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Separable extensions of noncommutative rings

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1. Introduction. Separable extensions of noncommutative rings were introduced in 1966 by K. Hirata and K. Sugano [4]. In [1] Hirata isolated a special class of separable extensions, now known as H-separable extensions. These have been studied extensively in a series of papers over the last fifteen years, notably by Hirata and Sugano, themselves.

A ring A is an *H-separable* extension of a subring R if $A\otimes_R A$ is isomorphic as A, A-bimodule to a direct summand of A^n, for some positive integer n. An H-separable extension is *separable*; i.e. the multiplication map $A\otimes_R A\rightarrow A$ splits. In the case of algebras over commutative rings, H-separable extensions are closely related to Azumaya algebras. In this case, A is an H-separable extension of R if A is an Azumaya algebra over a (commutative) epimorphic extension of R.

If A is a ring with subring R we denote by C the center of A and $\Delta=A^R$, the centralizer of R in A. Then A is an H-separable extension of R if and only if Δ is finitely generated and projective as C-module, and the map ϕ: $A\otimes_R A\rightarrow \mathrm{Hom}_C(\Delta, A)$ defined by $\phi(a\otimes b)(d)=adb$, for a, $b\in A$, $d\in\Delta$, is an isomorphism. There are similarly defined maps $\Delta\otimes_C A\rightarrow \mathrm{Hom}({}_RA, {}_RA)$, $A\otimes_C\Delta\rightarrow \mathrm{Hom}(A_R, A_R)$, and $\Delta\otimes_C\Delta\rightarrow \mathrm{Hom}({}_RA_R, {}_RA_R)$, all of which are isomorphisms when A is H-separable over R. (See [12].)

In Sections 3 and 4 of this paper we generalize H-separability in two directions. We call A a *strongly separable* extension of R if $A\otimes_R A\cong K\oplus L$, where $\mathrm{Hom}_{A,A}(K, A)=(0)$ and L is a direct summand of A^n, for some positive integer n. H-separability is the case where $K=(0)$. Strong separability is equivalent to separability for algebras over a commutative ring, but not in general. We show that A is strongly separable over R if and only if Δ_C is finitely generated and projective and the map ϕ defined above is a split epimorphism. The three maps above which are isomorphisms in the H-separable case are split monomorphisms when strong separability is assumed.

If σ is an automorphism of A, denote by A_σ the A, A-bimodule which as left A-module is just A but whose right A-module structure is "twisted" by σ. Then A is a *psuedo-Galois* extension of R if there is a finite set S of R-automorphisms of A such that $A\otimes_R A$ is a direct summand of $\sum_{\sigma\in S}\oplus A_\sigma{}^n$, some positive integer n. H-separability is the case $S=\{1\}$. When A is a Galois extension of R, it is pseudo-Galois, and this is the motivation for the name.

Assume A is a pseudo-Galois extension of R and that for all $\sigma, \tau \in \mathrm{Aut}_R(A)$ any nonzero A, A-bimodule map from A_σ to A_τ is an isomorphism. Then there is a positive integer n and a finite subset S of $\mathrm{Aut}_R(A)$ containing exactly one element from each coset of the subgroup I of inner automorphisms such that $A\otimes_R A \cong \sum_{\sigma\in S} \oplus \mathrm{Hom}_C(\Delta, A_\sigma)$, and $\mathrm{Hom}_C(\Delta, A_\sigma)$ is isomorphic to a direct summand of A_σ^n, each $\sigma\in S$. Under these assumptions, A is strongly separable over R.

In Section 2 we show that if A is an H-separable extension of R which is generated over R by the centralizer Δ of R, and if R contains the center C of A, then Δ is an Azumaya algebra over C and $A\cong\Delta\otimes_C R$. This conclusion has been obtained for H-separable extensions under other hypotheses by Hirata [2].

2. Assume A is a separable extension of R and let M be a left or right A-module. Sugano [12] has shown that if M is projective (injective) as R-module then it is also projective (injective) as A-module. An immediate consequence of this is that a separable extension of a semisimple artinian ring is also semisimple artinian. Sugano has shown further that if A is flat as left or right R-module, then A is quasi-Frobenius if R is. A related result is the following.

PROPOSITION 2.1. *Let A be a separable extension of R such that A is flat as left* (*resp. right*) *R-module. Then A is left* (*resp. right*) *perfect if R is.*

PROOF. Recall that a ring is left perfect if every flat left module is projective. Assume R is left perfect and M is a flat left A-module. We show that ${}_RM$ is flat. Let $(0)\rightarrow N\rightarrow N'$ be an exact sequence of right R-modules. Then $(0)\rightarrow N\otimes_R A\rightarrow N'\otimes_R A$ is exact because ${}_RA$ is flat. Thus $(0)\rightarrow N\otimes_R A\otimes_A M\rightarrow N'\otimes_R A\otimes_A M$ is exact, by the flatness of ${}_AM$. So $(0)\rightarrow N\otimes_R M\rightarrow N'\otimes_R M$ is exact, and ${}_RM$ is flat. Then ${}_RM$ is projective because R is perfect, and ${}_AM$ is projective by the result mentioned above. Therefore A is left perfect.

If Δ is an Azumaya algebra over its center C and R is a central C-algebra, then it is easy to see that $A=\Delta\otimes_C R$ is an H-separable extension of R. Furthermore, the centralizer of R in A is Δ. The following Theorem is a converse to this observation.

THEOREM 2.2. *Let A be an H-separable extension of a subring R such* *that A is generated over R by its centralizer Δ in A. Assume that the center C of A is contained in R. Then Δ is an Azumaya algebra over C, and $A \cong \Delta \otimes_C R$.*

PROOF. First, define ϕ: $\Delta \otimes_C A \to A \otimes_R A$ by $\phi(d \otimes a) = d \otimes a \in A \otimes_R A$. This map is well-defined because $C \subseteq R$.

Since $A = \Delta R$, each element of $A \otimes_R A$ can be written in the form $\sum_i d_i \otimes b_i$, with $d_i \in \Delta$, $b_i \in A$.

Define ψ: $A \otimes_R A \to \Delta \otimes_C A$ by $\psi(\sum_i d_i \otimes b_i) = \sum_i d_i \otimes b_i \in \Delta \otimes_C A$. We need to show that ψ is well-defined. Assume $\sum_i d_i \otimes b_i = 0$ in $A \otimes_R A$. Since A is H-separable over R, $A \otimes_R A \cong \mathrm{Hom}_C(\Delta, A)$ under the map $a \otimes b \mapsto [d \mapsto adb]$, and $\Delta \otimes_C A \cong \mathrm{Hom}({}_R A, {}_R A)$ under the map $d \otimes b \mapsto [x \mapsto dxb]$. From $\sum_i d_i \otimes b_i = 0$ in $A \otimes_R A$ we have $\sum_i d_i d b_i = 0$, for all $d \in \Delta$. Let $x \in A$, and write $x = \sum_j r_j e_j$, $r_j \in R$, $e_j \in \Delta$. Then $\sum_i d_i x b_i = \sum_{i,j} d_i r_j e_j b_i = \sum_j r_j \sum_i d_i e_j b_i = 0$. Thus $\sum d_i \otimes b_i$ determines the zero element of $\mathrm{Hom}({}_R A, {}_R A)$, and so $\sum d_i \otimes b_i = 0$ in $\Delta \otimes_C A$. It follows that ψ is a well-defined map. Clearly, ϕ and ψ are inverse isomorphisms, $A \otimes_R A \cong \Delta \otimes_C A$.

Since A is H-separable over R, Δ_C is finitely generated and projective, hence flat. Thus,

$(0) \to R \to A$ exact yields $(0) \to \Delta \otimes_C R \to \Delta \otimes_C A \cong A \otimes_R A$ exact. So the natural map $\Delta \otimes_C R \to A \otimes_R A$ is injective. The multiplication map f: $\Delta \otimes_C R \to A$, $d \otimes r \mapsto dr$, is surjective by hypothesis. We show f is also injective. Assume $\sum_i d_i r_i = 0$, $d_i \in \Delta$, $r_i \in R$. Then under the injective map $\Delta \otimes_C R \to A \otimes_R A$, $\sum_i d_i \otimes r_i \mapsto \sum_i d_i r_i \otimes 1 = 0$. Hence, $\sum_i d_i \otimes r_i = 0$ in $\Delta \otimes_C R$, and f is injective. This proves $A \cong \Delta \otimes_C R$.

From $A \cong \Delta \otimes_C R$ it follows easily that C is the center of Δ. Also, C is a direct summand of Δ_C (see, for example, Hirata [1], p. 112), which implies that ${}_R R_R$ is a direct summand of ${}_R(\Delta \otimes_C R)_R = {}_R A_R$.

So we can apply Prop. 4.7 of [2] to conclude that Δ is an Azumaya algebra over C.

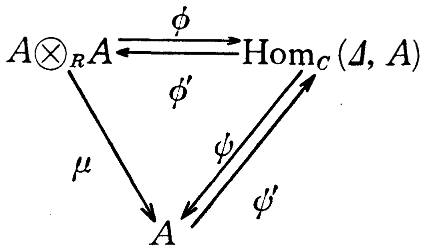
3. *Strongly separable extensions.* Many results which hold for H-separable extensions can be extended in weakened form to a much larger class of separable extnsions, which we call *strongly separable.*

DEFINITION 3.1. A is said to be *strongly separable* over R provided Δ is finitely generated and projective as C-module, and the map ϕ: $A \otimes_R A \to \mathrm{Hom}_C(\Delta, A)$ is surjective and splits.

An H-separable extension is strongly separable, and we show now that a strongly separable extension is separable. We will also see that for an algebra over a commutative ring, strong separability and separability are equivalent. We will present an example to show that this equivalence does not hold in general.

PROPOSITION 3. 2. *If A is strongly separable over R than A is separable over R.*

PROOF. Since Δ_C is finitely generated and projective, C is a direct summand of Δ_C (see, for example, Hirata [1], p. 112). Thus the map ψ: $\mathrm{Hom}_C(\Delta, A)\rightarrow A$, $f\mapsto f(1)$, splits as A, A-bimodule map. Let ψ' be the splitting map. Also, let ϕ' be the splitting map for ϕ. We have the commutative diagram



and it is seen that the map μ is split by $\phi'\circ\psi'$. Hence A is separable over R.

PROPOSITION 3. 3. *If A is a separable algebra over a commutative ring R then A is strongly separable over R.*

PROOF. We have $R\subseteq C\subseteq A$; hence $\Delta=A$. Since A is separable over R it is an Azumaya algebra over C. Hence A_C is faithfully projective and finitely generated, and $A\otimes_C A$ is isomorphic to $\mathrm{Hom}_C(A, A)=\mathrm{Hom}_C(\Delta, A)$. Also, C is separable over R; so the sequence $C\otimes_R C\rightarrow C\rightarrow(0)$ is split exact. Tensoring on the left and right with A over C, we obtain the split exact sequence $A\otimes_R A\rightarrow A\otimes_C A\rightarrow(0)$. The diagram

$$\begin{array}{ccc} A\otimes_R A & \longrightarrow & A\otimes_C A \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}_C(A, A) & \end{array}$$

is commutative. So the sequence $A\otimes_R A\rightarrow\mathrm{Hom}_C(A, A)$ splits, and A is strongly separable over R. This completes the proof.

The following lemma is well-known and is stated here without proof.

LEMMA 3. 4. *Let S and T be rings; let U be a right S-module, V an S, T-bimodule, and W a left T-module. There are canonical maps:*

$$U\otimes_S\mathrm{Hom}_T(V, W)\longrightarrow\mathrm{Hom}_T\big(\mathrm{Hom}_S(U, V), W\big),\quad u\otimes f\mapsto\big[g\mapsto g(u)f\big],$$

and

$$\mathrm{Hom}_S(V, U)\otimes_T W \longrightarrow \mathrm{Hom}_S\left(\mathrm{Hom}_T(W, V), U\right),\ f\otimes w \mapsto \left[g\mapsto f(wg)\right].$$

If U_S is finitely generated and projective, the first map is an isomorphism; if ${}_TW$ is finitely generated and projective, the second map is an isomorphism.

A is H-separable over R if and only if $A\otimes_R A$ is a bimodule direct summand of A^n, for some positive integer n. The following result gives an analogous characterization of strong separability.

THEOREM 3.5. *Let R be a subring of a ring A. Then the following conditions are equivalent:*

(1) *A is strongly separable over R.*

(2) *There exist $d_i\in\Delta$, $\sum_j a_{ij}\otimes b_{ij}\in(A\otimes_R A)^A$, $1\leq i\leq n$, such that $d=\sum_{i,j} d_i a_{ij} d b_{ij}$ for any $d\in\Delta$.*

(3) *$A\otimes_R A=K\oplus M$, where $\mathrm{Hom}_{A,A}(K, A)=(0)$ and M is isomorphic to an A, A-direct summand of A^n.*

PROOF. ((1)$\Rightarrow$(3)) Assume A is strongly separable over R. Then there is a split exact sequence $C^n\to\Delta\to(0)$ of C-modules, because Δ_C is finitely generated and projective. This yields a split exact sequence $(0)\to\mathrm{Hom}_C(\Delta, A)\to\mathrm{Hom}_C(C^n, A)\cong A^n$ of A, A-bimodules. Let $K=\ker(\phi)$. Since

$$(0)\longrightarrow K\longrightarrow A\otimes_R A\xrightarrow{\ \phi\ }\mathrm{Hom}_C(\Delta, A)\longrightarrow(0)$$

splits, K is a direct summand of $A\otimes_R A$ such that $A\otimes_R A/K\cong\mathrm{Hom}_C(\Delta, A)$. We need to show that $\mathrm{Hom}_{A,A}(K, A)=(0)$.

We apply Lemma 3.4 with $S=C$, $T=A\otimes_C A$, $U=\Delta$, $V=A$, $W=A$, noting that Δ_C is finitely generated and projective as required. Then

$$\Delta\otimes_C\mathrm{Hom}_{A\otimes_C A}(A, A)\cong\mathrm{Hom}_{A\otimes_C A}\left(\mathrm{Hom}_C(\Delta, A), A\right).$$

But $\Delta\otimes_C\mathrm{Hom}_{A\otimes_C A}(A, A)\cong\Delta\otimes_C C\cong\Delta$. By hypothesis, $A\otimes_R A\cong\mathrm{Hom}_C(\Delta, A)\oplus K$. Thus we have the following sequence of isomorphisms:

$$\begin{aligned}\Delta&\cong\mathrm{Hom}_{A\otimes_C A}(A\otimes_R A, A)\cong\mathrm{Hom}_{A\otimes_C A}\left(\mathrm{Hom}_C(\Delta, A), A\right)\oplus\mathrm{Hom}_{A\otimes_C A}(K, A)\\&\cong\Delta\oplus\mathrm{Hom}_{A\otimes_C A}(K, A).\end{aligned}$$

By tracing these isomorphisms through, one checks that the composite is the identity map on Δ. Hence

$$\mathrm{Hom}_{A\otimes_C A}(K, A)=\mathrm{Hom}_{A,A}(K, A)=(0).$$

((3)$\Rightarrow$(2)) Writing $A\otimes_R A=K\oplus M$, $M\oplus B\cong A^n$, we get an A, A-map from $A\otimes_R A$ into A^n by projecting $A\otimes_R A$ onto M and injecting M into A^n. Let $1\otimes 1\mapsto u\in M$, $u\mapsto(d_i)\in A^n$. Note that $1\otimes r=r\otimes 1$, all $r\in R$, implies $d_i\in\Delta$, each i. Let $e_i\in A^n$ be the element whose ith coordinate is $1\in A$ and whose other coordinates are zero. Let $m_i+b_i\mapsto e_i$ under the isomorphism $M\oplus B\to A^n$. Then $\sum d_i m_i+d_i b_i\mapsto(d_i)$. Thus $u-\sum d_i m_i-\sum d_i b_i\mapsto 0$ in A^n. It follows that $u-\sum d_i m_i=0$ in M, and $\sum d_i b_i=0$ in B.

Under the projection $A^n\to M$, $e_i\mapsto m_i$; so $ae_i=e_i a\mapsto am_i=m_i a$, all $a\in A$. Thus $m_i\in(A\otimes_R A)^A$, all i. Write $m_i=\sum_j a_{ij}\otimes b_{ij}\in A\otimes_R A$. Then $1\otimes 1-u\in K$, and $1\otimes 1-u=1\otimes 1-\sum_i d_i m_i=1\otimes 1-\sum_{i,j} d_i a_{ij}\otimes b_{ij}$.

Now $K\to A\otimes_R A\xrightarrow{\phi}\mathrm{Hom}_C(\Delta, A)\to A^n$, and the first and third maps are injective. Since $\mathrm{Hom}_{A,A}(K, A)=(0)$, we must have $K\subseteq\ker(\phi)$. Therefore $0=\phi(1\otimes 1-u)=\phi(1\otimes 1)-\phi(\sum_{i,j} d_i a_{ij}\otimes b_{ij})$. This says $d=\sum_{i,j} d_i a_{ij} d b_{ij}$, all $d\in\Delta$.

((2)⇒(1)) Note that $\sum_j a_{ij}\otimes b_{ij}\in(A\otimes_R A)^A$ implies $\sum_j a_{ij} d b_{ij}\in C$, for all i and all $d\in\Delta$. Let $f_i\in\mathrm{Hom}_C(\Delta, C)$ be defined by $f_i(d)=\sum_j a_{ij} d b_{ij}$, $d\in\Delta$, for each i. Then $d=\sum_i d_i f_i(d)$, all $d\in\Delta$. It is well-known that this implies Δ_C is finitely generated and projective.

Define ψ: $\mathrm{Hom}_C(\Delta, A)\to A\otimes_R A$ by $f\mapsto\sum_{i,j} f(d_i)\, a_{ij}\otimes b_{ij}$. For all $a\in A$,

$$\psi(af)=\sum_{i,j} af(d_i)\, a_{ij}\otimes b_{ij}=a\psi(f)\,, \quad\text{and}$$
$$\psi(fa)=\sum_{i,j}(fa)(d_i)\, a_{ij}\otimes b_{ij}=\sum_{i,j} f(d_i)\, aa_{ij}\otimes b_{ij}$$
$$=\sum_{i,j} f(d_i)\, a_{ij}\otimes b_{ij} a=\psi(f)\, a\,.$$

Hence ψ is an A, A-map. Furthermore,

$$\phi\psi(f)(d)=\sum_{i,j} f(d_i)\, a_{ij} d b_{ij}=f(\sum_{i,j} d_i a_{ij} d b_{ij})=f(d)\,,$$

since $\sum_j a_{ij} d b_{ij}\in C$, all i. Hence $\phi\circ\psi$ is the identity map on $\mathrm{Hom}_C(\Delta, A)$; i. e. ψ splits the exact sequence $A\otimes_R A\xrightarrow{\phi}\mathrm{Hom}_C(\Delta, A)\to(0)$. It follows that A is strongly separable over R, and the Theorem is proved.

We are indebted to K. Sugano for an example of a separable ring extension that is not strongly separable. The following example is a variant of the one which he provided.

EXAMPLE 3.6. Let $G=\{e, p, p^2\}$, a three element group, and let K be the Galois field with three elements. Let $S=KG$, the group algebra of G over K. For $s=ae+bp+cp^2\in S$, define $\bar{s}=ae+bp^2+cp$. The map $s\mapsto\bar{s}$ is an automorphism of S.

Let A be the 2×2 matrix ring over S, $R=\left\{\begin{bmatrix} s & 0 \\ 0 & \bar{s}\end{bmatrix} \mid s\in S\right\}\subseteq A$, and $T=\{a(e+p+p^2)\mid a\in K\}\subseteq S$. Since S is commutative, the center C of A is $C=\left\{\begin{bmatrix} s & 0 \\ 0 & s\end{bmatrix} \mid s\in S\right\}$. It is straightforward to verify that the centralizer Δ of R in A is $\Delta=\left\{\begin{bmatrix} s_1 & t_1 \\ t_2 & s_2\end{bmatrix} \mid s_1, s_2\in S,\ t_1, t_2\in T\right\}$. Let $\sigma=e+p+p^2\in S$. As C-module, $C\begin{bmatrix} 0 & \sigma \\ 0 & 0\end{bmatrix}$ is a direct summand of Δ. Thus if Δ_C were projective, $C\begin{bmatrix} 0 & \sigma \\ 0 & 0\end{bmatrix}$ would be also, and $S\sigma$ would be projective as S-module. That this is not the case is seen as follows.

Let $\varepsilon: S\to S\sigma$, $s\mapsto s\sigma$, a surjective map of S-modules. If $S\sigma$ is projective, there is a splitting map τ. If $\tau(\sigma)=u$, $S=\ker(\varepsilon)\oplus Su$. Then $\varepsilon(\sigma u)=\sigma\varepsilon(u)=\sigma\varepsilon\tau(\sigma)=\sigma^2$. But $\sigma^2=0$. So $\sigma u\in\ker(\varepsilon)\cap Su=(0)$; $\sigma u=0$. Then $u\in\ker(\varepsilon)\cap Su$; i.e. $u=0$, a contradiction.

Since Δ_C is not projective, A is not strongly separable over R. However A is separable over R. The element $\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}\otimes\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}\otimes\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}$ is in the A-center of $A\otimes_R A$ and is mapped to the unity element of A by $\mu: A\otimes_R A\to A$.

We now return to our general setting where R is a subring of A, Δ is the centralizer of R in A and $\phi: A\otimes_R A\to \mathrm{Hom}_C(\Delta, A)$. The proof of the following Lemma is straightforward and is omitted.

LEMMA 3.7. *Let* $K=\ker(\phi)$ *and* $S=\{s\in A \mid s\otimes 1-1\otimes s\in K\}$. *Then* S *and* Δ *are centralizers of each other in* A.

With S as defined in the Lemma we have

PROPOSITION 3.8. *If A is strongly separable over R then A is strongly separable over S. If S is separable over R then S is strongly separable over R.*

PROOF. Since $A^S=\Delta$, and Δ_C is finitely generated and projective by hypothesis, to prove the first statement we need only show that the map ϕ: $A\otimes_S A\to\mathrm{Hom}_C(\Delta, A)$ splits. Let ϕ' be the spliting map of ϕ: $A\otimes_R A\to \mathrm{Hom}_c(\Delta, A)$, and let $f: A\otimes_R A\to A\otimes_S A$ be the natural map defined because $R\subseteq S$. Then $f\circ\phi'$ is a splitting map for ϕ.

Assume S is separable over R and let C' denote the center of S. Then $C'=\Delta\cap S$, since $A^S=\Delta$. Furthermore, $\Delta'=S^R=C'$. So $\Delta'_{c'}$ is trivially finitely generated and projective. Also $\mathrm{Hom}_{C'}(\Delta', S)\cong S$; so the splitting of $S\otimes_R S\to \mathrm{Hom}_{C'}(\Delta', S)$ is equivalent to the splitting of $S\otimes_R S\to S\to(0)$.

The following proposition is the analogue for strong separability of 1.5 of [12].

PROPOSITION 3.9. *If A is strongly separable over R, then each of the following maps is a split monomorphism*:

(i) $\Delta\otimes_C A\rightarrow \mathrm{Hom}({}_RA, {}_RA)$, $d\otimes a\mapsto[x\mapsto dxa]$,

(ii) $A\otimes_C\Delta\rightarrow \mathrm{Hom}(A_R, A_R)$, $a\otimes d\mapsto[x\mapsto axd]$,

(iii) $\Delta\otimes_C\Delta\rightarrow \mathrm{Hom}_{R,R}(A, A)$, $d_i\otimes d_2\mapsto[x\mapsto d_i x d_2]$.

PROOF. (i) Using Lemma 3.4 we obtain $\Delta\otimes_C A\cong \mathrm{Hom}({}_A\mathrm{Hom}_C(\Delta\ A), {}_AA)$. Since A is strongly separable, $A\otimes_R A\cong K\oplus \mathrm{Hom}_C(\Delta, A)$. Applying these isomorphisms and the Adjoint Functor Theorem, we have

$$\mathrm{Hom}({}_RA, {}_RA)\cong \mathrm{Hom}\left({}_RA, {}_R\mathrm{Hom}({}_AA, {}_AA)\right)\cong \mathrm{Hom}\left({}_A(A\otimes_R A), {}_AA\right)\cong$$

$$\mathrm{Hom}({}_AK, {}_AA)\oplus \mathrm{Hom}\left({}_A\mathrm{Hom}_C(\Delta, A), {}_AA\right)\cong$$

$$\mathrm{Hom}({}_AK, {}_AA)\oplus\Delta\otimes_C A\xrightarrow{\ \pi\ }\Delta\otimes_C A\,,$$

where the last map π is the projection map arising from the direct sum decomposition. Tracing through these maps one checks that the composite is a splitting map for the map in (i).

(ii) The proof is similar to the proof of (i).

(iii) In the proof of part (i) above the isomorphism of $\mathrm{Hom}({}_RA, {}_RA)$ onto $\mathrm{Hom}({}_A(A\otimes_R A), {}_AA)$ maps $\mathrm{Hom}({}_RA_R, {}_RA_R)$ onto $\mathrm{Hom}({}_A(A\otimes_R A)_R, {}_AA_R)$. So we have

$$\mathrm{Hom}_{R,R}(A, A)\cong \mathrm{Hom}({}_AK_R, {}_AA_R)\oplus \mathrm{Hom}\left({}_A\mathrm{Hom}_C(\Delta, A)_R, {}_AA_R\right).$$

Since $\Delta\cong \mathrm{Hom}({}_AA_R, {}_AA_R)$, we can apply Lemma 3.4 to obtain

$$\Delta\otimes_C\Delta\cong \mathrm{Hom}_{A,R}(A, A)\otimes_C\Delta\cong \mathrm{Hom}_{A,R}\left(\mathrm{Hom}_C(\Delta, A), A\right).$$

This proves (iii).

PROPOSITION 3.10. *Assume A is strongly separable over R, $A\otimes_R A\cong \mathrm{Hom}_C(\Delta, A)\oplus K$. Then for every A, A-bimodule M, $M^R\cong\Delta\otimes_C M^A\oplus \mathrm{Hom}_{A,A}(K, M)$. In particular, $(A\otimes_R A)^R\cong\Delta\otimes_C(A\otimes_R A)^A\oplus \mathrm{Hom}_{A,A}(K, A\otimes_R A)$.*

PROOF. From Lemma 3.4 we have

$$\Delta\otimes_C M^A\cong\Delta\otimes_C \mathrm{Hom}_{A,A}(A, M)\cong \mathrm{Hom}_{A,A}\left(\mathrm{Hom}_C(\Delta, A), M\right).\quad \text{Then}$$

$$M^R\cong \mathrm{Hom}_{A,A}(A\otimes_R A, M)\cong \mathrm{Hom}_{A,A}\left(\mathrm{Hom}_C(\Delta, A), M\right)\oplus \mathrm{Hom}_{A,A}(K, M)$$
$$\cong\Delta\otimes_C M^A\oplus \mathrm{Hom}_{A,A}(K, M).$$

4. *Automorphisms.* If σ is an automorphism of a ring A we let A_σ denote the A, A-bimodule such that as left A-module A_σ is just A, but where the right module structure is "twisted" by σ, $x\cdot a=x\sigma(a)$ for $x\in A_\sigma$, $a\in A$.

If R is a subring of A then A is a *Galois extension* of R if there is a finite group G of automorphisms of A such that $R=A^G$, and such that there exists x_i, y_i, $1\leqslant i\leqslant n$, for which

$$\sum_i x_i\sigma(y_i)=\begin{cases}0 \text{ if } \sigma\neq 1\\ 1 \text{ if } \sigma=1\,.\end{cases}$$

If G is a finite group of R-automorphisms of A there is an A, A-bimodule map $h: A\otimes_R A\rightarrow AG$, defined by $a\otimes b\rightarrow\sum_{\sigma\in G} a\sigma b=\sum_{\sigma\in G} a\sigma(b)\sigma$. Here, AG is the twisted group algebra of G over A. It can be shown that if $R=A^G$ then A is a Galois extension of R if and only if h is an isomorphism.

The twisted group algebra AG is a direct sum $\sum_{\sigma\in G}\oplus A\sigma$, and, for each σ, $A\sigma$ is A, A-bimodule isomorphic to A_σ. Thus when A is a Galois extension of R with Galois group G, $A\otimes_R A\cong\sum_{\sigma\in G}\oplus A_\sigma$. This motivates the following definition.

Definition 4.1. A is a *pseudo-Galois* extension of R if there is a finite set S of R-automorphisms of A and a positive integer n such that $A\otimes_R A$ is isomorphic to a direct summand of $\sum_{\sigma\in S} A_\sigma{}^n$.

If in Definition 4.1 $S=\{1\}$, then the condition is that $A\otimes_R A$ is isomorphic to a direct summand of A^n, for some positive integer n. This is just the condition that A be H-separable over R. Thus H-separable extensions and Galois extensions are pseudo-Galois.

In Definition 4.1 we will assume that $A_\sigma\not\cong A_\tau$ if $\sigma\neq\tau$, σ, $\tau\in S$.

Assume that σ and τ are automorphisms of a ring A and let $\mu: A_\sigma\rightarrow A_\tau$ be an A, A-bimodule map. Let $\mu(1)=x$. Then, for each $a\in A$,

$$\sigma(a)\,x=\sigma(a)\,\mu(1)=\mu\sigma(a)=\mu(1\cdot a)=\mu(1)\cdot a=x\tau(a)\,.$$

Conversely, if $x\in A$ such that $\sigma(a)\,x=x\tau(a)$ for all $a\in A$, there is a unique bimodule map $\mu: A_\sigma\rightarrow A_\tau$ such that $\mu(1)=x$. The map μ is an isomorphism if and only if x is a unit in A, and in this case $\tau\sigma^{-1}$ is an inner automorphism of A. Conversely, if $\tau\sigma^{-1}(a)=x^{-1}ax$, for some unit x in A then $\sigma(a)\,x=x\tau(a)$ and there is a unique isomorphism $\mu: A_\sigma\rightarrow A_\tau$ such that $\mu(1)=x$. Let

$$J_{\sigma,\tau}=\left\{x\in A\,|\,\sigma(a)\,x=x\tau(a),\text{ all } a\in A\right\}.$$

Then $J_{\sigma,\tau}$ is a C-module and $J_{\sigma,\tau}\cong\mathrm{Hom}_{A,A}(A_\sigma, A_\tau)$; $A_\sigma\cong A_\tau$ if and only if

$\sigma\tau^{-1}$ is an inner automorphism of A. We will denote $J_{1,\sigma}$ by J_{σ}. Then $J_{\sigma,\tau}=J_{\sigma\tau^{-1}}$.

In part of what follows we will assume that any nonzero A, A-bimodule map from A_{σ} to A_{τ} is an automorphism. This condition holds, for example, if A is a simple ring.

PROPOSITION 4.2. *Let A be a pseudo-Galois extension of R, and assume that if σ, $\tau\in \mathrm{Aut}_R(A)$ that any nonzero bimodule map from A_{σ} to A_{τ} is an isomorphism. Then*

(i) *$\mathrm{Hom}_C(\Delta, A_{\sigma})$ is isomorphic to a direct summand of A_{σ}^n, each $\sigma\in S$.*

(ii) *If I is the group of inner R-automorphisms of A then S contains exactly one element from each coset of I in* $\mathrm{Aut}_R(A)$.

(iii) $A\otimes_R A\cong \sum_{\sigma\in S}\oplus \mathrm{Hom}_C(\Delta, A_{\sigma})$.

PROOF. Let $\sum_{\sigma\in S}\oplus A_{\sigma}^n\cong A\otimes_R A\oplus B$. Then

$$\mathrm{Hom}_{A,A}(A\otimes_R A, A)\oplus \mathrm{Hom}_{A,A}(B, A)\cong \mathrm{Hom}_{A,A}(\sum_{\sigma\in S}\oplus A_{\sigma}^n, A)\,;$$

i.e. $\Delta\oplus B'\cong \sum_{\sigma\in S}\oplus\ \mathrm{Hom}_{A,A}(A_{\sigma}^n, A)$. There must exist $\sigma_0\in S$ such that $\mathrm{Hom}_{A,A}(A_{\sigma_0}, A)\neq(0)$. Hence $A_{\sigma_0}\cong A$, and $\mathrm{Hom}_{A,A}(A_{\sigma_0}^n, A)\cong C^n$. For $\sigma\neq\sigma_0$, $\mathrm{Hom}_{A,A}(A_{\sigma}, A)=(0)$; hence $\Delta\oplus B'\cong C^n$.

Let $\tau\varepsilon \mathrm{Aut}_R(A)$. Then $A_{\tau}^n\cong \mathrm{Hom}_C(C^n, A_{\tau})\cong \mathrm{Hom}_C(\Delta, A_{\tau})\oplus \mathrm{Hom}_C(B', A_{\tau})$. Define ϕ_{τ}: $A\otimes_R A\rightarrow \mathrm{Hom}_C(\Delta, A_{\tau})$ by $\phi_{\tau}(a\otimes b)=[d\mapsto ad\tau(b)]$. Then ϕ_{τ} is an A, A-map, where $\mathrm{Hom}_C(\Delta, A_{\tau})$ is an A, A-bimodule via the action on A_{τ}. We then have a sequence of bimodule maps

$$\sum_{\sigma\in S}\oplus A_{\sigma}^n\longrightarrow A\otimes_R A\xrightarrow{\phi_{\tau}}\mathrm{Hom}_C(\Delta, A_{\tau})\longrightarrow A_{\tau}^n\,,$$

whose composition is nonzero. Hence there exists $\sigma'\in S$ such that $A_{\sigma'}\cong A_{\tau}$. Then S contains exactly one element from each coset of I in $\mathrm{Aut}_R(A)$.

Let p denote the split injective mapping of $A\otimes_R A$ to $\sum_{\sigma\in S}\oplus A_{\sigma}^n$ assumed to exist because A is a psuedo-Galois extension of R, and let p' denote the splitting map. Foe each $\tau\in S$ let u_{τ}, $v_{\tau}\in A\otimes_R A$ be chosen so that $p(1\otimes 1)=u'_{\tau}+v'_{\tau}$ where $u'_{\tau}\in A_{\tau}^n$ and $v'_{\tau}\in\sum_{\sigma\neq\tau}\oplus A_{\sigma}^n$ and such that $u_{\tau}=p'(u_{\tau}')$ and $v_{\tau}=p'(v_{\tau}')$. Since $\mathrm{Hom}_C(\Delta, A_{\tau})$ is isomorphic to a sub-bimodule of A_{τ}^n we must have $\phi_{\tau}(v_{\tau})=0$. Thus $\phi_{\tau}(1\otimes 1)=\phi_{\tau}(u_{\tau})$.

For $1\leq i\leq n$ let e_i denote the element of A_{τ}^n whose i^{th} coordinate is 1 and whose j^{th} coordinate is 0, $j\neq i$, $1\leq j\leq n$. Then $u'_{\tau}=\sum_{i=1}^n d_i e_i$, $d_i\in A$. For each $r\in R$, $r\otimes 1=1\otimes r$; so $\sum_{i=1}^n rd_ie_i=\sum_{i=1}^n d_ire_i$. Thus $rd_i=d_ir$, $1\leq i\leq n$. It follows that $d_i \in \Delta$, each i.

Let $p'(e_i) = \sum_j a_{ij} \otimes b_{ij} \in A \otimes_R A$, $1 \leq i \leq n$. Note that $\tau(a) e_i = e_i \cdot a$ implies that $\tau(a) \sum_j a_{ij} \otimes b_{ij} = \sum_j a_{ij} \otimes b_{ij} a$, all $a \in A$. Mapping with ϕ_τ we obtain

$$\tau(a) \sum_j a_{ij} d\tau(b_{ij}) = \sum a_{ij} d\tau(b_{ij}) \tau(a), \text{ all } a \in A, \ d \in \Delta .$$

Thus $\sum_j a_{ij} d\tau(b_{ij}) \in C$, all $d \in \Delta$. Now $\phi_\tau(1 \otimes 1) = \phi_\tau(u_\tau) = \phi_\tau(\sum_{i,j} d_i a_{ij} \otimes b_{ij})$. Hence $d = \sum_{i,j} d_i a_{ij} d\tau(b_{ij})$, all $d \in \Delta$.

Let us now define ψ_τ: $\mathrm{Hom}_C(\Delta, A_\tau) \to A \otimes_R A$ by $\psi_\tau(f) = \sum_{i,j} f(d_i) a_{ij} \otimes b_{ij}$. ψ_τ is clearly a left A-module map. If $a \in A$, then

$$\begin{aligned} \psi_\tau(fa) &= \sum_{i,j} (fa)(d_i)\, a_{ij} \otimes b_{ij} = \sum_{i,j} f(d_i)\, \tau(a)\, a_{ij} \otimes b_{ij} \\ &= \sum_{i,j} f(d_i)\, a_{ij} \otimes b_{ij} a = \psi_\tau(f)\, a . \end{aligned}$$

Hence ψ_τ is a bimodule map.

We now show that ψ_τ splits ϕ_τ. Since $\sum_j a_{ij} d\tau(b_{ij}) \in C$, $1 \leq i \leq n$; for each $f \in \mathrm{Hom}_C(\Delta, A_\tau)$ we have

$$\sum_{i,j} f(d_i)\, a_{ij} d\tau(b_{ij}) = f\Big(\sum_{i,j} d_i a_{ij} d\tau(b_{ij})\Big) = f(d) .$$

Then $\phi_\tau \circ \psi_\tau$ is the identity map on $\mathrm{Hom}_C(\Delta, A_\tau)$. Also, it is straightforward that $\psi_\tau \circ \phi_\tau(u_\tau) = u_\tau$. Since $p(1 \otimes 1) = \sum_{\tau \in S} u'_\tau$, we have $1 \otimes 1 = \sum u_\tau = \sum \psi_\tau \circ \phi_\tau(u_\tau)$ $= \sum \psi_\tau \circ \phi_\tau(1 \otimes 1)$; and thus $\sum \psi_\tau \circ \phi_\tau$ is the identity map on $A \otimes_R A$. Therefore $A \otimes_R A \cong \sum_{\sigma \in S} \oplus \mathrm{Hom}_C(\Delta, A_\sigma)$.

COROLLARY 4.3. *Let A be a pseudo-Galois extension of R, and assume for each σ, $\tau \in \mathrm{Aut}_R(A)$ that any nonzero bimodule map from A_σ to A_τ is an isomorphism. Then A is strongly separable over R.*

PROOF. Assume $A \otimes_R A \cong \sum_{\sigma \in S} \oplus \mathrm{Hom}_C(\Delta, A_\sigma)$. Let $K = \sum_{\sigma \neq 1} \oplus \mathrm{Hom}_C(\Delta, A_\sigma)$, and apply Theorem 3.5.

We have seen that H-separable extensions are pseudo-Galois. There appears to be no general relationship between strongly separable extensions and pseudo-Glaois extensions. The following proposition for strongly separable extensions has a conclusion similar to, but weaker than, that of Proposition 4.2.

Recall that for each R-automorphism σ of A, ϕ_σ: $A \otimes_R A \to \mathrm{Hom}_C(\Delta, A_\sigma)$ is defined by $\phi_\sigma(a \otimes b) = [d \mapsto ad\sigma(b)]$. Also, let I denote the subgroup of inner automorphisms in $\mathrm{Aut}_R(A)$.

PROPOSITION 4. 4. *Let A be a strongly separable extension of R. Then for each R-automorphism σ of A the map ϕ_σ is a split epimorphism. Assume further that I is of finite index in* $\mathrm{Aut}_R(A)$ *and that if σ and τ are R-automorphisms of A then any nonzero bimodule map from A_σ to A_τ is an isomorphism. Then there exists a set S of R-automorphisms of A containing exactly one element from each coset of I in* $\mathrm{Aut}_R(A)$ *such that* $\sum_{\sigma\in S}\oplus\mathrm{Hom}_C(\Delta, A_\sigma)$ *is isomorphic to a direct summand of $A\otimes_R A$.*

PROOF. As in the proof of Theorem 3. 5, we can find elements $d_i\in\Delta$, a_{ij}, $b_{ij}\in A$ such that for each $d\in\Delta$, $d=\sum_{i,j}d_i a_{ij} d b_{ij}$, and such that $\sum_j a_{ij}\otimes b_{ij}\in(A\otimes_R A)^A$ and $\sum_j a_{ij} d b_{ij}\in C$, each i. We define ψ_σ: $\mathrm{Hom}_C(\Delta, A_\sigma)\to A\otimes_R A$ by $\psi_\sigma(f)=\sum_{i,j}f(d_i)\, a_{ij}\otimes\sigma^{-1}(b_{ij})$. ψ_σ is clearly a map of left A-modules.

We note that $1\otimes\sigma^{-1}$ is a well-defined map of abelian groups from $A\otimes_R A$ to $A\otimes_R A$. Since, for each $a\in A$,

$\sum_j \sigma(a)\, a_{ij}\otimes b_{ij}=\sum_j a_{ij}\otimes b_{ij}\sigma(a)$, we can apply $1\otimes\sigma^{-1}$ to obtain

$\sum_j \sigma(a)\, a_{ij}\otimes\sigma^{-1}(b_{ij})=\sum_j a_{ij}\otimes\sigma^{-1}(b_{ij})\, a$, each i.

We now show that ψ_σ is a right A-module map. For each $a\in A$, $f\in\mathrm{Hom}_C(\Delta, A_\sigma)$,

$$\begin{aligned}\psi_\sigma(fa)&=\sum_{i,j}(fa)(d_i)\, a_{ij}\otimes\sigma^{-1}(b_{ij})=\sum_{i,j}f(d_i)\,\sigma(a)\, a_{ij}\otimes\sigma^{-1}(b_{ij})\\&=\sum_{i,j}f(d_i)\, a_{ij}\otimes\sigma^{-1}(b_{ij})\, a=\psi_\sigma(f)\, a\,.\end{aligned}$$

Next we show that ψ_σ splits ϕ_σ. For each $f\in\mathrm{Hom}_C(\Delta, A_\sigma)$,

$$\begin{aligned}\phi_\sigma\circ\psi_\sigma(f)(d)&=\sum_{i,j}f(d_i)\, a_{ij} d\sigma\left(\sigma^{-1}(b_{ij})\right)=\sum_{i,j}f(d_i)\, a_{ij} d b_{ij}\\&=f(\sum_{i,j}d_i a_{ij} d b_{ij})=f(d)\,.\end{aligned}$$

Now, assume that if σ and τ are R-automorphisms of A then any nonzero bimodule map from $A\sigma$ to $A\tau$ is an isomorphism.

For each $\sigma\in\mathrm{Aut}_R(A)$ we write $A\otimes_R A=K_\sigma\oplus L_\sigma$ where $L_\sigma\cong\mathrm{Hom}_C(\Delta, A_\sigma)$. Since Δ_C is finitely generated and projective, L_σ is isomorphic to a direct summand of A_σ^n, some positive integer n. If σ, $\tau\in\mathrm{Aut}_R(A)$ and $A_\sigma\not\cong A_\tau$, then there is no nonzero bimodule map from A_τ to A_σ; hence $L_\tau\subseteq K_\sigma$.

Let S be a subset of $\mathrm{Aut}_R(A)$ containing exactly one element from each coset of I, and let $K'=\bigcap_{\sigma\in S}K_\sigma$. Then $A\otimes_R A=\sum_{\sigma\in S}\oplus L_\sigma\oplus K'$. This completes the proof of the Proposition.

We now drop the hypothesis that for σ, $\tau \in \mathrm{Aut}_R(A)$, any nonzero bimodule map from A_σ to A_τ is an isomorphism. Recall that for $\sigma \in \mathrm{Aut}_R(A)$, $J_\sigma = \{x \in A \mid ax = x\sigma(a)$, for all $a \in A\}$. J_σ is a C-module and $\mathrm{Hom}_{A,A}(A, A_\sigma) \cong J_\sigma$ under the map $f \mapsto f(1)$.

Hirata [3] has shown that if A is an H-separable extension of R then J_σ is finitely generated and projective of rank 1, and is free if and only if σ is an inner automorphism. In the following we generalize to strongly separable extensions.

We assume throughout the rest of this section that A has no nontrivial central idempotents.

PROPOSITION 4.5. *Assume A is strongly separable over R; i.e.* $(0) \to K \to A \otimes_R A \to \mathrm{Hom}_C(\Delta, A) \to (0)$ *is a split exact sequence. Then* $J_\sigma \neq (0)$ *if and only if* $\mathrm{Hom}_{A,A}(K, A_\sigma) = (0)$. *In this case J_σ is finitely generated and projective of rank* 1.

PROOF. From Lemma 3.4,
$\mathrm{Hom}_{A,A}(\mathrm{Hom}_C(\Delta, A), A_\sigma) \cong \Delta \otimes_C \mathrm{Hom}_{A,A}(A, A_\sigma) \cong \Delta \otimes_C J_\sigma$. Hence

$$\Delta \cong \mathrm{Hom}_{A,A}(A \otimes_R A, A_\sigma) \cong \mathrm{Hom}_{A,A}\left(\mathrm{Hom}_C(\Delta, A), A_\sigma\right) \oplus \mathrm{Hom}_{A,A}(K, A_\sigma)$$
$$\cong \Delta \otimes_C J_\sigma \oplus \mathrm{Hom}_{A,A}(K, A_\sigma).$$

From this we see first that $J_\sigma = (0)$ implies $\mathrm{Hom}_{A,A}(K, A_\sigma) \neq (0)$. Next since C is a direct summand of Δ, we conclude from the above that J_σ is a direct summand of Δ. So J_σ is finitely generated and projective. Finally let $t = \mathrm{rank}(J_\sigma)$, $n = \mathrm{rank}(\Delta)$. Then, from the above, we have $n = n \cdot t + \mathrm{rank}(\mathrm{Hom}_{A,A}(K, A_\sigma))$. If $J_\sigma \neq (0)$, we must have $t = 1$ and $\mathrm{Hom}_{A,A}(K, A_\sigma) = (0)$. This completes the proof.

Let $\sigma \in \mathrm{Aut}_R(A)$. Then $\bar{\sigma} = 1 \otimes \sigma$: $A \otimes_R A \to A \otimes_R A$ is an automorphism of A as left A-module, but is not a bimodule map in general. However, if M is a sub-bimodule of $A \otimes_R A$ then $\bar{\sigma}(M)$ is again a sub-bimodule.

Now assume A is a strongly separable extension of R. For each $\sigma \in \mathrm{Aut}_R(A)$, $A \otimes_R A = K_\sigma \oplus L_\sigma$, where $L_\sigma \cong \mathrm{Hom}_C(\Delta, A_\sigma)$ and $K_\sigma = \ker(\phi_\sigma)$. Let $L = L_1$.

LEMMA 4.6. *Assume A is a strongly separable extension of R, and let* $\sigma \in \mathrm{Aut}_R(A)$. *If* $\mathrm{Hom}_{A,A}(A, A_\sigma) \neq (0)$, *then* $\mathrm{Hom}_{A,A}(A_\sigma, A) \neq (0)$; *so* $J_\sigma \neq (0)$ *implies* $J_{\sigma^{-1}} \neq (0)$. *Further,* $\mathrm{Hom}_{A,A}(A_\sigma, A) \neq (0)$ *implies* $K = K_\sigma$, $L \cong L_\sigma$.

PROOF. If $\mathrm{Hom}_{A,A}(A, A_\sigma) \neq (0)$ then $\mathrm{Hom}_{A,A}(K, A_\sigma) = (0)$, by Proposition 4.5. Hence the projection of K to L_σ arising from the direct sum decomposition $A \otimes_R A = K_\sigma \oplus L_\sigma$ must be the zero map, since L_σ is isomorphic to a direct summand of A_σ^n, for some positive integer n. Thus $K \subseteq K_\sigma$. The projection of L_σ to L arising from the direct sum decomposition $A \otimes_R A = K \oplus L$ must be nonzero. Since L is isomorphic to a direct summand of A^m, for some positive integer m, this gives rise to a nonzero bimodule map from A_σ to A. Thus $J_{\sigma^{-1}} \cong \mathrm{Hom}_{A,A}(A_\sigma, A) \neq (0)$. The argument showing $K \subseteq K_\sigma$ can now be used to show $K_\sigma \subseteq K$, giving $K = K_\sigma$. Thus $A \otimes_R A \cong K \oplus L \cong K_\sigma \oplus L_\sigma = K \oplus L_\sigma$, from which it follows that $L \cong L_\sigma$.

PROPOSITION 4. 7. *Let A be a strongly separable extension of R and let $\sigma \in \mathrm{Aut}_R(A)$ such that $J_\sigma \neq (0)$. Then $J_{\sigma^{-1}} \neq (0)$ and $J_{\sigma^{-1}} \cong J_\sigma^* = \mathrm{Hom}_C(J_\sigma, C)$.*

PROOF. Let $\beta: L \to L_\sigma$ be an isomorphism, guaranteed by the Lemma, and let $\alpha = \beta^{-1}$. Since C is isomorphic to a direct summand of Δ, A is isomorphic to a direct summand of $\mathrm{Hom}_C(\Delta, A) \cong L$. Hence there exist maps $\gamma: A \to L$ and $\delta: L \to A$ such that $\delta \circ \gamma = 1_A$. L_σ is isomorphic to a direct summand of A_σ^n, some positive integer n; so there exist maps $f_i: L_\sigma \to A_\sigma$, $g_i: A_\sigma \to L_\sigma$, $1 \leq i \leq n$, such that $\sum_i g_i \circ f_i = 1_{L_\sigma}$.

Let $\bar{f}_i = f_i \beta \gamma: A \to A_\sigma$, $\bar{g}_i = \delta \alpha g_i: A_\sigma \to A$. Then $\sum_i \bar{g}_i \circ \bar{f}_i = 1_A$. Thus the map

$$J_{\sigma^{-1}} \otimes_C J_\sigma \cong \mathrm{Hom}_{A,A}(A_\sigma, A) \otimes_C \mathrm{Hom}_{A,A}(A, A_\sigma) \longrightarrow \mathrm{Hom}_{A,A}(A, A) \cong C,$$

defined by $g \otimes f \mapsto g \circ f$, for $g \in \mathrm{Hom}_{A,A}(A_\sigma, A)$ and $f \in \mathrm{Hom}_{A,A}(A, A_\sigma)$, is surjective. It follows that $J_{\sigma^{-1}} \cong J_\sigma^*$. This completes the proof.

We summarize the above as follows. Let

$$G = \left\{ \sigma \in \mathrm{Aut}_R(A) \,\middle|\, J_\sigma \neq (0) \right\} = \left\{ \sigma \in \mathrm{Aut}_R(A) \,\middle|\, K_\sigma = K \right\}.$$

Then G is a subgroup of $\mathrm{Aut}_R(A)$. If $\sigma, \tau \in \mathrm{Aut}_R(A)$ then $\mathrm{Hom}_{A,A}(A_\sigma, A_\tau) \cong \mathrm{Hom}_{A,A}(A, A_{\tau\sigma^{-1}}) \neq (0)$ if and only if σ and τ are in the same right coset modulo G.

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