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On vanishing or recurrent Bochner curvature tensor

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The Bochner curvature tensor has been introduced by S. Bochner ([2]) in Kaehlerian manifolds with respect to the complex coordinates, as an analogy of the conformal curvature tensor. S. Tachibana gave it a real expression ([12]) and showed that a reducible Kaehlerian manifold M with vanishing Bochner curvature tensor is locally the Riemannian product of two Kaehlerian manifolds with constant holomorphic sectional curvatures $H(\geqq 0)$ and $-H$, respectively, and in this case the scalar curvature of M is constant ([13]). On the other hand, since it is known ([8]) that a Kaehlerian manifold M with vanishing Bochner curvature tensor and constant scalar curvature is locally symmetric, if M is irreducible, then M is Einsteinian and thus of constant holomorphic sectional curvature. Therefore, taking account of de Rham decomposition of Kaehlerian manifolds ([6], II), we can determine complete (simply connected) Kaehlerian manifolds with vanishing Bochner curvature tensor and constant scalar curvature. In the present paper, we shall show, by using the theory of fibred spaces with projectable Kaehlerian structure of M. Ako ([1]), that a real $n(\geqq 4)$-dimensional Kaehlerian manifold with vanishing Bochner curvature tensor has constant scalar curvature.

Recurrent geometry has been introduced by A. G. Walker ([14]) and he has determined recurrent manifolds, that is, Riemannian manifolds with recurrent Riemannian curvature tensor (see also [6], I). Projective recurrent manifolds, that is, Riemannian manifolds with recurrent projective curvature tensor, were determined by M. Matsumoto ([7]). Conformally recurrent manifolds, that is, Riemannian manifolds with recurrent conformal curvature tensor, were determined by A. Gebarowski ([4]) in the case that the mainfolds are reducible, and by [9] in the case that the manifolds are irreducible and of dimension greater than four. In the present paper, we shall study Kaehlerian manifolds with recurrent Bochner curvature tensor and give some results in reducible and irreducible cases (Theorems 3 and 4).

§ 1. Preliminaries.

(I) Fibred spaces. Let M and N be two manifolds of dimensions n

and $n-1$, respectively, and suppose that there is given a mapping σ of maximum rank $n-1$. In this case, σ is called a submersion. Moreover, we suppose that there is defined in M the vector field C such that $C\in\ker\dot{\sigma}$, $\dot{\sigma}$ being the differential of σ, and a positive definite Riemannian metric g satisfying

$$g(C, C)=1.$$

If we introduce in M a 1-form η defined by the equation

$$\eta(X)=g(C, X),$$

X being an arbitrary vector in M, we have

$$\eta(C)=1.$$

The set $(M, N, \sigma;\ C, g)$ satisfying the conditions above is called a fibred space with Riemannian metric g.

Let $\mathscr{T}_s^r$ be the space of all tensor fields of type (r, s) in M. We put $\mathscr{T}=\Sigma_{r,s}\mathscr{T}_s^r$. Then a linear endomorphism $T\to T^H$ of $\mathscr{T}$ is defined by

$$\begin{aligned}
&f^H=f && \text{for } f\in\mathscr{T}_0^0,\\
&X^H=X-\eta(X)C && \text{for } X\in\mathscr{T}_0^1,\\
&w^H=w-w(C)\,\eta && \text{for } w\in\mathscr{T}_1^0,\\
&(S\otimes T)^H=S^H\otimes T^H && \text{for } S,\ T\in\mathscr{T}.
\end{aligned}$$

When an element T of $\mathscr{T}$ satisfies the condition $(\mathscr{L}_C(T^H))^H=0$, we say that T is projectable, where $\mathscr{L}_C$ denotes the Lie differentiation with respect to C.

Let M be a Kaehlerian manifold with the complex structure F and the Kaehlerian metric g. If both F and g are projectable, we call $(M, N, \sigma;\ C, g)$ or M a fibred space with projectable Kaehlerian structure ([10]). Then, M. Ako proved

LEMMA 1 ([1]). *Let M be a fibred space with projectable Kaehlerian structure. We denote by D_1 the distribution spanned by C and FC and by D_2 the distribution of all vector fields orthogonal to D_1. Then D_1 and D_2 are both involutive distributions and their integral manifolds M_1 and M_2 are Kaehlerian submanifolds of M which are totally geodesic and M is locally the Riemannian product of M_1 and M_2.*

(II) Recurrency. Let M be a Riemannian manifold with the Riemannian connection ∇. A tensor field T on M is said to be recurrent if there exists on M a non-zero 1-form ω such that

$$\nabla T=\omega\otimes T.$$

In this case, the 1-form ω is called a recurrence vector of T.

(III) Bochner curvature tensor. Let M be a real $n(\geqq 4)$-dimensional Kaehlerian manifold with the complex structure F_i^h and the Kaehlerian metric g_{ji}. If we denote by $K_{kji}{}^h$, K_{ji} and K the Riemannian curvature tensor, the Ricci tensor and the scalar curvature respectively, the Bochner curvature tensor B_{kjih} is given by

$$(1)\qquad \begin{aligned} B_{kjih} = K_{kjih} &-(g_{kh}K_{ji}-g_{jh}K_{ki}+K_{kh}g_{ji}-K_{jh}g_{ki}+F_{kh}S_{ji}-F_{jh}S_{ki} \\ &+S_{kh}F_{ji}-S_{jh}F_{ki}-2F_{kj}S_{ih}-2S_{kj}F_{ih})/(n+4)+K(g_{kh}g_{ji} \\ &-g_{jh}g_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih})/(n+2)(n+4)\,, \end{aligned}$$

where $S_{ji}=-K_{jk}F_i^k=-\frac{1}{2}F^{kh}K_{khji}=K_{kjih}F^{kh}$ is a closed 2-form. B_{kjih} satisfies

$$(2)\qquad \begin{aligned} \nabla_h B_{kji}{}^h = \frac{n}{n+4}\Big\{&\nabla_k K_{ji}-\nabla_j K_{ki}-\frac{1}{2(n+2)}(u_k g_{ji}-u_j g_{ki}+v_k F_{ji} \\ &-v_j F_{ki}-2F_{kj}v_i)\Big\}, \end{aligned}$$

where $u_k=\nabla_k K$, $v_k=F_k{}^j u_j$ and ∇_k is the Riemannian connection.

M is called of constant holomorphic sectional curvature, if we have

$$K_{kjih}=\frac{K}{n(n+2)}(g_{kh}g_{ji}-g_{jh}g_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih})\,.$$

In this case, K is constant if $n\geqq 4$.

§ 2. Vanishing Bochner curvature tensor.

If we differentiate covariantly (2), we have

$$(3)\qquad \begin{aligned} \frac{n+4}{n}\nabla_l\nabla_h B_{kji}{}^h &= \nabla_l\nabla_k K_{ji}-\nabla_l\nabla_j K_{ki} \\ &-\frac{1}{2(n+2)}(\nabla_l u_k g_{ji}-\nabla_l u_j g_{ki}+\nabla_l v_k F_{ji}-\nabla_l v_j F_{ki}-2F_{kj}\nabla_l v_i)\,. \end{aligned}$$

Interchanging the indices l, k, j in (3) cyclically and adding the resulting two equations to (3) we get

$$(4)\qquad \begin{aligned} -\frac{n+4}{n}(\nabla_l\nabla_h B_{kji}{}^h+\nabla_k\nabla_h B_{jli}{}^h+\nabla_j\nabla_h B_{lki}{}^h) &= K_{kji}{}^h K_{lh}+K_{jli}{}^h K_{kh} \\ +K_{lki}{}^h K_{jh}+\frac{1}{2(n+2)}\Big\{&(\nabla_k v_j-\nabla_j v_k)F_{li}+(\nabla_j v_l-\nabla_l v_j)F_{ki} \\ +(\nabla_l v_k-\nabla_k v_l)F_{ji}&-2F_{kj}\nabla_l v_i-2F_{jl}\nabla_k v_i-2F_{lk}\nabla_j v_i\Big\}\,. \end{aligned}$$

Transvecting F^{lk} with (4) we obtain

(5) $$\frac{n+4}{n} F^{lk}\nabla_l\nabla_h B_{kji}{}^h = -K_{kjih}S^{kh} + S_i{}^h K_{jh} - \frac{1}{2(n+2)}(-\nabla^k u_k F_{ji} - (n-1)\nabla_j v_i + \nabla_i v_j) .$$

Taking the symmetric part of (5) we have

$$2(n+2)(n+4) F^{lk}\nabla_l\nabla_h(B_{kji}{}^h + B_{kij}{}^h) = n(n-2)(\nabla_j v_i + \nabla_i v_j) ,$$

where we have used the fact that $S_i{}^h K_{ih} = -K_{ik}F^{hk}K_{jh} = K_{jh}F^{kh}K_{ik} = -S_j{}^k K_{ik}$. Thus, if we have

$$F^{lk}\nabla_l\nabla_h(B_{kji}{}^h + B_{kij}{}^h) = 0 ,$$

and $n \geqq 4$, then we obtain

(6) $$\mathcal{L}_v g_{ji} = \nabla_j v_i + \nabla_i v_j = 0 ,$$

where $\mathcal{L}_v$ is the Lie differentiation with respect to v^i.

On the other hand, because we have

$$\mathcal{L}_v F_{ji} = \nabla_j u_i - \nabla_i u_j = 0 ,$$

we see that

$$\mathcal{L}_v F_i{}^h = 0 ,$$

that is, v^i is a contravariant analytic vector. Since a vector $u^h = F_i{}^h v^i$ for a contravariant analytic vector v^i is also contravariant analytic ([15]), we have proved

THEOREM 1. *In a real $n(\geqq 4)$-dimensional Kaehlerian manifold, if we have*

$$F^{lk}\nabla_l\nabla_k(B_{kji}{}^h + B_{kij}{}^h) = 0 ,$$

then the vector $u^h = \nabla_i K g^{ih}$ is contravariant analytic.

NOTE. Theorem 1 gives a generalization of the result of M. Matsumoto ([8]) in a compact case.

In the rest of this section, we consider only a real $n(\geqq 4)$-dimensional Kaehlerian manifold with vanishing Bochner curvature tensor. In this case, v^i is Killing and contravariant analytic.

Let P be a point at which $v^i \neq 0$. We consider around P. If we take a local unit vector $C^i = fv^i$ around P, we obtain an open neighborhood U of P such that there exist the orbit space U/C by C^i and the submersion σ from U onto U/C. Then, we can easily verify that the fibred space $(U, U/C, \sigma, C^i, g_{ji})$ has the projectable Kaehlerian structure $(F_i{}^h, g_{ji})$ as follows.

CLAIM. *$(\mathcal{L}_C g^H)^H = 0$, and $(\mathcal{L}_C F^H)^H = 0$.*

PROOF. For a tensor T, $T - T^H$ is called a non-horizontal part of T.

In the following, we adopt a symbol * as non-horizontal parts of some tensors.

Since $(g^H)_{ji}=g_{ji}-C_jC_i$, we have

$$\begin{aligned}\mathscr{L}_C(g^H)_{ji} &= \mathscr{L}_C g_{ji}-(\mathscr{L}_C C_j)\, C_i - C_j\,(\mathscr{L}_C C_i)\\ &= \nabla_j C_i+\nabla_i C_j+*\\ &= f(\nabla_j v_i+\nabla_i v_j)+*=*\,.\end{aligned}$$

Thus, we have $(\mathscr{L}_C g^H)^H=0$.

Next, if we put $D^h=F_i^h C^i=fu^h$, we easily see

$$(F^H)_i^h=F_i^h-(C_iD^h-D_iC^h)\,.$$

Then we have

$$\begin{aligned}\mathscr{L}_C(F^H)_i^h &= \mathscr{L}_C F_i^h-(\mathscr{L}_C C_i)\, D^h+*\\ &= F_j^h\nabla_i C^j-F_i^j\nabla_j C^h-(C^j\nabla_j C_i+C_j\nabla_i C^j)D^h+*\\ &= F_j^h(\nabla_i f v^j+f\nabla_i v^j)-F_i^j(\nabla_j f v^h+f\nabla_j v^h)-f^2v^j(\nabla_j f v_i+f\nabla_j v_i)\, u^h+*\\ &= f\mathscr{L}_v F_i^h+\nabla_i f u^h-f^3v^j\nabla_j v_i u^h+*\\ &= (\nabla_i f+f^3v^j\nabla_i v_j)\, u^h+*=\left\{\nabla_i f+\frac{1}{2}f^3\nabla_i(v^jv_j)\right\}u^h+*\,.\end{aligned}$$

Therefore, we obtain $(\mathscr{L}_C F^H)^H=0$, if we have the equation $\nabla_i f+\frac{1}{2}f^3\nabla_i(v^jv_j)=0$. Since $1=C^jC_j=f^2v^jv_j$, we have $f^2=(v^jv_j)^{-1}$. By differentiating this covariantly, we get

$$2f\nabla_i f=-f^4\nabla_i(v^jv_j)\,,$$

and thus

$$\nabla_i f+\frac{1}{2}f^3\nabla_i(v^jv_j)=0\,.$$

By Lemma 1 and the above claim, U is locally the Riemannian product of two Kaehlerian manifolds and thus the scalar curvature is constant, that is, $v^i=0$. But this is a contradication.

Therefore, we have proved

THEOREM 2. *A real $n\,(\geqq 4)$-dimensional Kaehlerian manifold with vanishing Bochner curvature tensor has the constant scalar curvature.*

§ 3. Recurrent Bochner curvature tensor.

If a Riemannian manifold is of class C^ω and has the Riemannian metric of class C^ω, we say the manifold to be analytic. A Kaehlerian manifold is analytic and has the complex structure of class C^ω.

In the seqnel, we consider a Kaehlerian manifold $M(F_i^h, g_{ji})$ with recurrent Bochner curvature tensor B_{kjih}, that is, we have

(7) $$\nabla_l B_{kjih} = p_l B_{kjih}\,,$$

LEMMA 2 (cf. [9], [11]). *We see that B_{kjih} vanishes identically or p_l is gradient. In the case that p_l is gradient, if $B_{kjih}=0$ at a point, then B_{kjih} vanishes identically.*

Hereafter we consider the case that B_{kjih} does not vanish. In this case, if we put

$$p = \frac{1}{2}\cdot \log\,(B_{kjih}\,B^{kjih})\,,$$

then p is the analytic function on M and we have

$$\nabla_l p = p_l\,.$$

(I) Reducible case. We consider the case that M is the Riemannian product of two Kaehlerian manifolds M_1 and M_2 of real dimensions r and $n-r$ ($r \leqq n-r$), respectively, and the complex structure F_i^h on M is given by

$$(F_i^h) = \begin{bmatrix} F_b^a, & 0 \\ 0, & F_\beta^\alpha \end{bmatrix},$$

with respect to the local coordinates (x^a, x^α), where (x^a) and (x^α) are local coordinates, and F_b^a and F_β^α the complex structures of M_1 and M_2, respectively. The indices a, b and α, β run over the ranges $\{1, 2, \cdots, r\}$ and $\{r+1, r+2, \cdots, n\}$, respectively.

Then, by the method of A. Gębarowski ([4]), we can easily verify

LEMMA 3. *Around a point of M at which $p_i \neq 0$, one of the following is valid*:

(i) *one of M_1 and M_2 is flat and the other is recurrent,*

(ii) *M_1 and M_2 are of constant holomorphic sectional curvature.*

But, by analyticity, we see that Lemma 3 is true on M. In the case (i), M is recurrent. Let us consider the case (ii). In this case, in order that B_{kjih} is recurrent, it is necessary and sufficient that

$$(n-r)(n-r+2)\,\nabla_a K_1 = (n-r)(n-r+2)\,K_1 + r(r+2)\,K_2 p_a,$$
$$r(r+2)\,\nabla_\alpha K_2 = (n-r)(n-r+2)\,K_1 + r(r+2)\,K_2 p_\alpha\,,$$

or

$$\nabla_i\Big\{(n-r)(n-r+2)\,K_1 + r(r+2)\,K_2\Big\} = \Big\{(n-r)(n-r+2)\,K_1 + r(r+2)\,K_2\Big\}p_i\,,$$

where K_1 and K_2 are the scalar curvatures of M_1 and M_2, respectively. If both K_1 and K_2 are constant, we have, because there exists on M a point at which $p_i \neq 0$,

$$(n-r)(n-r+2)K_1 + r(r+2)K_2 = 0,$$

which shows that $B_{kjih}=0$. But, since we consider the case that $B_{kjih} \neq 0$ at every point of M, one of K_1 and K_2 is non-constant. Therefore, by the assumption that $r \leqq n-r$, r must be equal to 2.

THEOREM 3. *Let M be a real $n\,(\geqq 4)$-dimensional Kaehlerian manifold with recurrent Bochner curvature tensor, which does not banish. If M is the Riemannian product of two Kaehlerian manifolds M_1 and M_2 of real dimensions r and $n-r$ $(r \leqq n-r)$, respectively, then M is recurrent or M_1 and M_2 are of constant holomorphic sectional curvature. In the latter case, $r=2$ and we have*

$$(8) \qquad \nabla_i \rho = \rho p_i,$$

for the function $\rho = n(n-2)K_1 + 8K_2$ which does never vanish, where K_1 and K_2 are the scalar curvatures of M_1 and M_2, respectively, and p_i is the recurrence vector of the Bochner curvature tensor of M.

REMARK. Since $n(n-2)K_1 + 8K_2 = (n-2)(n-4)K_1 + 8K$, in the case $n=4$ (8) shows that M has recurrent scalar curvature with the same recurrence vector as that of the Bochner curvature tensor.

(II) Irreducible case. In a previous paper ([9]), we have shown that a real $n\,(>4)$-dimensional irreducible Kaehlerian manifold with recurrent Bochner curvature tensor has vanishing Bochner curvature tensor. In this part, we study the case $n=4$. In the sequel, M is a real 4-dimensional irreducible Kaehlerian manifold with recurrent Bochner curvature tensor which does not vanish.

LEMMA 4. *There is the equation,*

$$(9) \qquad K_{mlk}{}^{t} B_{tjih} + K_{mlj}{}^{t} B_{ktih} + K_{mli}{}^{t} B_{kjth} + K_{mlh}{}^{t} B_{kjit} = 0.$$

(9) follows from the Ricci identity for B_{kjih} and the fact that p_i is gradient.

LEMMA 5. *We have*

$$(10) \qquad S_k{}^{t} B_{tjih} + S_j{}^{t} B_{ktih} + S_i{}^{t} B_{kjth} + S_h{}^{t} B_{kjit} = 0.$$

Transvecting F^{ml} with (9) we get (10).

LEMMA 6. *If we put $D_s = \nabla_s - p_s$, we have*

$$(11) \qquad D_s K_{lk} B_{mjih} + D_s K_{lj} B_{kmih} + D_s K_{li} B_{kjmh} + D_s K_{lh} B_{kjim}$$
$$- D_s K_{mk} B_{ljih} - D_s K_{mj} B_{klih} - D_s K_{mi} B_{kjlh} - D_s K_{mh} B_{kjil}$$

$$
\begin{aligned}
&+D_s K_m{}^t (B_{tjik} g_{lk} + B_{ktih} g_{lj} + B_{kjth} g_{li} + B_{kjit} g_{lh}) \\
&-D_s K_l{}^t (B_{tjih} g_{mk} + B_{ktih} g_{mj} + B_{kjth} g_{mi} + B_{kjit} g_{mh}) \\
&+F_m{}^t (D_s S_{lk} B_{tjih} + D_s S_{lj} B_{ktih} + D_s S_{li} B_{kjth} + D_s S_{lh} B_{kjit}) \\
&-F_l{}^t (D_s S_{mk} B_{tjih} + D_s S_{mj} B_{ktih} + D_s S_{mi} B_{kjth} + D_s S_{mh} B_{kjit}) \\
&+D_s S_l{}^t (B_{tjih} F_{lk} + B_{ktih} F_{lj} + B_{kjth} F_{li} + B_{kjit} F_{lh}) \\
&-D_s S_m{}^t (B_{tjih} F_{mk} + B_{ktih} F_{mj} + B_{kjth} F_{mi} + B_{kjit} F_{mh}) \\
&-D_s K/6 \cdot \{B_{mjih} g_{lk} + B_{kmih} g_{lj} + B_{kjmh} g_{li} + B_{kjim} g_{lh} \\
&-B_{ljih} g_{mk} - B_{klih} g_{mj} - B_{kjlh} g_{mi} - B_{kjil} g_{mh} \\
&+F_m{}^t (B_{tjih} F_{lk} + B_{ktih} F_{lj} + B_{kjth} F_{li} + B_{kjit} F_{lh}) \\
&-F_l{}^t (B_{tjih} F_{mk} + B_{ktih} F_{mj} + B_{kjth} F_{mi} + B_{kjit} F_{mh})\} = 0 .
\end{aligned}
$$

PROOF. Differentiating covariantly (9) and taking account of (7) and (9) we have

$$(12) \qquad \nabla_s K_{mlk}{}^t B_{tjih} + \nabla_s K_{mlj}{}^t B_{ktih} + \nabla_s K_{mli}{}^t B_{kjth} + \nabla_s K_{mlh}{}^t B_{kjit} = 0 .$$

Next, multiplying (9) by p_s we have

$$(13) \qquad p_s K_{mlk}{}^t B_{tjih} + p_s K_{mlj}{}^t B_{ktih} + p_s K_{mli}{}^t B_{kjth} + p_s K_{mlh}{}^t B_{kjit} = 0 .$$

Subtracting (13) from (12) and substituting (1) and (7) into the resulting equation we obtain (11), where we have used the facts that

$$B_{tjih} F_k{}^t + B_{ktih} F_j{}^t = 0 ,$$

and

$$D_s S_k{}^t B_{tjih} + D_s S_j{}^t B_{ktih} + D_s S_i{}^t B_{kjth} + D_s S_h{}^t B_{kjit} = 0 ,$$

which is induced from (7) and (10).

LEMMA 7 (cf. [9]). *We have*

$$B_{kjih} B^{kji}{}_l = \frac{1}{4} e^{2p} g_{hl} .$$

Transvecting $g^{mh} B^{kjil}$ with (11) and taking account of Lemma 7, we get

$$e^{2p} D_s K = 0 ,$$

that is,

$$\nabla_s K = p_s K .$$

Thus, we have shown

THEOREM 4. *A real 4-dimensional irreducible Kaehlerian manifold with recurrent Bochner curvature tensor which does not vanish, has the* *recurrent scalar curvature with the same recurrence vector as that of the Bochner curvature tensor.*

As a conformal analogy of Theorem 4, we obtain

PROPOSITION. *A 4-dimensional analytic, irreducible, conformally recurrent Riemannian manifold which is not conformally flat, has the recurrent scalar curvature with the same recurrence vector as that of the conformal curvature tensor.*

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