. The theorems corresponding to the above of Kubo and Hasegawa and Nakane in Sasakian manifolds have been obtained in [3].

We give preliminaries in § 1 and examples described above in §§ 2 and 3, respectively.

§ **1. Preliminaries.** In this section, we recall some well-known facts for later use.

Let M be a Riemannian manifold. A set (P, Q) of two linear transformation fields P and Q on M is called an almost product structure on M if P and Q satisfy

$$P^2=P\,,\ Q^2=Q\,,\ PQ=QP=0 \quad\text{and}\quad P+Q=I\,,$$

where I and 0 denote the identity and zero transformation fields on M, respectively.

LEMMA 1 ([8]). *A Riemannian manifold M with an almost product structure (P, Q) is locally Riemannian product of two integral manifolds of two distributions determined by P and Q if and only if*

$$\nabla(P-Q)=0\,,$$

where ∇ denotes the Riemannian connection.

We denote by $H(X, Y)$ the sectional curvature for the 2-plane spanned by two mutually orthogonal unit vectors X and Y in the Riemannian manifold M. In the rest of this section, we only consider a Kaehlerian manifold M.

LEMMA 2 ([1]). *In M, the Bochner curvature tensor vanishes if and only if there exists a hybrid quadratic form L such that*

$$H(X, FX) = -8L(X, X),$$

for any unit vector X, where F is the complex structure on M.

An orthonormal basis $\{e_i, e_{i^*}=Fe_i\}$ $\left(i=1, 2, \cdots, \frac{1}{2}\dim M;\ i^*=\frac{1}{2}\dim M+i\right)$ is called an F-basis.

LEMMA 3 ([4]). *In M of real dimension $\geqq 4$, if the Bochner curvature tensor vanishes, then we obtain*

$$H(e_i, e_{i^*})+H(e_j, e_{j^*}) = +8\, H(e_i, e_j),\ (i\neq j),$$

for every F-basis $\{e_i, e_{i^*}\}$ $(i, j=1, 2, \cdots, \frac{n}{2};\ i^*=\frac{n}{2}+i,\ j^*=\frac{n}{2}+j)$.

LEMMA 4 ([6]). *In M with constant scalar curvature, if the Bochner curvature tensor vanishes, then the Ricci tensor is parallel.*

Note that, in this case, each eigenvalue of the Ricci tensor is locally constant.

§ **2. A counter example in a Kaehlerian case.** (a) Let $M(F, g)$ be a real 4-dimensional Kaehlerian manifold with zero scalar curvature and vanishing Bochner curvature tensor. $\{e_1, e_2, e_{1^*}=Fe_1, e_{2^*}=Fe_2\}$ being an F-basis of eigenvectors of the Ricci tensor, we have

$$H(e_1, e_{1^*})+H(e_2, e_{2^*}) = 8H(e_1, e_2),$$

by Lemma 3, and

$$H(e_1, e_2) = H(e_1, e_{2^*}) = H(e_{1^*}, e_2) = H(e_{1^*}, e_{2^*}),$$

where H is the sectional curvature. Then, the Ricci tensor R is given by

$$R(e_1, e_1) = R(e_{1^*}, e_{1^*}) = 10H(e_1, e_2) - H(e_2, e_{2^*}),$$
$$R(e_2, e_2) = R(e_{2^*}, e_{2^*}) = 2H(e_1, e_2) + H(e_2, e_{2^*}),$$

the other components being zero, and the scalar curvature trace R is given by

$$0 = \text{trace}\, R = R(e_1, e_1)+R(e_{1^*}, e_{1^*})+R(e_2, e_2)+R(e_{2^*}, e_{2^*}) = 24H(e_1, e_2).$$

Therefore, we have

$$H(e_1, e_{1^*})+H(e_2, e_{2^*}) = 0.$$

We may put $c=H(e_1, e_{1^*})\geqq 0$. Then, we have

$$R(e_1, e_1)=R(e_{1^*}, e_{1^*})=c,\ R(e_2, e_2)=R(e_{2^*}, e_{2^*})=-c,$$

that is, c and $-c$ are eigenvalues corresponding to the eigenvectors e_1, e_{1^*} and e_2, e_{2^*}, respectively. Hence, c is constant.

We assume that M is not locally flat, so $c>0$. If we put

$$P=\frac{1}{2}\left(\frac{1}{c}S+I\right), \quad Q=\frac{1}{2}\left(-\frac{1}{c}S+I\right),$$

where S denotes the Ricci transformation, while I is the identity transformation, then the set (P, Q) is an almost product structure on M, and P and Q are the projectors on the eigenspaces of R corresponding to c and $-c$, respectively. Therefore, by Lemma 1, M is locally the Riemannian product of $M(c)$ and $M(-c)$ which are 2-dimensional integral manifolds of the distributions of eigenspaces of R corresponding to c and $-c$, respectively, since we have

$$\nabla(P-Q)=0,$$

∇ being the Riemannian connection of g. Both $M(c)$ and $M(-c)$ admit Kaehlerian structures (F_1, g_1) and (F_2, g_2) induced from (F, g) on M and are of constant curvature c and $-c$, respectively. If (x^1, x^2) (resp. (y^1, y^2)) is a local coordinates in $M(c)$ (resp. in $M(-c)$), then we have

$$(\partial/\partial x^i)F_2=0, \qquad (\partial/\partial y^i)F_1=0 (i=1, 2),$$

since $\nabla F=0$.

Conversely, given real 2-dimensional Kaehlerian manifolds $M(c)$ and $M(-c)$ of constant curvature c and $-c$, respectively, for a positive constant c, the Riemannian product $M(c)\times M(-c)$ has the naturally defined Kaehlerian structure (F, g). Then, setting

$$L\big((X_1, Y_1), (X_2, Y_2)\big)=\frac{c}{8}\big(g_2(Y_1, Y_2)-g_1(X_1, X_2)\big),$$

for any vectors X_1, X_2 tangent to $M(c)$ and Y_1, Y_2 tangent to $M(-c)$, where g_1 and g_2 are Kaehlerian metrics of $M(c)$ and $M(-c)$, respectively, L is a hybrid quadratic form on $M(c)\times M(-c)$ and we have

$$H(X, FX)=-8L(X, X),$$

for any unit vector X tangent to $M(c)\times M(-c)$, where H is the sectional curvature for g. Therefore, by Lemma 2, we see that $M(c)\times M(-c)$ has zero scalar curvature and vanishing Bochner curvature tensor. It is easy to verify that g is not locally flat.

Thus, by giving real 2-dimensional Kaehlerian manifolds $M(c)$ and $M(-c)$ of constant curvature c and $-c$, respectively, for any positive constant c, we can obtain a real 4-dimensional Kaehlerian manifold with zero scalar curvature and vanishing Bochner curvature tensor which is not locally flat.

(b) Let M be a real 2-dimensional Kaehlerian manifold. Then we can take a coordinate neighborhood $\{U; (x^1. x^2)\}$ in which the complex structure F of M has the following numeral components

$$F=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then, the Kaehlerian metric g of M is given by

$$g=e^{2p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

for a function p in M, because

$$g(FX, FY)=g(X, Y),$$

for any vectors X and Y in M, that is, g is conformal to a locally flat metric. Hence, with respect to the local coordinates (x^1, x^2), we have

$$K_{kjih}=e^{2p}(-\delta_{kh}C_{ji}+\delta_{jh}C_{ki}-C_{kh}\delta_{ji}+C_{jh}\delta_{ki}), \quad (h, i, j, k=1, 2),$$

where K_{kjih} is the covariant components of the curvature tensor of g and

$$C_{ji}=\partial_j p_i-p_j p_i+1/2\cdot\left((p_1)^2+(p_2)^2\right)\delta_{ji}, \quad p_i=\partial_i p, \quad \partial_i=\partial/\partial x^i.$$

We assume that g is of constant curvature c, c being arbitrarily given constant. Then, we have

$$ce^{4p}=K_{1221}=e^{2p}(-C_{22}-C_{11}),$$

that is,

$$(*) \quad \partial_1 p_1+\partial_2 p_2=-ce^{2p}.$$

Conversely, for a differentiable solution p of the partial differential equation $(*)$, defining

$$F=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g=e^{2p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

on a connected definition domain in (x^1, x^2)-plane, we have a real 2-dimensional Kaehlerian manifold of constant curvature c.

Hence, we need only give a solution of the partial differential equation

$(*)$, which is, for example, given by

$$p=\begin{cases}\frac{1}{2}\sqrt{c}\cdot x^1-\log(1+e^{\sqrt{c}\cdot x^1}), & \textit{for } c>0,\\ \frac{1}{2}\sqrt{-c}\cdot x^1-\log(1-e^{\sqrt{-c}\cdot x^1}),\ (x^1<0), & \textit{for } c<0.\end{cases}$$

(c) Thus, we obtain an example of real 4-dimensional Kaehlerian mainfold $M(F, g)$ with zero scalar curvature and vanishing Bochner curvature tensor which is not locally flat; R being a 1-dimensional manifold consisting of all real numbers, M is defined by

$$M=\left\{(x^1, x^2, x^3, x^4);\ x^1, x^2, x^3, x^4\in R,\ x^3<0\right\},$$

and the Kaehlerian structure (F, g) is given by

$$F=\begin{pmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix},\quad g=\begin{pmatrix}a & 0 & 0 & 0\\ 0 & a & 0 & 0\\ 0 & 0 & b & 0\\ 0 & 0 & 0 & b\end{pmatrix},$$

where $a=e^{2p}$, $b=e^{2q}$,

$$p=\frac{1}{2}\sqrt{c}\cdot x^1-\log(1+e^{\sqrt{c}\cdot x^1}),\quad q=\frac{1}{2}\sqrt{c}\cdot x^3-\log(1-e^{\sqrt{c}\cdot x^3}),$$

for arbitrarily given positive constant c.

§ **3. A Sasakian case.** We begin this section with the following lemmas.

LEMMA 5 ([7]). *A $(2n+1)$-dimensional $(n\geqq 1)$ Sasakian manifold $\overline{M}$ has a system of local corrdinate (x^i, s) $(i=1, 2, \cdots, 2n)$ with the following properties.*

(1) *Each $M=M(s)$ determined by fixing s is a Kaehlerian manifold which admits a 1-form v satisfying*

$$\frac{1}{2}dv(X, Y)=g(FX, Y),$$

for any vectors X and Y in M, (F, g) being the Kaehlerian structure on M. The set (F, g, v) does not depend on s.

(2) *With respect to the local coordinate (x^i, s), the Sasakian structure $(\phi, \xi, \eta, \bar{g})$ is given by*

$$\phi=\begin{pmatrix}F & 0\\ -F^*v & 0\end{pmatrix},\quad \begin{matrix}\xi=(0\ \ 1),\\ \eta=(v\ \ 1),\end{matrix}\quad \bar{g}=\begin{pmatrix}g+v\otimes v & v\\ v & 1\end{pmatrix},$$

where F^*v is a 1-form on M defined by

$$F^*v(X)=v(FX),$$

for any vector X in M.

LEMMA 6. *If a $(2n+1)$-dimensional Sasakian manifold $\bar{M}$ has the vanishing contact Bochner curvature tensor (resp. constant scalar curvature $-2n$), then the Kaehlerian manifold M appearing in Lemma 5 has the vanishing Bochner curvature tensor (resp. zero scalar curvature).*

PROOF. Refer to [2] and [7].

Let $\bar{M}$ be a 5-dimensional Sasakian manifold with constant scalar curvature -4 and vanishing contact Bochner curvature tensor which is not of constant ϕ-holomorphic sectional curvature -3. Then, $\bar{M}$ has local coordinates (x^i, s) as in Lemma 5 and M given in Lemma 5 is a real 4-dimensional Kaehlerian manifold with zero scalar curvature and vanishing Bochner curvature tensor which is not locally flat ([7]), and admits a 1-form v satisfying

$$\frac{1}{2}dv(X, Y)=g(FX, Y),$$

for any vectors X and Y in M, because of Lemma 6.

Thus we obtain an example of a 5-dimensional Sasakain manifold $\bar{M}(\phi, \xi, \eta, \bar{g})$ with constant scalar curvature -4 and vanishing contact Bochner curvature tensor which is not of constant ϕ-holomorphoc sectional curvature -3, as follows.

$$\bar{M}=\left\{(x^1, x^2, x^3, x^4, s)\,;\ x^1, x^2, x^3, x^4, s\in R,\ x^3<0\right\},$$

$$\phi=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ v_2 & 0 & v_4 & 0 & 0 \end{pmatrix},\quad \bar{g}=\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a+v_2v_2 & 0 & v_2v_4 & v_2 \\ 0 & 0 & b & 0 & 0 \\ 0 & v_2v_4 & 0 & b+v_4v_4 & v_4 \\ 0 & v_2 & 0 & v_4 & 1 \end{pmatrix}$$

$$\xi=(0, 0, 0, 0, 1),\quad \eta=(0, v_2, 0, v_4, 1),$$

where

$$v_2=-\frac{2}{\sqrt{c}\,(1+e^{\sqrt{c}x^1})},\quad v_4=\frac{2}{\sqrt{c}\,(1-e^{\sqrt{c}x^3})},$$

for arbitrarily given positive constant c, and a and b are the functions given in § 2.

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