Hokkaido Mathematical Journal Vol. 12 (1983) p. 237-243

A generalization of monodiffric function

By Shih Tong TU

(Received November 22, 1982)

1. Introduction

The purpose of this paper is to introduce the generalized monodiffric functions, namely, p-monodiffric functions, and to prove some interesting properties of p monodiffric functions. When $p=1$, our results reduce to the classical theory of monodiffric functions which have been developed by Berzsenyi [1, 2], Kurowski [3] and the present author [4, 5].

2. Definition and Notation

Let $\boldsymbol{C}$ be the complex plane, $D=\{z\in\boldsymbol{C}|z=x+iy\}$ where x, $y\in\{pj|j=0,1,2,\cdots\}$ and $0<p\leq 1$ and f: $D\rightarrow\boldsymbol{C}$.

DEFINITION 1. *The p monodiffric residue of f at z is the value*

$$M_{p}f(z)=(i-1)f(z)+f(z+ip)-if(z+p)\,. \qquad (2.1)$$

DEFINITION 2. *The function f is said to be p monodiffric at z if $M_{p}f(z)=0$. The function f is said to be p monodiffric in D if it is p monodiffric at any point in D (denoted by $f\in M_{p}(D)$).*

DEFINITION 3. *The p monodiffric derivative f' of f is defined by*

$$f'(z)=\frac{1}{2p}\left[(i-1)f(z)+f(z+p)-if(z+ip)\right]. \qquad (2.2)$$

We also use the symbols df/dz or $D_{z}f$ to represent f'. It is easy to see that $f'(z)$ can be formulated in the following forms:

$$f'(z)=\frac{f(z+p)-f(z)}{p} \quad \text{or} \quad f'(z)=\frac{1}{ip}\left[f(z+ip)-f(z)\right], \qquad (2.3)$$

if $f\in M_{p}(D)$ at z.

DEFINITION 4. *The line integral of f from z to $z+hp$ is defined by*

$$\int_{z}^{z+hp}f(t)\,dt=\begin{cases} hpf(z) & \text{if } h=1 \text{ or } i \\ -\int_{z+hp}^{z}f(t)\,dt & \text{if } h=-1 \text{ or } -i\,. \end{cases} \qquad (2.4)$$

More generally, if $\Omega=\{a=z_0, z_1, \cdots, z_n=b\}$ is a discrete curve in D, then the line integral of f from a to be along Ω is defined by

$$\int_{\Omega} f(t)\,dt=\int_a^b f(t)\,dt=\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k} f(t)\,dt \tag{2.5}$$

3. Property

The following properties follow directly from the above definitions.

PROPOSITION 1. *The line integral* $\int_a^b f(t)\,dt$ *is independent of path in* D *for every* $a, b\in D$, *if and only if,* $f\in M_p(D)$.

PROPOSITION 2. *If* $a\in D$ *and* f *is* p *monodiffric in* D, *then the function* F *defined by* $F(z)=\int_a^z f(t)\,dt$ *for* $z\in D$, *is also* p *monodiffric in* D, *and* $F'(z)=f(z)$ *for* $z\in D$.

PROPOSITION 3. *If* $f\in M_p(D)$, *then* $\int_a^b f'(t)\,dt=f(b)-f(a)$.

4. The p monodiffric exponential function

In [6] Isaacs introduced the monodiffric exponential function $E(z)=(1+a)^x(1+ia)^y$ for $z=x+iy$ and $a\in \boldsymbol{C}$. We extend it to p monodiffric as follows: The p monodiffric exponential function $e_p^{a,z}$ is defined by $e_p^{a,z}=(1+ap)^j(1+iap)^k$ for $z=(j+ik)p$, where j and k are integers. It is not difficult to prove the following results.

PROPOSITION 4. (a) $\frac{d^n}{dz^n}e_p^{a,z}=a^n e_p^{a,z}$, *where* $\frac{d^n}{dz^n}$ *means n'th* p *monodiffric derivative.*

$$\text{(b)}\quad \frac{d^n}{dz^n}e_p^{a,z}\in M_p(D)\quad \textit{for } n=0, 1, 2, \cdots. \tag{4.1}$$

THEOREM 1. *The solution of the* p *monodiffric difference equation*

$$\frac{dF}{dz}-aF(z)=0 \qquad \textit{with}\quad F(0)=c$$

is given by the p *monodiffric function*

$$F(z)=ce_p^{a,z} \qquad \textit{for every}\quad z\in D,$$

where c *is an arbitrary constant.*
In general, we have

THEOREM 2. *Let $a_1, a_2, \cdots, a_n$ be distinct roots of*

$$a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = 0\,, \tag{4.2}$$

then the general solution to the n'th order p monodiffric linear homogeneous difference equation

$$F^{(n)}(z) + c_{n-1}F^{(n-1)}(z) + \cdots + c_n F'(z) + c_0 F(z) = 0 \tag{4.3}$$

is $F(z) = \sum_{k=1}^{n} B_k e_p^{a_k, z}$

where the coefficients B_k ($k=1, 2, \cdots, n$) are arbitrary constants.

PROOF. Let $F(z) = e_p^{a,z}$. Then from Proposition 4, we have

$$(a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0)\, e_p^{a,z} = 0\,.$$

Since, $a_1, a_2, \cdots, a_n$ are distinct roots of (4.2), we obtain that $e_p^{a_k, z}$ ($k=1, 2, \cdots, n$) is a solution of (4.3). The general solution of (4.3) is $F(z) = \sum_{k=1}^{n} B_k e_p^{a_k, z}$, where B_k ($k=1, 2, \cdots, n$) are arbitrary constants.

5. The p monodiffric homogeneous difference equation of the n'th order

In [4], the author shown that the monodiffric homogeneous difference equation of the n'th order $\sum_{k=0}^{n} (-1)^k C_k^n f(z+n-k)(1-a)^k = 0$ has monodiffric general solution (In [4], Theorem 2, page 48). Now we want to generalize this result to p monodiflric equation. We begin with the following propositions:

PROPOSITION 5.

$$\text{(a)}\quad \frac{d}{da} e_p^{a,z} = (1+ap)^{j-1}(1+iap)^{k-1}\{z + ia(j+k)p^2\}$$
$$\textit{for} \quad z = (j+ik)p\,, \tag{5.1}$$

$$\text{(b)}\quad \frac{d}{da} e_p^{a,z} \in M_p(D) \tag{5.2}$$

where $\dfrac{d}{da} e_p^{a,z} = \lim_{h \to a} \dfrac{e_p^{(a+h),z} - e_p^{a,z}}{h}$ *for fixed point $z \in D$.*

A proof is given by a straightforward calculation.

PROPOSITION 6. $F(z) = \dfrac{d}{da} e_p^{a,z}$ *is a solution of*

$$(D_z - a)^2 F(z) = 0\,. \tag{5.3}$$

and is also a solution of $(D_z-a)^m F(z)=0$ *for any integer* $m \geqq 2$.

PROOF. Since, $F(z) \in M_p(D)$ we obtain $F'(z)=\frac{1}{p}\left[F(z+p)-F(z)\right]$ and $F''(z)=\frac{1}{p}\left[F'(z+p)-F'(z)\right]=\frac{1}{p^2}\left[F(z+2p)-2F(z+p)+F(z)\right]$.

Now $(D_z-a)^2 F(z)=F''(z)-2aF'(z)+a^2 F(z)$

$$=\frac{1}{p^2}\left[F(z+2p)-2(1+ap)\,F(z+p)+(1+ap)^2 F(z)\right], \tag{5. 4}$$

substituting (5. 1) into the right-hand side of (5. 4), we have $(D_z-a)^2 F(z)=0$. Therefore, $\frac{d}{da} e_p^{a,z}$ is a solution of (5. 3). Furthermore, by the straightforward calculation, we get

$$F'(z)=(1+ap)^{j-1}(1+iap)^{k-1}\left[1+(z+p+ip)\,a+i(j+k+1)\,a^2p^2\right].$$

It is easy to verify that $M_p F'(z)=0$, i. e., $F'(z) \in M_p(D)$ and $(D_z-a)^m F(z)=(D_z-a)^{m-2}(D_z-a)^2 F(z)=0$ for $m \geqq 2$.

PROPOSITION 7. *Let* $H(z)=\frac{d^2}{da^2} e_p^{a,z}$ *for* $z=(j+ik)\,p$. *Then we have*

(a) $$H(z)=(1+ap)^{j-2}(1+iap)^{k-2}\left\{z^2+(k-j)\,p^2+2iz(j+k-1)\,ap^2-(j+k)(j+k-1)\,a^2p^4\right\}, \tag{5. 5}$$

(b) $$H(z) \in M_p(D), \tag{5. 6}$$

(c) $$(D_z-a)^3 H(z)=0, \tag{5. 7}$$

(d) $$(D_z-a)^m H(z)=0 \quad \textit{for} \quad m \geqq 3. \tag{5. 8}$$

PROOF. For fix z, we differentiate $\frac{d}{da} e_p^{a,z}$ with respect to a directly, the conclusion of (a) follows. Now we shall prove (b). Rewriting $M_p H(z)=(i-1)\,H(z)+H(z+ip)-iH(z+p)$ into the form $M_p H(z)=(1+ap)^{j-2}(1+iap)^{k-2}$ $[A+Ba+Ca^2+Da^3]$ where the branket [ ] is the form of the polynomial in a and A, B, C and D are constants, then we obtain $A=0$, $B=0$, $C=0$ and $D=0$, and $(D_z-a)^3 H(z)=H(z+3p)-3(1+ap)\,H(z+2p)+3(1+ap)^2 H(z+p)-(1+ap)^3 H(z)$.

To prove (c), we rewrite $(D_z-a)^3 H(z)$ into the form $(D_z-a)^3 H(z)=(1+ap)^{j+1}(1+iap)^{k-2}[Ez^2+Fz+G]$, then $E=0$, $F=0$ and $G=0$. The proof of (d) is obvious. This completes the proof. p monodiffric homogeneous difference equation of the n'th order is of the form $(D_z-a)^n f(z)=0$ or

$$\sum_{k=1}^{n}(-1)^k C_k^n (1+ap)^k f\big(z+(n-k)p\big)=0\,, \tag{5.9}$$

where $C_k^n=\dfrac{n!}{(n-k)!\;k!}$.

From the results of Proposition 6 and 7, we have the general solutions of (5.9) for $n=2$ and $n=3$ respectively as follows:

PROPOSITION 8.

(a) *p monodiffric homogeneous difference equation of the second order*

$$\sum_{k=0}^{2}(-1)^k C_k^2 f\big(z+(2-k)p\big)(1+ap)^k=0$$

has p monodiffric general solution of the form

$$f(z)=c_0 e_p^{a,z}+c_1\frac{d}{da}e_p^{a,z}\,.$$

(b) *p monodiffric homogeneous difference equation of the third order*

$$\sum_{k=0}^{3}(-1)^k C_k^3 f\big(z+(3-k)p\big)(1+ap)^k=0$$

has p monodiffric general solution of the form

$$f(z)=c_0 e_p^{a,z}+c_1\frac{d}{da}e_p^{a,z}+c_2\frac{d^2}{da^2}e_p^{a,z}\,,$$

where the coefficients c_i $(i=0, 1, 2)$ are arbitrary constants.

With the observation of the above Proposition 8, we have the following more general result.

PROPOSITION 9. $\dfrac{d^n}{da^n}e_p^{a,z}\in M_p(D)$ (5.10)

PROOF. Let $E(a, z)=e_p^{a,z}$, $E_a^{(n)}(a, z)=\dfrac{d^n}{da^n}e_p^{a,z}$ for $n\in N$.

From Proposition 6 and 7, (5.10) is true for $n=1$ and $n=2$. Suppose it holds for $n=k$, then $M_p E_a^{(k)}(a, z)=0$, so that

$$(i-1)\,E_a^{(k)}(a, z)+E_a^{(k)}(a, z+ip)-iE_a^{(k)}(a, z+p)=0\,,$$

$$(i-1)\,E_a^{(k)}(a+h, z)+E_a^{(k)}(a+h, z+ip)-iE_a^{(k)}(a+h, z+p)=0\,.$$

Substracting the first from the second of above equalities and dividing by h, we have

$$(i-1)\frac{E_a^{(k)}(a+h, z)-E_a^{(k)}(a, z)}{h}+\frac{E_a^{(k)}(a+h, z+ip)-E_a^{(k)}(a, z+ip)}{h}$$
$$-i\frac{E_a^{(k)}(a+h, z+p)-E_a^{(k)}(a, z+p)}{h}=0\,.$$

Tending h to 0, we get

$$(i-1)\,E_a^{(k+1)}(a, z)+E_a^{(k+1)}(a, z+ip)-iE_a^{(k+1)}(a, z+p)=0\,.$$

Thus, $M_p E_a^{(k+1)}(a, z)=0$ for fixed $z\in D$.

PROPOSITION 10. $(D_z-a)^n \frac{d^{n-1}}{da^{n-1}} e_p^{a,z}=0$ *for* $n=1, 2, 3, \cdots$.

PROOF. It is true for $n=1$. Suppose it is true for $n=k$, i. e.

$$(D_z-a)^k \frac{d^{k-1}}{da^{k-1}} e_p^{a,z}=0\,.$$

Fixing z and differentiating with respect to a, we have

$$(D_z-a)^k \frac{d^k}{da^k} e_p^{a,p}-k(D_z-a)^{k-1} \frac{d^{k-1}}{da^{k-1}} e_p^{a,z}=0\,.$$

Applying D_z-a, we have

$$(D_z-a)^{k+1} \frac{d^k}{da^k} e_p^{a,z}=k(D_z-a)^k \frac{d^{k-1}}{da^{k-1}} e_p^{a,z}=0\,.$$

By induction, the proof is complete. In summary of the above developments we have

THEOREM 4. *p monodiffric homogeneous difference equation of the n'th order $\sum_{k=0}^{n}(-1)^k C_k^n (1+ap)^k f(z+(n-k)p)=0$ has p monodiffric general solution $f(z)=\sum_{k=0}^{n-1} c_k \frac{d^k}{da^k} e_p^{a,z}$, where the coefficients c_k $(k=0, 1, \cdots, n-1)$ are arbitrary constants.*

THEOREM 5. *The general solution to the homogeneous p monodiffric difference equation of the n'th order*

$$F^{(n)}(z)+c_{n-1}F^{(n-1)}(z)+\cdots+c_1 F'(z)+c_0 F(z)=0$$

is $F(z)=\sum_{k=1}^{p}\sum_{j=0}^{m_k-1} B_{k,j} \frac{d^j}{da_k^j} e_p^{a_k,z}$,

where $a_1, a_2, \cdots, a_p$ with multiplicities $m_1, m_2, \cdots, m_p$ respectively are the roots of $a^n+c_{n-1}a^{n-1}+\cdots+c_1 a+c_0=0$ and the coefficients $B_{k,j}$ are arbitrary constants.

References

[1] G. BERZSENYI: Line integrals for monodiffric functions, J. Math. Anal. Appl. 30 (1970), 99–112.

[2] G. BERZSENYI: Convolution products of monodiffric functions, J. Math. Anal. Appl. 37 (1972), 271–287.

[3] G. J. KUROWSKI: Further results in the theory of monodiffric functions, Pacific J. Math. 18 (1966), 139–147.

[4] S. T. TU: Higher order monodiffric difference equations, Hokkaido Math. J. 7 (1978), 43–48.

[5] S. T. TU: Monodiffric function theory and formal power series, J. Math. Anal. Appl. 84 (1981), 595–613.

[6] R. P. ISAACS: Monodiffric functions, Natl. Bur. Stand. Appl. Math. Ser. 18 (1952), 257–266.

Department of Mathematics
Chung Yuan Christian University