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Notes on Beurling's theorem

To Professor Mitsuru Ozawa on the occasion of his 60th birthday

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For some harmonic function on a Riemann surface with Kuramochi boundary, fine limits exist on the boundary except for a set of capacity zero (Beurling type theorem, (1), (2)). The purpose of the present paper is to improve a result in (2).

Let R be an open Riemann surface and $\{R_n\}_{n=0}^{\infty}$ be an exhaustion of R. Let R^* be the Kuramochi compactification of R and Δ_1 be the set of minimal points of $\Delta=R^*-R$. For any $p\in\Delta_1$, denote by $\mathfrak{G}_p$ the family of open sets G in R such that $R-G$ is N-thin at p. Let u be a harmonic function on R. For any $p\in\Delta_1$, then N-fine cluster set $u^N(p)$ is defined by $u^N(p)=\cap\{\overline{u(G)}: G\in\mathfrak{G}_p\}$, where the closure $\overline{u(G)}$ is taken in extended real numbers. Let F be a closed set in R with piecewise analytic boundary ∂F and G be an open set in R containing F with piecewise analytic boundary. Suppose there is a Dirichlet finite function f in $G-F$ with boundary values 1 on ∂F and 0 on ∂G. Denote by $\omega(\partial F, z, G-F)$ the unique function which gives the smallest Dirichlet integral among the functions like f. Let E be a closed set in Δ. Set $E_k=\left\{z\in R: d(z, E)\leqq\frac{1}{k}\right\}$, where d is a Kuramochi metric. Let E_k' be a closed set in R with piecewise analytic boundary such that $E_{k+1}\subset E_k'\subset E_k-\partial E_k$. Then $\omega(E\cap B(F), z, G)$ is defined by $\lim_{k\to\infty}\omega(\partial(E_k'\cap F), z, G-E_k'\cap F)$. Set $\omega(E\cap B(F), z)=\omega(E\cap B(F), z, R-R_0)$, $\omega(E, z)=\omega(E\cap B(R-R_1), z)$ and $\omega(B(F), z)=\omega(\Delta\cap B(F), z)$. A Borel set A on Δ is said to be a capacity zero if $\omega(E, z)=0$ for any closed subset E of A.

Let u be a harmonic function on R. For any open set G in R, denote by $D_G(u)$ the Dirichlet integral of u on G. Let y be a real number. If there is a number $\delta>0$ such that $D_{(a<u<b)}(u)=\infty$ for any interval (a, b) in $(y-\delta, y+\delta)$, then we call y an I-point. Denote by $\mathcal{E}=\mathcal{E}(u)$ the set of I-points. Then $\mathcal{E}$ is an open subset of real numbers. For any component $e=(c, d)$ of $\mathcal{E}$, denote by e_n the closed interval $\left[c-\frac{1}{n}, d+\frac{1}{n}\right]$.

DEFINITION 1. *A harmonic function u on R is said to be almost Dirichlet finite, if* $\lim_{n\to\infty}\omega(B(u^{-1}(e_n)), z)=0$ *on R for any component e of $\mathcal{E}$.*

DEFINITION 2. *A harmonic function u on R is said to be quasi-Dirichlet finite, if* $D_{(-n<u<n)}(u)<\infty$ *for any* n.

Z. Kuramochi (2) proved the following.

THEOREM. *Let u be a quasi-Dirichlet finite harmonic function on R. If*

$$\lim_{n\to\infty} \omega\Big(B(|u|\geqq n), z\Big)=0,$$

then $S=\{p\in\Delta_1 : \operatorname{diam} u^N(p)>0\}$ *is a set of capacity zero.*

Our result is the following.

THEOREM 1. *If u is an almost Dirichlet finite harmonic function on R, then* $S=\{p\in\Delta_1 : \operatorname{diam} u^N(p)>0\}$ *is a set of capacity zero.*

If u is quasi-Dirichlet finite, $\mathcal{E}(u)=\phi$. Hence any quasi-Dirichlet finite harmonic function is almost Dirichlet finite. By Theorem 1, we obtain the following improvement of Theorem.

COROLLARY. *If u is quasi-Dirichlet finite, then S is a set of capacity zero.*

1. The proof of Theorem 1.

LEMMA 1. *If* $\omega(E\cap B(F), z, G)>0$, *then* $E_G=\{p\in E\cap\Delta_1 : G\in\mathfrak{G}_p\}$ *is a set of positive capacity.*

PROOF. Let μ be the canonical measure of $\omega(E, z)$. Then, by Lemma 4 in (1),

$$\int_{E\cap\Delta_1} N(\cdot, p)\, d\mu(p)>\int_{E\cap\Delta_1} N(\cdot, p)_{R-G}\, d\mu(p)$$

on G and so $\mu(E_G)>0$. Since the energy $\int_{E\cap\Delta_1}\omega(E, p)\, d\mu(p)$ of μ is finite, E_G is a set of positive capacity.

LEMMA 2. *Let u be a harmonic function on R such that* $D_{(\alpha<u<\beta)}(u)<\infty$. *Then, for any closed set E in Δ with* $\omega(E, z)>0$, *either* $E_u^{\alpha}=\{p\in E\cap\Delta_1 : u^N(p)\subset[\alpha, +\infty]\}$ *or* ${}^{\beta}E_u=\{p\in E\cap\Delta_1 : u^N(p)\subset[-\infty, \beta]\}$ *is a set of positive capacity.*

PROOF. Set $c=\dfrac{\alpha+\beta}{2}$. Since

$$\omega(E, z)\leqq\omega\Big(E\cap B(u\leqq c), z\Big)+\omega\Big(E\cap B(u\geqq c), z\Big),$$

it follows that either $\omega(E\cap B(u\leqq c), z)>0$ or $\omega(E\cap B(u\geqq c), z)>0$. Suppose now $\omega(E\cap B(u\leqq c), z)>0$. Consider the function $u_0=1-\dfrac{u-c}{\beta-c}$. Then $D_{(\alpha<u<\beta)}(u_0)<\infty$. Hence $\omega(E\cap B(u\leqq c), z, (u<\beta))$ is well-defined and, by Dirichlet principle,

$$D\Big(\omega\Big(E\cap B(u\leqq c), z, (u<\beta)\Big)\Big)\geqq D\Big(\omega\Big(E\cap B(u\leqq c), z\Big)\Big).$$

This shows that $\omega(E\cap B(u\leqq c), z, (u<\beta))>0$. Hence, by Lemma 1, $E_{(u<\beta)}$ is of positive capacity. This shows that ${}^{\beta}E_u$ is of positive capacity. If $\omega(E\cap B(u\geqq c), z)>0$, we see similarly that E_u^{α} is of positive capacity.

LEMMA 3. *Let u be an almost Dirichlet finite harmonic function. Let E be a closed set in Δ and (a, b) be an open interval. If $u^N(p)\supset(a, b)$ for any point $p\in E\cap\Delta_1$, then $\omega(E, z)\equiv 0$ on R.*

PROOF. Suppose $\omega(E, z)>0$. Let us first assume $(a, b)-\mathcal{E}(u)\neq\phi$. Then there exists a closed interval $[\alpha, \beta]$ contained in (a, b) such that $D_{(\alpha<u<\beta)}(u)<\infty$. Hence, by Lemma 1, $E^{\alpha}\neq\phi$ or ${}^{\beta}E\neq\phi$ for u. But this contradicts $[\alpha, \beta]\subset(a, b)$.

Next we consider the case when $(a, b)\subset\mathcal{E}$. Since $\mathcal{E}$ is open, there is a component $e=(c, d)$ of $\mathcal{E}$ such that $(a, b)\subset(c, d)$. Since $u^{-1}(e_n)\neq R$ for some n, it follows that either $c>-\infty$ or $d<\infty$. Then

$$\omega(E, z)\leqq\omega\Big(E\cap B(u\leqq c-\varepsilon), z\Big)$$
$$+\omega\Big(E\cap B(c-\varepsilon\leqq u\leqq d+\varepsilon), z\Big)+\omega\Big(E\cap B(u\geqq d+\varepsilon), z\Big)$$

for any $\varepsilon>0$. Since $\omega(B(u^{-1}(e_n)), z)\geqq\omega(E\cap B(c-\varepsilon\leqq u\leqq d+\varepsilon), z)$ $\left(\varepsilon<\dfrac{1}{n}\right)$, it follows that

$$\lim_{\varepsilon\to 0}\omega\Big(E\cap B(c-\varepsilon\leqq u\leqq d+\varepsilon), z\Big)=0.$$

Hence either $\omega(E\cap B(u\leqq c-\varepsilon), z)>0$ or $\omega(E\cap B(u\geqq d+\varepsilon), z)>0$ for some $\varepsilon>0$. Suppose now $\omega(E\cap B(u\leqq c-\varepsilon), z)>0$. Then $c>-\infty$ and $c\notin\mathcal{E}$. Hence there is a closed interval $[\alpha, \beta]$ contained $(c-\varepsilon, c)$ such that $D_{(\alpha<u<\beta)}(u)<\infty$. Since $\omega(E\cap B(u\leqq c-\varepsilon), z)>0$, by Lemma 2, $u^N(p_1)\subset[-\infty, \beta]\subset[-\infty, c)$ for some point $p_1\in E\cap\Delta_1$. If $\omega(E\cap B(u\geqq d+\varepsilon), z)>0$, then $d<\infty$ and $u^N(p_2)\subset(d, \infty]$ for some $p_2\in E\cap\Delta_1$. These contradict $(a, b)\subset(c, d)$.

PROOF OF THEOREM 1. Let $\{a_k\}$ be the set of rational numbers. Set

$$A_{n,k}=\left\{p\in\Delta_1:\ u^N(p)\supset\left[a_k-\frac{1}{n},\ a_k+\frac{1}{n}\right]\right\}$$

for any pair of k and n. Then $A_{n,k}$ is a Borel set. Set $A=\bigcup_{n,k}A_{n,k}$. Since

$u_N(p)$ is a closed interval of extended real numbers for any $p\in S$, we have $S\subset A$. Suppose A has positive capacity. Then there exists a closed set E contained in $A_{n,k}$ for some pair of n and k such that $\omega(E, z)>0$. But this contradicts Lemma 3.

EXAMPLE. *There is a quasi-Dirichlet finite harmonic function on* $R=\{|z|<1\}$ *such that* $\{p\in\Delta_1:\ u^N(p)=\{\infty\}\}$ *is a set of positive capacity.*

Let $F_n(n=1,2,\cdots)$ be a finite sum of closed intervals on $\Delta=\Delta_1=\{|z|=1\}$ such that $F_n\supset F_{n+1}$ and $F=\cap F_n$ has linear measure zero and positive capacity. Set $H_n=\left\{z:\ |z|=1,\ \min_{w\in F_n}|z-w|>\frac{1}{n}\right\}$ and $\tilde{w}_n(z)=\frac{1}{2\pi}\int_{H_n^c}\frac{1-|z|^2}{|e^{i\theta}-z|^2}d\theta$. Then $\lim_{n\to\infty}\tilde{w}_n(z)=0$. Let w_n be a harmonic function on R which has the boundary values 1 on F_n and 0 on $\bar{H}_n$ and whose normal derivative vanishes on $\Delta-F_n-\bar{H}_n$. Then $\tilde{w}_n\geqq w_n\geqq w_{n+1}$. On choosing a subsequence, if necessary, we may assume $\tilde{w}_n(0)<\frac{1}{n^2}$. Set $u(z)=\sum w_n(z)$. Since $\lim_{z\to\zeta}u(z)\geqq\lim_{z\to\zeta} nw_n(z)=n$ for any $\zeta\in F_n$, we have $u^N(\zeta)=\lim_{z\to\zeta}u(z)=\infty$ for any $\zeta\in F$. Take any positive integer m and take n_0 such that $2m<n_0$. Set $G_0=\left(w_{n_0}<\frac{1}{2}\right)$. Then $G_0\supset(u<m)$ and $\bar{G}_0\cap\Delta\cap F_{n_0}=\phi$. Take n_1 such that $\bar{G}_0\cap\Delta\subset H_{n_1}$. By $w_n(0)<\frac{1}{n^2}$, $\sum_{n\geqq n_1}w_n(z)$ has boundary values 0 on H_{n_1}. Then $\sum_{n\geqq n_1}w_n(z)$ is harmonic on $\bar{G}_0$ and so $D_{G_0}(\sum_{n\geqq n_1}w_n)<\infty$. Hence we have

$$D_{(u<m)}(u)\leqq D_{G_0}(u)\leqq D(\sum_{n\leqq n_1}w_n)+D_{G_0}(\sum_{n\geqq n_1}w_n)<\infty\,.$$

THEOREM 2 (*Riesz type theorem*). *Let u be a harmonic function on R. Suppose there exist quasi-Dirichlet finite functions u_1 and u_2 such that $u_1\leqq u\leqq u_2$ on R and $\infty>\sup u_1>\inf u_2>-\infty$. Then $u^N(p)\neq const$ on Δ_1 except on a set of capacity zero.*

PROOF. Set $\inf u_2=\alpha_0$ and $\sup u_1=\beta_0$. Take any real numbers α and β such that $\alpha_0<\alpha<\beta<\beta_0$. Let $w_{n,n+i}$ be the harmonic function in $R_{n+i}-((u_1\geqq\beta)-R_n)$ which has the boundary values 0 on $\partial R_{n+i}-(u_1\geqq\beta)$ and 1 on $\partial((u_1\geqq\beta)-R_n)\cap R_{n+i}$. Since $u_1\leqq\beta+\beta_0 w_{n,n+i}$ on $R_{n+i}-((u_1\geqq\beta)-R_n)$, we have $\lim_{n\to\infty}\lim_{i\to\infty}w_{n,n+i}>0$. This shows $\omega(B(u_1\geqq\beta), z)>0$. Since $D_{(\alpha<u_1<\beta)}(u_1)<\infty$, $\Delta_{u_1}^{\alpha}$ has positive capacity by Lemma 2. Next take any real numbers α' and β' such that $\alpha_0<\beta'<\alpha'<\alpha$. Then we have $\omega((u_2\leqq\beta'), z)>0$ similarly. And since $D_{(\beta'<u_2<\alpha')}(u_2)<\infty$, ${}^{\alpha'}\Delta_{u_2}$ has positive capacity by Lemma 2. Since $u_1\leqq u\leqq u_2$, $\Delta_{u_1}^{\alpha}\subset\Delta_u^{\alpha}$ and ${}^{\alpha'}\Delta_{u_2}\subset{}^{\alpha'}\Delta_u$. This shows that $u^N(p)\neq const$ except for a set of capacity zero.

COROLLARY. *Let u be a non-constant Dirichlet finite harmonic function on R. Then* $u^N(p) \neq$ *const on* Δ_1 *except for a set of capacity zero.*

PROOR. Take positive and Dirichlet finite harmonic functions $u_i (i=1, 2)$ such that $u=u_1-u_2$ on R. Let $\{u_{i,n}\}_n$ be sequences of positive, bounded and Dirichlet finite harmonic functions such that $u_{i,n} \uparrow u_i$ $(n \uparrow \infty)$ on R. Then $u_{i,n}-u_2 \leqq u \leqq u_1-u_{2,n}$ on R. Take n_0 such that

$$\infty > \sup_R (u_{1,n_0}-u_2) > \inf_R (u_1-u_{2,n_0}) > -\infty .$$

References

[1] Z. KURAMOCHI: On Beurling's and Fatou's theorems, Lecture Notes in Mathematics 58, Springer, 1968, 43-69.

[2] Z. KURAMOCHI: On Fatou's and Beurling's Theorems, Hokkaido Math. J., vol. 11 (1982), 262-278.

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