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Hypoellipticity for a class of pseudo-differential operators

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§ 0. Introduction

In the present paper we shall consider a class of pseudo-differential operators P on a manifold X whose characteristic set Σ is the union of two closed conic submanifolds Σ_1 and Σ_2. This class is denoted by $OPL_k^{m,M_1,M_2}(X;\Sigma_1,\Sigma_2)$. Under some quasi-transversality and involutiveness, we shall give a necessary and sufficient condition for hypoellipticity of P by constructing the parametrix.

When $\Sigma_1=\Sigma_2$, our class nearly coincides with $OPL_k^{m,M_1+M_2}(X;\Sigma_1)$ introduced by Helffer [5] or Sjöstrand [10]. Moreover in the case where $M_1=2$, $k=2$ and Σ_1 is involutive, Boutet de Monvel [2] gives a necessary and sufficient condition for existence of a parametrix of P in $OPS^{-m,-2}$ (more general class than ours) which is also equivalent to the hypoellipticity for P with loss of 1-derivative. For general M_1, [5] gives a necessary and sufficient condition for hypoellipticity of P with loss of $M_1/2$-derivatives.

When Σ_1 and Σ_2 intersect transversally and Σ_i $(i=1,2)$, $\Sigma_1\cap\Sigma_2$ are involutive, Aramaki [1] constructs parametrices for the operators of a slightly different class.

The plan of this paper is as follows: In § 1, we introduce a class of pseudo-differential operators and study the symbol calculus and the associatad invariances of P using the technique developed by [5], [1]. Finally we give the main theorem (Theorem 1.10). § 2 is the preparations for the proof of our theorem. Mainly we consider the class $OPS^{m,M_1,M_2}(X;\Sigma_1,\Sigma_2)$ which is a generalization of the class $OPS^{m,M_1}(X;\Sigma_1)$ introduced by [2]. In § 3, we give the proof of the Theorem 1.10. § 4 is devoted to a study of the special case of type $P=P_1\cdot P_2+P_3$. Finally in § 5, we apply the results of § 4 to the system of the type

$$\boldsymbol{P}=\begin{pmatrix} P_1 & A \\ B & P_2 \end{pmatrix}$$

where A and B are lower order terms.

§ 1. A class of operators and the associated invariances

Let X be a paracompact C^{∞} manifold of dimension n and $T^{*}X-\{0\}$ be the cotangent bundle minus the zero section.

DEFINITION 1.1. *Let Σ_1 and Σ_2 be closed conic submanifolds of codimensions μ_1 and μ_2 in $T^{*}X-\{0\}$ respectively and let $m\in\boldsymbol{R}$, M_1, $M_2\in\boldsymbol{Z}^{+}$ (non-negative integers), $k\geq 2$ an integer. Then the space $OPL_k^{m,M_1,M_2}(X;\Sigma_1,\Sigma_2)$ is the set of all pseudo-differential operators $P\in L^{m}(X)$ (for the notation, see Hörmander* [7], [8]) *such that for every local coodinate system $V\subset X$, P has a symbol of the form*

(1.2) $p(x,\xi)\sim\sum_{j=0}^{\infty}p_{m-j/k}(x,\xi)$ *with $p_{m-j/k}(x,\xi)$ positively-homogeneous of degree $m-j/k$ with respect to ξ (j integral) and satisfy:*

(1.2) *For every $K\subset\subset V$, there exists a constant $C_K>0$ such that*

$$\frac{|p_{m-j/k}(x,\xi)|}{|\xi|^{m-j/k}}\leq C_K\sum_{\substack{k_1+k_2=j\\0\leq k_1\leq M_1\\0\leq k_2\leq M_2}}d_{\Sigma_1}(x,\xi)^{M_1-k_1}d_{\Sigma_2}(x,\xi)^{M_2-k_2},$$

$0\leq j\leq M_1+M_2$, for all $(x,\xi)\in K\times(\boldsymbol{R}^{n}-\{0\})$ and $|\xi|\geq 1$. Here $d_{\Sigma_i}(x,\xi)=\inf_{(y,\eta)\in\Sigma_i}\left(|y-x|+\left|\eta-\frac{\xi}{|\xi|}\right|\right)$ are the distances from $\left(x,\frac{\xi}{|\xi|}\right)$ to Σ_i, $i=1,2$.

We also introduce the set $OPL_{k,c}^{m,M_1,M_2}(X;\Sigma_1,\Sigma_2)\subset OPL_k^{m,M_1,M_2}(X;\Sigma_1,\Sigma_2)$ for which the $p_{m-j/k}$ in (1.1) *can be taken to be zero when j/k is not an integer.*

REMARK 1.2. $OPL_k^{m,M_1,M_2}(X;\Sigma_1,\Sigma_2)$ reduces to $OPL_k^{m,M_1}(X;\Sigma_1)$ of [5] when $M_2=0$ and to $OPL^{m,M_1}(X;\Sigma_1)$ of [10] when $M_2=0$ and $k=2$.

It is clear that if $\Sigma_1\cap\Sigma_2$ is a submanifold, we have

$$OPL_k^{m,M_1,M_2}(X;\Sigma_1,\Sigma_2)\subset OPL_k^{m,M_1+M_2}(X;\Sigma_1\cap\Sigma_2).$$

The class of symbols satisfying (1.1) and (1.2) in an open cone $U\subset T^{*}X-\{0\}$ is denoted by $L_k^{m,M_1,M_2}(U;\Sigma_1,\Sigma_2)$.

By a routine consideration (c.f. [7], [8]) we set the followings:

PROPOSITION 1.3. *Let $P_1\in OPL_k^{m_1,M_1}(X;\Sigma_1)$ and $P_2\in OPL_k^{m_2,M_2}(X;\Sigma_2)$ where one of the factors is properly supported. Then we have*

$$P_1\cdot P_2\in OPL_k^{m_1+m_2,M_1,M_2}(X;\Sigma_1,\Sigma_2).$$

PROPOSITION 1.4. $L_k^{m,M_1,M_2}(T^{*}X-\{0\};\Sigma_1,\Sigma_2)\subset L_k^{m+1/k,M_1+1,M_2}(T^{*}X-\{0\};\Sigma_1,\Sigma_2)\cap L_k^{m+1/k,M_1,M_2+1}(T^{*}X-\{0\};\Sigma_1,\Sigma_2)$.

Let $\Sigma_1\cap\Sigma_2$ be a submanifold. If q_1 and q_2 are elements in L_k^{m,M_1,M_2}

$(U;\ \Sigma_1, \Sigma_2)$ where U is a conic neighbourhood of $\rho\in\Sigma_1\cap\Sigma_2$, we define the following equivalence relation:

$q_1\equiv q_2$ in U if and only if $q_1-q_2\in L_k^{m,M_1+M_2+1}(U;\ \Sigma_1\cap\Sigma_2)$.

PROPOSITION 1.5. *Let p be a symbol in $L_k^{m,M_1,M_2}(T^*X-\{0\};\ \Sigma_1, \Sigma_2)$ and let $\rho\in\Sigma_1\cap\Sigma_2$. Then there exists a conic neighbourhood U of ρ such that $q\in L_k^{m,M_1,M_2}(U;\ \Sigma_1, \Sigma_2)/L_k^{m,M_1+M_2+1}(U;\ \Sigma_1\cap\Sigma_2)$ defined by*

$$(1.3)\qquad q\equiv\exp\left(-\frac{1}{2i}\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial\xi_j}\right)\right)\cdot p$$

$$=\sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\left(\frac{1}{2i}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\frac{\partial}{\partial\xi_j}\right)^t\cdot p$$

is invariant under a locally homogeneous canonical transformation: χ; $U\to T^*\boldsymbol{R}^n-\{0\}$.

REMARK 1.6. If $p\in L_k^{m,M_1,M_2}(U;\ \Sigma_1, \Sigma_2)$ and $k>2$, the class q given by the formula (1.3) coincides with

$$\sum_{j=0}^{M_1+M_2}p_{m-j/k}\ \text{modulo}\ L_k^{m,M_1+M_2+1}(U;\ \Sigma_1\cap\Sigma_2)\,.$$

Now let U be a conic neighbourhood of $\rho\in\Sigma_1\cap\Sigma_2$. Let

$$q=\sum_{j=0}^{M_1+M_2}q_{m-j/k}\in L_k^{m,M_1,M_2}(U;\ \Sigma_1, \Sigma_2)/L_k^{m,M_1+M_2+1}(U;\ \Sigma_1\cap\Sigma_2)$$

be a symbol associated with p in U. Define a (M_1+M_2-j)-linear form, denoted by $\tilde q_{m-j/k}(\rho)$, on $T_\rho(T^*X-\{0\})$ by: For $Y_1, Y_2, \cdots, Y_{M_1+M_2-j}\in T_\rho(T^*X-\{0\})$,

$$(1.4)\qquad \tilde q_{m-j/k}(\rho)\,(Y_1, Y_2, \cdots, Y_{M_1+M_2-j})$$

$$=\frac{1}{(M_1+M_2-j)!}\,\tilde Y_1\tilde Y_2\cdots\tilde Y_{M_1+M_2-j}q_{m-j/k})\,(\rho)$$

where $\tilde Y$ means a vector field extending Y to a neighbourhood of ρ. It is clear that $\tilde q_{m-j/k}(\rho)$ is independent of the choice of the representative of q.

DEFINITION 1.7. *For every $\rho\in\Sigma_1\cap\Sigma_2$, we define*

$$(1.5)\qquad \tilde q(\rho, Y)=\sum_{j=0}^{M_1+M_2}\tilde q_{m-j/k}(\rho)\,(Y, Y, \cdots, Y)\ \textit{for all}\ Y\in T_\rho\big(T^*X-\{0\}\big)\,.$$

If $\rho\in\Sigma_1\backslash\Sigma_2$, p belongs to $L_k^{m,M_1}(U;\ \Sigma_1)$ for some conic neighbourhood U of ρ. So if we apply Proposition 1.5 to p with $\Sigma_1=\Sigma_2$, $M_2=0$, we can also define $\tilde q(\rho, Y)$ for $\rho\in\Sigma_1\backslash\Sigma_2$. Similarly define $\tilde q(\rho, Y)$ for $\rho\in\Sigma_2\backslash\Sigma_1$. Therefore, for every $\rho\in\Sigma=\Sigma_1\cup\Sigma_2$, we can define

$$\Gamma_\rho=\left\{\tilde q(\rho, Y);\ Y\in T_\rho\big(T^*X-\{0\}\big)\right\}.$$

REMARK 1.8. When $M_2=0$, $M_1=k=2$, we have $q_m=p_m$ and $q_{m-1}=p_{m-1}-\frac{1}{2i}\sum_{l=1}^{n}\frac{\partial}{\partial x_l}\frac{\partial}{\partial \xi_l}p_m$. In this case $\tilde{q}(\rho, Y)$ is the sum of the transversal hessian of p_m and the subprincipal symbol of P at ρ.

PROPOSITION 1.9. *Let q_1 and q_2 be the symbols associated with p_1 and p_2 respectively. Then the symbol q associated with the composition of p_1 and p_2 is given by the formula:*

$$q\equiv\left(\exp\left(\frac{1}{2i}\sum_{l=1}^{n}\frac{\partial}{\partial x_l}\frac{\partial}{\partial \xi_l}\right)\cdot q_1\right)\#\left(\exp\left(\frac{1}{2i}\sum_{l=1}^{n}\frac{\partial}{\partial x_l}\frac{\partial}{\partial \xi_l}\right)\cdot q_2\right)$$

where $\#$ designs the composition of the symbols.

Next we describe the hypotheses on Σ_1 and Σ_2. Let Σ_1 and Σ_2 be closed conic submanifolds in $T^*X-\{0\}$ of codimension μ_1 and μ_2 respectively.
$(H.1)$ Σ_1 and Σ_2 intersect quasi-transversally. That is, $\Sigma_1\cap\Sigma_2$ is a closed conic submanifold such that for every point $\rho\in\Sigma_1\cap\Sigma_2$,

$$T_\rho(\Sigma_1\cap\Sigma_2)=T_\rho\Sigma_1\cap T_\rho\Sigma_2 .$$

Locally this means: If the codimension of $\Sigma_1\cap\Sigma_2$ is equal to $(\mu_1+\mu_2)-\nu_0$, there exist positively-homogeneous functions

$$u_1^{(1)},\cdots,u_{\nu_1}^{(1)},\ u_1^{(0)},\cdots,u_{\nu_0}^{(0)},\ u_1^{(2)},\cdots,u_{\nu_2}^{(2)},$$

$du_j^{(i)}$ $(j=1,2,\cdots,\nu_i,\ i=1,0,2)$ being linearly independent such that

$$\Sigma_1 \text{ is defined by } u_1^{(1)}=\cdots=u_{\nu_1}^{(1)}=u_1^{(0)}=\cdots=u_{\nu_0}^{(0)}=0,$$

$$\Sigma_2 \qquad \text{by } u_1^{(0)}=\cdots=u_{\nu_0}^{(0)}=u_1^{(2)}=\cdots=u_{\nu_2}^{(2)}=0,$$

and

$$\Sigma_1\cap\Sigma_2 \qquad \text{by } u_1^{(1)}=\cdots=u_{\nu_1}^{(1)}=u_1^{(0)}=\cdots=u_{\nu_0}^{(0)}=u_1^{(2)}=\cdots=u_{\nu_2}^{(2)}=0 .$$

Here $\nu_1=\mu_1-\nu_0$, $\nu_2=\mu_2-\nu_0$ (≥ 0).
$(H.2)$ Σ_1, Σ_2 and $\Sigma_1\cap\Sigma_2$ are involutive, *i.e.* if $u_1^{(1)},\cdots,u_{\nu_1}^{(1)},u_1^{(0)},\cdots,u_{\nu_0}^{(0)},u_1^{(2)},\cdots,u_{\nu_2}^{(2)}$ are as above, then

$$\{u_j^{(i)},u_{j'}^{(i)}\}=\{u_k^{(0)},u_{k'}^{(0)}\}=\{u_j^{(i)},u_k^{(0)}\}=0 \qquad \text{on } \Sigma_i\ (i=1,2)$$

and

$$\{u_j^{(1)},u_l^{(2)}\}=0 \qquad \text{on } \Sigma_1\cap\Sigma_2 .$$

$(H.3)$ The radial vector $\sum_{l=1}^{n}\xi_l\frac{\partial}{\partial \xi_l}$ is linearly independent of $H_{u_j^{(i)}}$, $j=1,\cdots,\nu_i$, $i=1,0,2$. Here we denote by H_f the Hamilton vector field and by $\{f,g\}$ their Poisson bracket for C^∞ functions f, g on $T^*X-\{0\}$.

Then we obtain the following:

THEOREM 1.10. *Let P be in $OPL_k^{m,M_1,M_2}(X;\ \Sigma_1, \Sigma_2)$ and be elliptic outside $\Sigma=\Sigma_1\cap\Sigma_2$. Assume that (H. 1), (H. 2) and (H. 3) are satisfied. Then P is hypoelliptic at $\rho\in\Sigma_1\cap\Sigma_2$ with loss of $(M_1+M_2)/k$-derivatives if and only if*

(1.6) $\qquad \Gamma_\rho$ *does not meet the origin.*

Here we say that P is hypoelliptic at ρ with loss of $(M_1+M_2)/k$-derivatives if $u\in\mathscr{D}'(X)$ and $Pu\in H^s$ at ρ implies $u\in H^{s+m-(M_1+M_2)/k}$ at ρ.

We also obtain a sufficient condition for the usual hypoellipticity:

COROLLARY 1.11. *Let P be in $OPL_k^{m,M_1,M_2}(X;\ \Sigma_1, \Sigma_2)$ and be elliptic outside $\Sigma=\Sigma_1\cup\Sigma_2$. Then P is hypoelliptic with loss of $(M_1+M_2)/k$-derivatives, if for every $\rho\in\Sigma=\Sigma_1\cup\Sigma_2$, Γ_ρ does not meet the origin and moreover:*

(A) *When $k>2$, (H. 1) and (H. 3) are satisfied (note that (H. 2) is unnecessary).*

(B) *When $k=2$, (H. 1), (H. 2) and (H. 3) are satisfied.*

Here we say that P is hypoelliptic with loss of $(M_1+M_2)/k$-derivatives if for all open set O in X, $u\in\mathscr{D}'(X)$ and $Pu\in H^s_{loc}(\mathrm{O})$ implies $u\in H^{s+m-(M_1+M_2)/k}_{loc}(\mathrm{O})$.

Finally we give a simple example:

EXAMPLE 1.12. Let $P=D_{x_1}^2D_{x_2}^2+aD_{x_1}^2D_{x_3}+2bD_{x_1}D_{x_2}D_{x_3}+cD_{x_2}^2D_{x_3}+d(D_{x_1}^2+D_{x_2}^2+D_{x_3}^2)$ in $\boldsymbol{R}^3$ where a, b, c and d are complex numbers. Let $\Delta_d=\{\lambda d+\mu;\ \lambda\geq 1,\ \mu\geq 0\}$ and Δ be the set of values of the quadratic form corresponding to the symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Then P is hypoelliptic at $\rho=(x_1^0, x_2^0, x_3^0, 0, 0, \xi_3^0)$ with loss of 2-derivatives if and only if $(\mathrm{sgn}\,(\xi_3^0))\cdot\Delta+\Delta_d$ does not meet the origin.

In fact, if we set $\Sigma_1=\{\xi_1=0\}$ and $\Sigma_2=\{\xi_2=0\}$, $P\in OPL_{2,c}^{4,2,2}(\boldsymbol{R}^3;\ \Sigma_1, \Sigma_2)$. Then

$$\Gamma_\rho=\left\{|\xi_3^0|^2\eta_1^2\eta_2^2+|\xi_3^0|\xi_3^0(a\eta_1^2+2b\eta_1\eta_2+c\eta_2^2)+d\left(|\xi_3^0|^2(\eta_1^2+\eta_2^2)+(\xi_3^0)^2\right);\ (\eta_1, \eta_2)\in\boldsymbol{R}^2\right\},$$

therefore Theorem 1.10 leads to the conclusion.

§ 2. The preparations for the proof of Theorem 1.10 and Corollary 1.11

In this section we introduce the class of operators in which we construct the parametrix of $P\in OPL_k^{m,M_1,M_2}(X;\ \Sigma_1, \Sigma_2)$. (c.f. Helffer [6])

Let U be an open cone in $T^*X-\{0\}=S^*X\times\boldsymbol{R}^+$ where S^*X is the cosphere bundle of X. We denote by $u=(u^{(1)}, u^{(0)}, u^{(2)}, v, r)$ the variables in U. Let Σ_1 and Σ_2 be the subcones defined by

$$\Sigma_1=\{u^{(1)}=u^{(0)}=0\},\ \Sigma_2=\{u^{(0)}=u^{(2)}=0\}$$

where

$$u^{(i)}=(u_1^{(i)}, \cdots, u_{\nu_i}^{(i)}) \qquad (i=1, 0, 2)$$

$$v=(v_1, \cdots, v_{(2n-1)-(\nu_1+\nu_0+\nu_2)})$$

and $u_j^{(i)}$ $(j=1, \cdots, \nu_i,\ i=1, 0, 2)$, v_l $(l=1, \cdots, (2n-1)-(\nu_1+\nu_0+\nu_2))$ are functions of positively-homogeneous of degree 0. We set

$$\rho_{\Sigma_i}=\left\{\sum_{j=1}^{\nu_i}|u^{(i)}|^2+\sum_{j=1}^{\nu_0}|u^{(0)}|^2+r^{-2/k}\right\}^{1/2}, \qquad (i=1, 2)\,.$$

Definition 2.1. *Let m, M_1, $M_2\in\boldsymbol{R}$. Then we denote by $S^{m,M_1,M_2}(U;\ \Sigma_1, \Sigma_2)$ the set of all C^∞ functions $a(u)$ on U such that for any $j\in\boldsymbol{Z}_+$ and any multi-indeces $\alpha_1\in(\boldsymbol{Z}_+)^{\nu_1}$, $\alpha_0\in(\boldsymbol{Z}_+)^{\nu_0}$, $\alpha_2\in(\boldsymbol{Z}_+)^{\nu_2}$, $\beta\in(\boldsymbol{Z}_+)^{(2n-1)-(\nu_1+\nu_0+\nu_2)}$, we have*

$$\left|\left(\frac{\partial}{\partial u^{(1)}}\right)^{\alpha_1}\left(\frac{\partial}{\partial u^{(0)}}\right)^{\alpha_0}\left(\frac{\partial}{\partial u^{(2)}}\right)^{\alpha_2}\left(\frac{\partial}{\partial v}\right)^{\beta}\left(\frac{\partial}{\partial r}\right)^{j}a\right|$$
$$\lesssim r^{m-j}\sum_{k_1+k_2=|\alpha_0|}\rho_{\Sigma_1}{}^{M_1-|\alpha_1|-k_1}\rho_{\Sigma_2}{}^{M_2-|\alpha_2|-k_2}\,.$$

Here we use the notation $f\lesssim g$ if for any subcone $U'\subset U$ with compact basis and any $\varepsilon>0$, there exists a constant $C>0$ such that $0\leq f\leq Cg$ in U' when $r>\varepsilon$.

Remark 2.2. (1) Note that we can also express the above definition in the invariant fashion. (c.f. [2], [6])
(2) $S^{m,M_1,M_2}(T^*X-\{0\}\ ;\ \Sigma_1, \Sigma_2)\subset S_{\rho,\delta}^{m-((M_1)_-+(M_2)_-)/k}$ with $\rho=1-1/k$ and $\delta=1/k$ where $(s)_-=\inf(0, s)$ for real s. In fact, since $\rho_{\Sigma_i}{}^{-1}\leq r^{1/k}$, the right hand side in the definition is estimated by

$$r^{m-j}\rho_{\Sigma_1}{}^{M_1}\rho_{\Sigma_2}{}^{M_2}r^{(|\alpha_1|+|\alpha_0|+|\alpha_2|)/k}\,.$$

Here by definition of ρ_{Σ_i}, we have $\rho_{\Sigma_i}{}^{M_i}\lesssim r^{-(M_i)_-/k}$.

Note that if $k>2$, we have $\delta<\rho$ and if $k=2$, $\rho=\delta=1/2$.

The following three propositions follow from a routine consideration. (c.f. [2])

Proposition 2.3. *For M_1, $M_2\geq 0$ integers, we have*

$$L_k^{m,M_1,M_2}(U;\ \Sigma_1, \Sigma_2)\subset S^{m,M_1,M_2}(U;\ \Sigma_1, \Sigma_2)\,.$$

Proposition 2.4. *If $p_1\in S^{m,M_1,M_2}(U;\ \Sigma_1, \Sigma_2)$ and $p_2\in S^{m',M_1',M_2'}(U;\ \Sigma_1, \Sigma_2)$,* *then* $p_1 \cdot p_2 \in S^{m+m', M_1+M_1', M_2+M_2'}(U;\ \Sigma_1, \Sigma_2)$.

PROPOSITION 2.5. *If* $p \in S^{m, M_1, M_2}(U; \Sigma_1, \Sigma_2)$ *and satisfies*

$$|p| \gtrsim r^m \rho_{\Sigma_1}{}^{M_1} \rho_{\Sigma_2}{}^{M_2},$$

then

$$p^{-1} \in S^{-m, -M_1, -M_2}(U;\ \Sigma_1, \Sigma_2).$$

§ 3. Proofs of Theorem 1.10 and Corollary 1.11

(1) *Sufficiency of Theorem* 1.10 *and Corollary* 1.11

(*A*) *The case* $k>2$. Let $\rho \in \Sigma_1 \cap \Sigma_2$ and U be a conic neighbourhood of ρ. By Remark 1.6, the class q defined by Proposition 1.5 has the following form:

$$q \sim \sum_{j=0}^{M_1+M_2} p_{m-j/k}(x, \xi).$$

Therefore by definition 1.7,

$$\tilde{q}(\rho, Y) = \sum_{j=0}^{M_1+M_2} \tilde{p}_{m-j/k}(\rho)(Y, \cdots, Y), \quad Y \in T_\rho\left(T^* X - \{0\}\right)$$

where the (M_1+M_2-j)-linear form $\tilde{p}_{m-j/k}(\rho)$ on $T_\rho(T^* X - \{0\})$ is defined in the same way as (1.4). Then our hypothesis (1.6) implies

$$p'(x, \xi) = \sum_{j=0}^{M_1+M_2} p_{m-j/k}(x, \xi) \neq 0 \qquad \text{at } \rho \in \Sigma_1 \cap \Sigma_2. \tag{3.1}$$

Thus for every $\rho \in \Sigma_1 \cap \Sigma_2$, there exists a conic neighbourhood U of ρ and constants C_1 and C_2 such that

$$\left|p(x, \xi)\right| \geq C_1 |\xi|^{m-(M_1+M_2)/k} \qquad \text{for all } (x, \xi) \in U \text{ and } |\xi| \geq C_2. \tag{3.2}$$

By Taylor's formula, in a conic neighbourhood of ρ, we can write:

$$p' = \sum_{j=0}^{M_1+M_2} \sum_{(\alpha_1, \alpha_0, \alpha_2)} a_{\alpha_1, \alpha_0, \alpha_2, j} (u^{(1)})^{\alpha_1} (u^{(0)})^{\alpha_0} (u^{(2)})^{\alpha_2}$$

where $(\alpha_1, \alpha_0, \alpha_2)$ in the summation range all multi-indices such that $|\alpha_1| + |\alpha_0| + |\alpha_2| = M_1 + M_2 - j$, $|\alpha_1| \leq M_1$, $|\alpha_2| \leq M_2$. Moreover $u^{(i)}$ $(i=1, 0, 2)$ are functions of positively-homogeneous of degree 0 defining Σ_i in $(H.1)$ and $a_{\alpha_1, \alpha_0, \alpha_2, j}$ of degree $m-j/k$. Since

$$\rho_{\Sigma_1 \cap \Sigma_2} = \left\{|u^{(1)}|^2 + |u^{(0)}|^2 + |u^{(2)}|^2 + r^{-2/k}\right\}^{1/2},$$

it is clear that if we assign to r the weight 1, to $(u^{(1)}, u^{(0)}, u^{(2)})$ the weight

$-1/k$, to v the weight 0, then p' and $r^m \rho_{\Sigma_1 \cap \Sigma_2}{}^{M_1+M_2}$ have the same degree $m-(M_1+M_2)/k$ of quasi-homogeneity. Thus by (3. 1) and Proposition 2. 5, we have

$$p'^{-1} \in S^{-m,-(M_1+M_2)}(U''\,;\ \Sigma_1 \cap \Sigma_2) \qquad \text{for some } U'' \subset U'.$$

Since $p^{(\alpha)}_{(\beta)} \in S^{m-|\alpha|, M_1+M_2-(|\alpha|+|\beta|)}(U''\,;\ \Sigma_1 \cap \Sigma_2)$, we see

$$p^{(\alpha)}_{(\beta)} \cdot p^{-1} \in S^{-|\alpha|,-(|\alpha|+|\beta|)}(U''\,;\ \Sigma_1 \cap \Sigma_2).$$

Therefore with some constants C_3, $C_4 > 0$,

$$\left|p^{(\alpha)}_{(\beta)}(x, \xi)\right| \leq C_3 |\xi|^{-(1-1/k)|\alpha|+(1/k)|\beta|} \left|p(x, \xi)\right| \tag{3. 3}$$

for all $(x, \xi) \in U''$ and $|\xi| \geq C_4$. Thus (3. 2) and (3. 3) show that Hörmander's condition [7 ; Theorem 4. 2] is satisfied, so P is hypoelliptic at $\rho \in \Sigma_1 \cap \Sigma_2$ with loss of $(M_1+M_2)/k$-derivatives. If $\rho \in \Sigma_1 \backslash \Sigma_2$, $L_k^{m,M_1,M_2}(U;\ \Sigma_1, \Sigma_2) = L_k^{m,M_1}(U;\ \Sigma_1)$ for some conic neighbourhood of ρ. So we can apply the above arguments with $\Sigma_1 = \Sigma_2$, $M_2 = 0$. It is similar to the case $\rho \in \Sigma_2 \backslash \Sigma_1$. Thus we complete the proof.

(B) *The case* $k=2$.

LEMMA 3. 1. *Assume that the closed conic submanifolds* Σ_1 *and* Σ_2 *satisfy* (H. 1), (H. 2) *and* (H. 3). *Then for every* $\rho \in \Sigma_1 \cap \Sigma_2$, *there exists a conic neighbourhood* U *of* ρ *and a homogeneous canonical transformation* $\chi:\ U \to T^* \boldsymbol{R}^n - \{0\}$ *such that*

$$\chi(\Sigma_i) = \{\xi^{(i)}_1 = \cdots = \xi^{(i)}_{\nu_i} = \xi^{(0)}_1 = \cdots = \xi^{(0)}_{\nu_0} = 0\}, \qquad i = 1, 2$$

where

$$(x, \xi) = (x^{(1)}, x^{(0)}, x^{(2)}, x', \xi^{(1)}, \xi^{(0)}, \xi^{(2)}, \xi') \in T^* \boldsymbol{R}^n - \{0\}$$

and

$$\begin{aligned} x^{(1)} &= (x_1, \cdots, x_{\nu_1}) & \xi^{(1)} &= (\xi_1, \cdots, \xi_{\nu_1}) \\ x^{(0)} &= (x_{\nu_1+1}, \cdots, x_{\nu_1+\nu_0}) & \xi^{(0)} &= \xi_{\nu_1+1}, \cdots, \xi_{\nu_1+\nu_0}) \\ x^{(2)} &= (x_{\nu_1+\nu_0+1}, \cdots, x_{\nu_1+\nu_0+\nu_2}) & \xi^{(2)} &= (\xi_{\nu_1+\nu_0+1}, \cdots, \xi_{\nu_1+\nu_0+\nu_2}). \end{aligned} \tag{3. 4}$$

PROOF. With the notations in (H. 1) if we define locally

$$\Sigma_{10} = \{u^{(1)}_1 = \quad = u^{(1)}_{\nu_1} = 0\}$$

$$\Sigma_{00} = \{u^{(0)}_1 = \quad = u^{(0)}_{\nu_0} = 0\}$$

$$\Sigma_{20} = \{u^{(2)}_1 = \quad = u^{(2)}_{\nu_2} = 0\},$$

it is easy to see that they intersect transversally and

$$\Sigma_1 = \Sigma_{10} \cap \Sigma_{00} \qquad \Sigma_2 = \Sigma_{20} \cap \Sigma_{00} .$$

Therefore under the hypotheses $(H. 1)$, $(H. 2)$ and $(H. 3)$, there exists locally a canonical transformation from some conic neighbourhood U of ρ into $T^*\boldsymbol{R}^n - \{0\}$ such that

$$\chi(\Sigma_{i0}) = \{\xi_1^{(i)} = \cdots = \xi_{\nu_i}^{(i)} = 0\} , \qquad i = 1, 0, 2 .$$

(c. f. [1], Grigis and Lascar [4], Duistermaat and Hörmander [3]) This completes the proof.

Since the hypotheses and the conclusions of Theorem 1. 10 and Corollary 1. 11 are invariant under the above canonical transformation, we are reduced to the case: $X = \boldsymbol{R}^n$ and

$$\Sigma_i = \{\xi_1^{(i)} = \cdots = \xi_{\nu_i}^{(i)} = \xi_1^{(0)} = \cdots = \xi_{\nu_0}^{(0)} = 0\} , \qquad i = 1, 2$$

with the notations in (3. 4). Then in a conic neighbourhood of $\rho \in \Sigma_1 \cap \Sigma_2$, we have

$$(3.5) \qquad P = \sum_{j=0}^{M_1+M_2} \sum_{(\alpha_1, \alpha_0, \alpha_2)} A_{\alpha_1, \alpha_0, \alpha_2, j} (D_{x^{(1)}})^{\alpha_1} (D_{x^{(0)}})^{\alpha_0} (D_{x^{(2)}})^{\alpha_2}$$

where $A_{\alpha_1, \alpha_0, \alpha_2, j}$ are classical pseudo-differential operators of order $m - (M_1 + M_2) + j/2$. Then the hypothesis (1. 6) implies

$$p'(x, \xi) = \sum_{j=0}^{M_1+M_2} p_{m-j/k}(x, \xi) \neq 0 \qquad \text{at } \rho ,$$

because the $\dfrac{\partial}{\partial x_j}$ are all tangent to $\Sigma_1 \cap \Sigma_2$. Therefore we have

$$q'(x, \xi) = p'^{-1}(x, \xi) \in S^{-m, -(M_1+M_2)}(U; \Sigma_1 \cap \Sigma_2)$$

similarly to the case (A). If we set $Q' = q'(x, D)$, the symbol of $Q' \cdot P$ is asymptotically equal to

$$1 + \sum_{|\alpha| \geq 1} \frac{1}{\alpha !} q'^{(\alpha)} D_x^\alpha p .$$

Again since the $\dfrac{\partial}{\partial x_j}$ are all tangent to $\Sigma_1 \cap \Sigma_2$, the second term in the right hand side belongs to $S^{-1/2, 0}(U; \Sigma_1 \cap \Sigma_2)$. Thus we have

$$Q' \cdot P = I - R' \qquad \text{with} \quad R' \in OPS^{-1/2, 0}(\boldsymbol{R}^n ; \Sigma_1 \cap \Sigma_2) .$$

Finally if we set $Q \sim \sum_{k=0}^{\infty} (R')^k \cdot Q$, then $Q \cdot P \sim I$.

If $\rho \in \Sigma_1 \backslash \Sigma_2$ or $\rho \in \Sigma_2 \backslash \Sigma_1$, it is similar to the case (A). This completes the proof.

(2) *Necessity of Theorem* 1.10

We suppose that Γ_ρ contains the zero for some point $\rho=(x^0, \xi^0)\in\Sigma_1\cap\Sigma_2$. We may assume the same form as (3.5) *i.e.*

$$P=\sum_{j=0}^{M_1+M_2}\sum_{(\alpha_1,\alpha_0,\alpha_2)} A_{\alpha_1,\alpha_0,\alpha_2,j}(D_{x^{(1)}})^{\alpha_1}(D_{x^{(0)}})^{\alpha_0}(D_{x^{(2)}})^{\alpha_2}$$

where $A_{\alpha_1,\alpha_0,\alpha_2,j}$ are of order $m-(M_1+M_2)+(1-1/k)j$. For brevity, we may assume $x^0=0$, $\xi^0=(0)^{(1)}, (0)^{(0)}, (0)^{(2)}, 0, \cdots, 0, 1)$ $(\xi_n^0=1)$. Then our hypothesis on $\tilde{q}(\rho, Y)$ means:

$$\sum_{j=0}^{M_1+M_2}\sum a_{\alpha_1,\alpha_0,\alpha_2,j}(0, \cdots, 0, \xi_n^0)\,(\eta^{(1)})^{\alpha_1}(\eta^{(0)})^{\alpha_0}(\eta^{(2)})^{\alpha_2}=0 \tag{3.6}$$

for some $(\eta^{(1)}, \eta^{(0)}, \eta^{(2)})\in \boldsymbol{R}^{(\nu_1+\nu_0+\nu_2)}$ where $a_{\alpha_1,\alpha_0,\alpha_2,j}$ are the principal symbols of $A_{\alpha_1,\alpha_0,\alpha_2,j}$. Here if we assign to $(\eta^{(1)}, \eta^{(0)}, \eta^{(2)})$ the weight 1, to ξ_n^0 the weight $k/(k-1)$, we can regard the left hand side in (3.6) as quasi-homogeneous symbol of degree $(km-(M_1+M_2))/(k-1)$ of type $(1, k(k-1))$. Then by [9; Lemma 7.1] (c.f. [1; Proposition 3.1]), there exists a distribution u such that the wave front set $WF(u)=\{(x^0, \lambda\xi^0);\ \lambda>0\}$ and $Pu\in H^s$ at ρ but $u\notin H^{s+m-(M_1+M_2)/k}$ at ρ. This completes the proof.

§ 4. The special case of type $P=P_1\cdot P_2+P_3$

Let $P_1\in OPL_k^{m_1,M_1}(X;\ \Sigma_1)$ and $P_2\in OPL_k^{m_2,M_2}(X;\ \Sigma_2)$ where one of the factors is properly supported and elliptic outside Σ_1 and Σ_2 respectively. By Proposition 1.3, $P_1\cdot P_2\in OPL_k^{m_1+m_2,M_1,M_2}(X;\ \Sigma_1, \Sigma_2)$. Then we shall consider the operator of the following type:

$$P=P_1\cdot P_2+P_3 \tag{4.1}$$

where

$$P_3\in OPL_k^{m_1+m_2-1/k,M_1-1,M_2}(X;\ \Sigma_1, \Sigma_2)\cup OPL_k^{m_1+m_2-1/k,M_1,M_2-1}(X;\ \Sigma_1, \Sigma_2).$$

PROPOSITION 4.1. *Assume that* Σ_1 *and* Σ_2 *satisfy* $(H.1)$, $(H.2)$ *and* $(H.3)$. *Then* $\tilde{q}_i(\rho, Y)$ *be the associated forms of* P_i $(i=1, 2, 3)$ *given by* (1.5). *Then for every* $\rho\in\Sigma_1\cap\Sigma_2$, *we have*

$$\tilde{q}(\rho, Y)=\tilde{q}_1(\rho, Y)\cdot\tilde{q}_2(\rho, Y)+\tilde{q}_3(\rho, Y),\quad Y\in T_\rho\bigl(T^*X-\{0\}\bigr).$$

PROOF. First we calculate the q in Proposition 1.5. Let q_i be the associated symbols given by (1.3) $(i=1, 2, 3)$ in a conic neighbourhood U of ρ. By our hypotheses $(H.1)$, $(H.2)$ and $(H.3)$,

$$p_1\# p_2=p_1\cdot p_2$$

and

$$q_i \equiv p_i \quad (i=1, 2, 3) \qquad \text{modulo } L_k^{m_1+m_2, M_1+M_2+1}(U;\ \Sigma_1 \cap \Sigma_2).$$

Thus by Proposition 1.9, we have

$$q \equiv q_1 \cdot q_2 + q_3 .$$

Since

$$q_{m_1+m_2-j/k} = \sum_{t+s=j} q_{1,m_1-t/k}\, q_{2,m_2-s/k} + q_{3,m_1+m_2-j/k},$$

we readily see

$$\tilde{q}(\rho, Y) = \tilde{q}_1(\rho, Y) \cdot \tilde{q}_2(\rho, Y) + \tilde{q}_3(\rho, Y).$$

Thus by Theorem 1.10 and Corollary 1.11, we have

THEOREM 4.1. *Assume that Σ_1 and Σ_2 satisfy (H.1), (H.2) and (H.3) and let $\rho \in \Sigma_1 \cap \Sigma_2$. Then P is hypoelliptic at ρ with loss of $(M_1+M_2)/k$-derivatives if and only if*

$$\tilde{q}_1(\rho, Y) \cdot \tilde{q}_2(\rho, Y) + \tilde{q}_3(\rho, Y) \neq 0 \qquad \textit{for all } Y \in T_\rho(T^*X - \{0\}).$$

Next for $\rho \in \Sigma_1 \backslash \Sigma_2$, it is easy to see

$$\tilde{q}(\rho, Y) = \tilde{q}_1(\rho, Y) \cdot p_{2,m_2}(\rho) + \tilde{q}_3(\rho, Y)$$

and for $\rho \in \Sigma_2 \backslash \Sigma_1$,

$$\tilde{q}(\rho, Y) = p_{1,m_1}(\rho) \cdot \tilde{q}_2(\rho, Y) + \tilde{q}_3(\rho, Y)$$

where p_{1,m_1} and p_{2,m_2} are the principal symbols of p_1 and p_2 respectively. Therefore we have

COROLLARY 4.3. *Under the hypotheses in the above theorem, if, for every $\rho \in \Sigma = \Sigma_1 \cup \Sigma_2$,*

$$\tilde{q}(\rho, Y) \neq 0 \qquad \textit{for all } Y \in T_\rho\big(T^*X - \{0\}\big),$$

P is hypoelliptic with loss of $(M_1+M_2)/k$-derivatives.

EXAMPLE 4.4. (1) Let $P = P_1 \cdot P_2 + P_3$ in $\boldsymbol{R}^4$ where

$$P_1 = D_{x_1}^2 + D_{x_2}^2 + aD_{x_4},$$

$$P_2 = D_{x_2}^2 + D_{x_3}^2 + bD_{x_4},$$

$$P_3 = c(D_{x_1}^2 + D_{x_2}^2 + D_{x_3}^2 + D_{x_4}^2)$$

and a, b and c be complex numbers.

Let $\rho = (x_1^0, x_2^0, x_3^0, x_4^0, 0, 0, 0, \xi_4^0)$ $(\xi_4^0 \neq 0)$. Put $\Delta_{a,b,\xi_4^0} = \{\lambda \operatorname{sgn}(\xi_4^0)(a+b) + \mu;\ \lambda, \mu \geq 0\}$. Then if

$$ab+c+\Delta_{a,b,\xi_4^0} \quad \text{does not meet the origin,}$$

P is hypoelliptic at ρ with loss of 2-derivatives.

In fact, if we set $\Sigma_1=\{\xi_1=\xi_2=0\}$, $\Sigma_2=\{\xi_2=\xi_3=0\}$, $P\in POL_{2,c}^{4,2,2}(\boldsymbol{R}^4;\ \Sigma_1, \Sigma_2)$. Then we have

$$\Gamma_\rho=\left\{\left(|\xi_4^0|(\eta_1^2+\eta_2^2)+a\xi_4^0\right)\left(|\xi_4^0|(\eta_2^2+\eta_3^2)+b\xi_4^0\right)+c(\xi_4^0)^2;\ (\eta_1, \eta_2\in\boldsymbol{R}^2\right\}.$$

(2) Let $P=(D_{x_1}^4+x_1^4(D_{x_2}^4+D_{x_3}^4)(D_{x_2}^4+x_2^4(D_{x_1}^4+D_{x_3}^4)+a(D_{x_1}^6+D_{x_2}^6+D_{x_3}^6)$ in $\boldsymbol{R}^3$ where a is a complex number. Let $\rho=(0, 0, x_3^0, 0, 0, \xi_3^0)$. Then P is hypoelliptic at ρ with loss of 2-derivatives if and only if

$$a+[0, +\infty) \quad \text{does not meet the origin.}$$

In fact, if we set $\Sigma_1=\{\xi_1=x_1=0\}$, $\Sigma_2=\{\xi_2=x_2=0\}$, $P\in OPL_{4,c}^{8,4,4}(\boldsymbol{R}^3;\ \Sigma_1, \Sigma_2)$. Then we have

$$\Gamma_\rho=\left\{|\xi_3^0|^6(\eta_1^4+y_1^4)(\eta_2^4+y_2^4)+c|\xi_3^0|^6;\ (y_1, y_2, \eta_1, \eta_2)\in\boldsymbol{R}^4\right\}.$$

§ 5. The case of system

For brevity, let $P_i\in OPL_2^{1,1}(X;\ \Sigma_i)$ be elliptic outside Σ_i $(i=1, 2)$, and let A, $B\in OPL_{1,0}^{1/2}(X)$ $(=L_{1,0}^{1/2}$, for the notation, see [6]). We consider the following system

$$\boldsymbol{P}=\begin{pmatrix} P_1 & A \\ B & P_2 \end{pmatrix}.$$

By using Corollary 4.3, we can prove

THEOREM 5.1. *Auusme that Σ_1 and Σ_2 satisfy (H.1), (H.2) and (H.3). Then $\boldsymbol{P}$ is hypoelliptic with loss of 1/2-derivative if, for $\rho\in\Sigma=\Sigma_1\cup\Sigma_2$ and for all $Y\in T_\rho(T^*X-\{0\})$,*

$$\tilde{q}_1(\rho, Y)\neq 0 \qquad \text{when} \quad \rho\in\Sigma_1\backslash\Sigma_2,$$

$$\tilde{q}_2(\rho, Y)\neq 0 \qquad \text{when} \quad \rho\in\Sigma_2\backslash\Sigma_1,$$

$$\tilde{q}_1(\rho, Y)\cdot\tilde{q}_2(\rho, Y)-a^0(\rho)\cdot b^0(\rho)\neq 0 \qquad \text{when} \quad \rho\in\Sigma_1\cap\Sigma_2.$$

Here a^0 and b^0 are the principal symbols of A and B respectively. In particular, the system is subelliptic with loss of 1/2-derivatives.

PROOF. If we set

$$\hat{\boldsymbol{P}}=\begin{pmatrix} P_2 & -A \\ -B & P_1 \end{pmatrix},$$

we have

$$\hat{\boldsymbol{P}}\cdot\boldsymbol{P}=\begin{pmatrix} P_2\cdot P_1-A\cdot B & [P_2, A] \\ [P_1\cdot B] & P_1\cdot P_2-B\cdot A \end{pmatrix}.$$

Since $P_2\cdot P_1-A\cdot B$, $P_1\cdot P_2-B\cdot A\in OPL^{2,1,1}_{2}(X;\ \Sigma_1, \Sigma_2)$, under our hypotheses, there exist left parametrices Q_1, $Q_2\in OPS^{-2,-1,-1}(X;\ \Sigma_1, \Sigma_2)$ such that

$$Q_1\cdot(P_2\cdot P_1-A\cdot B)\sim I\,,$$

$$Q_2\cdot(P_1\cdot P_2-B\cdot A)\sim I$$

where I is the identity operator.
Thus if we put

$$\boldsymbol{Q}''=\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix},$$

we have

$$\boldsymbol{Q}''\cdot\hat{\boldsymbol{P}}\cdot\boldsymbol{P}=\boldsymbol{I}-\boldsymbol{R}$$

where $\boldsymbol{I}$ is the identity matrix and

$$\boldsymbol{R}=\begin{pmatrix} 0 & R_1 \\ R_2 & 0 \end{pmatrix},$$

$R_1=Q_1=[P_2, A]$, $R_2=Q_2\cdot[P_1, B]\in OPL^{-3/2,-1,-1}(X;\ \Sigma_1\cap\Sigma_2)\subset OPL^{-1/2}_{1/2}(X)$. By the standard technique, we find that

$$\boldsymbol{Q}=\sum_{j=0}^{\infty}(\boldsymbol{R})^j\cdot\boldsymbol{Q}''\cdot\hat{\boldsymbol{P}}\in OPL^{-1/2}_{1/2}(X)$$

is a left parametrix of $\boldsymbol{P}$. This completes the proof.

EXAMPLE 5.2. Let

$$\boldsymbol{P}=\begin{pmatrix} D_{x_1}+a|D_{(x_2,x_3)}|^{1/2} & c|D_{(x_1,x_2,x_3)}|^{1/2} \\ d|D_{(x_1,x_2,x_3)}|^{1/2} & D_{x_2}+b|D_{(x_1,x_3)}|^{1/2} \end{pmatrix} \text{ in } \boldsymbol{R}^3$$

where

$$|D_{(x_2,x_3)}|=(D_{x_2}^2+D_{x_3}^2)^{1/2},$$

$$|D_{(x_1,x_3)}|=(D_{x_1}^2+D_{x_3}^2)^{1/2},$$

$$|D_{(x_1,x_2,x_3)}|=(D_{x_1}^2+D_{x_2}^2+D_{x_3}^2)^{1/2}$$

and a, b, c and d are complex numbers. Put $\varDelta_{a,b}=\{\lambda a+\mu b+\nu\ ;\ \lambda, \mu, \nu\in\boldsymbol{R}\}$. Then if

$$\operatorname{Im} a \neq 0\,,$$
$$\operatorname{Im} b \neq 0$$

and

$$ab-cd+\Delta_{a,b} \qquad \text{does not meet the origin,}$$

$\boldsymbol{P}$ is hypoelliptic with loss of 1/2-derivative.

In fact, we have

$$\begin{aligned}\Gamma_\rho &= \left\{|\xi^0_{(2,3)}|^{1/2}(\eta_1+a)\,;\ \eta_1\in\boldsymbol{R}\right\} \qquad \text{if } \rho\in\Sigma_1\backslash\Sigma_2\,,\\ &= \left\{|\xi^0_{(1,3)}|^{1/2}(\eta_2+b)\,;\ \eta_2\in\boldsymbol{R}\right\} \qquad \text{if } \rho\in\Sigma_2\backslash\Sigma_1\,,\\ &= \left\{|\xi^0_3|\Big((\eta_1+a)(\eta_2+b)-cd\Big)\,;\ (\eta_1,\eta_2)\in\boldsymbol{R}^2\right\} \qquad \text{if } \rho\in\Sigma_1\cap\Sigma_2.\end{aligned}$$

References

[1] ARAMAKI, J.: On a class of pseudo-differential operators and hypoellipticity, Hokkaido Math. J. Vol. IX No. 1 (1980), 46–58.

[2] BOUTET DE MONVEL, L.: Hypoelliptic operators with double characteristics and related pseudo-differential operators, Comm. Pure and Appl. Math. 27 (1974), 585–639.

[3] DUISTERMAAT, J. J. and HÖRMANDER, L.: Fourier integral operators II, Acta Math. 128 (1972), 183–269.

[4] GRIGIS, M. A. and LASCAR, R.: Équations locales d'un système de sous-variétés involutives, C. R. Acad. Sc. Paris 283 (1976), 503–506.

[5] HELFFER, B.: Invariant associés à une classe d'opérateurs pseudo-différentiels et applications à L'hypoellipticité, Ann. Inst. Fourier, Grenoble 26 (1976), 55–70.

[6] HELFFER, B.: Construction de parametrix pour des opérateurs pseudo-différentiels caracteristiques sur reunion de deux cones lisses, Bull. Soc. Math. France, Memoire, 51–52 (1977), 63–123.

[7] HÖRMANDER, L.: Pseudo-differential operators and hypoelliptic equations, Amer. Math. Soc. Symp. Pure Math. 10 (1966), Singular integrals 138–183.

[8] HÖRMANDER, L.: Fourier integral operators I, Acta Math. 127 (1971), 79–183.

[9] LASCAR, R.: Propagation des singularités des solutions d'équations pseudo-différentielles quasi homogénes, Ann. Inst. Fourier, Grenoble 27 (1977), 79–123.

[10] SJÖSTRAND, J.: Parametrices for pseudo-differential operators with multiple characteristics, Arkiv för Mat. 12 (1974), 85–130.

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