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On infinitesimal $C_{2\pi}$-deformations of standard metrics on spheres

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Introduction

Let M be a riemannian manifold and g its riemannian metric. Then we call M a C_l-manifold and g a C_l-metric if all of its geodesics are closed and have the common length l. As is well-known, the unit sphere S^n in the euclidian space $\boldsymbol{R}^{n+1}$ equipeed with the induced metric (the standard metric) g_0 is a $C_{2\pi}$-manifold.

Let us consider a one-parameter family $\{g_t\}$ of $C_{2\pi}$-metrics on S^n such that $g_0=g_t|_{t=0}$ is the standard one. Put

$$h=\frac{d}{dt}g_t|_{t=0}\,.$$

We shall call such a family $\{g_t\}$ a $C_{2\pi}$-deformation of the standard metric g_0, and h an infinitesimal $C_{2\pi}$-deformation of g_0. It is known that each infinitesimal $C_{2\pi}$-deformation h satisfies the so-called zero-energy condition, i. e.,

$$\int_0^{2\pi} h\left(\dot{\gamma}(s), \dot{\gamma}(s)\right) ds=0$$

for any geodesic $\gamma(s)$ of (S^n, g_0) parametrized by arc-length (cf. [1] p. 151). We denote by $\mathscr{K}^2$ the vector space of symmetric 2-forms on S^n which satisfy the zero-energy condition.

In his paper [3] Guillemin proved that in the case of S^2 any symmetric 2-form $h\in\mathscr{K}^2$ is necessarily an infinitesimal $C_{2\pi}$-deformation of g_0. On the other hand, for $C_{2\pi}$-deformations on S^n $(n\geq 3)$, the examples constructed by Weinstein ([1] p. 119) are all that we know up to now, and the corresponding infinitesimal $C_{2\pi}$-deformations form a rather small subset of $\mathscr{K}^2$.

The main purpose of this paper is to introduce and study a necessary condition for a symmetric 2-form $h\in\mathscr{K}^2$ to be an infinitesimal $C_{2\pi}$-deformation of the standard metric g_0 on the n-dimensional sphere S^n $(n\geq 3)$. This condition is called the second order condition, and is naturally obtained through the interpretation of the $C_{2\pi}$-property in terms of the symplectic geometry on the cotangent bundle T^*S^n (Proposition 1. 4).

Let $\iota_0: S^n \to \boldsymbol{R}^{n+1}$ be the natural embedding, and let $(x_1, \cdots, x_{n+1})$ be the canonical coordinate system of $\boldsymbol{R}^{n+1}$. In this paper we restrict our attention to the symmetric 2-forms of the form $h = (\iota_0{}^*f)\, g_0$, where f is a polynomial function on R^{n+1} (or a polynomial in the variables $x_1, \cdots, x_{n+1}$) and $h \in \mathscr{K}^2$. In general it is known for a function v on S^n that the symmetric 2-form vg_0 satisfies the zero-energy condition if and only if v is odd with respect to the antipodal map of S^n. Hence we may assume that f is an odd polynomial, i.e., polynomial whose terms of even degrees vanish.

The main result in this paper (Theorem 4.1) may be stated as follows;

THEOREM. Assume that the dimension n of the sphere under consideration is equal to or greater than 3. Let f be an odd polynomial in the variables $x_1, \cdots, x_{n+1}$. Then the symmetric 2-form $(\iota_0{}^*f)\, g_0 \in \mathscr{K}^2$ satisfies the second order condition if and only if f has one of the following forms:

$$\text{(i)} \quad f \equiv h_1 + h_3 + \sum_{i=2}^{m} (\sum_k a_k x_k)^{2i} (\sum_j b_{ij} x_j) \qquad \mathrm{mod}\left(1 - \sum_{i=1}^{n+1} x_i^2\right),$$

where h_1 and h_3 are homogeneous polynomials of degrees 1 and 3 respectively, and a_k and b_{ij} are real numbers;

$$\text{(ii)} \quad f \equiv h_1 + h_3 + cA^*h \qquad \mathrm{mod}\left(1 - \sum_{i=1}^{n+1} x_i^2\right),$$

where h_1 and h_3 are as in (i), $c \in \boldsymbol{R}$, $A \in O(n+1, \boldsymbol{R})$, and h is a polynomial of degree 21 in the variables (x_1, x_2) whose coefficients satisfy certain relations (for a strict form, see § 4).

As an immediate consequence of this theorem, we see that the zero-energy condition is no more sufficient for a symmetric 2-form to be an infinitesimal $C_{2\pi}$-deformation of g_0 in the case of S^n $(n \geqq 3)$.

The infinitesimal $C_{2\pi}$-deformations given by Weinstein are essentially of the form $(\iota_0{}^*u(\sum_k a_k x_k))\, g_0$, where $u = u(t)$ is any function in one variable t satisfying $u(-t) = -u(t)$. Hence we have a subclass of (i) consisting of odd polynomials f of the form

$$\text{(i)}' \quad f \equiv h_1 + \sum_{i=1}^{m} c_i (\sum_k a_k x_k)^{2i+1} \qquad \mathrm{mod}\left(1 - \sum_{i=1}^{n+1} x_i^2\right), \qquad c_i \in \boldsymbol{R},$$

and for these polynomials the symmetric 2-forms $(\iota_0{}^*f)\, g_0$ are really infinitesimal $C_{2\pi}$-deformations of g_0. For the other polynomials f satisfying (i) or (ii) we do not know whether $(\iota_0{}^*f)\, g_0$ is an infinitesimal $C_{2\pi}$-deformation or not.

Recently Tsukamoto [6] proved that the second order condition is not satisfied for a certain subclass of $\mathscr{K}^2$. It should be noted that this subclass is in some sense a complement of the subspace of $\mathscr{K}^2$ spanned by the Lie derivatives $\mathscr{L}_X g_0$, X being vector fields on S^n, and the symmetric 2-forms$\in$ $\mathscr{K}^2$ which are conformal to g_0 (see also [5]).

This paper consists of five sections. In § 1 we introduce the second order condition. In § 2 we deal with the laplacian acting on the functions on the unit cotangent bundle S^*S^n. We restrict this operator to the subspace consisting of functions which are constant along each orbit of the geodesic flow, and decompose it into a sum of eigenspaces. The second order condition for a symmetric 2-form $(\iota_0{}^*f)\, g_0$ is then interpretted as the vanishing of some eigenspace components of a function $G\iota^*F(f, f)$ which is suitably defined by f. In § 3 we prove Proposition 3. 1, which is the first step to Theorem 4. 1. The proof consists of two steps ; the explicit calculations for polynomials in two variables (x_1, x_2) and the reduction of the general case to two variables case. For this reduction we use some algebraic geometric properties of complex quadrics (Proposition 3. 11). This trick is also used extensively in subsequent sections. § 4 and § 5 are devoted to the proof of Theorem 4. 1. There appear a kind of polynomials of degree 21 as an exceptional case. This case is considered in detail in § 5.

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The contents of this paper were partially anounced in [4].

§ 1. The second order condition

Throughout the paper we assume the differentiability of class C^∞.

We first introduce some terminologies.

Set

$$\mathring{T}^*S^n = T^*S^n - \{0\text{-section}\}\ .$$

Let $\boldsymbol{R}^*$ (resp. $\boldsymbol{R}_+$) be the multiplicative group of non-zero real numbers (resp. of positive real numbers). We say that a function f on $\mathring{T}^*S^n$ is homogeneous (resp. positively homogeneous) of degree d if

$$f(s\lambda) = s^d f(\lambda)$$

for any $\lambda \in \mathring{T}^*S^n$ and any $s \in \boldsymbol{R}^*$ (resp. $s \in \boldsymbol{R}_+$). A vector field X on $\mathring{T}^*S^n$ is called homogeneous (resp. positively homogeneous) if it is invariant under the $\boldsymbol{R}^*$-action (resp. the $\boldsymbol{R}_+$-action).

Let α be the canonical 1-form on T^*S^n, which is defined by

$$\alpha(X) = \lambda(\pi_* X)\,, \qquad \lambda \in T^*S^n\,, \qquad X \in T_\lambda T^*S^n\,,$$

π being the projection $T^*S^n \to S^n$. As is well-known, the 2-form $d\alpha$ defines a symplectic structure on T^*S^n.

To each function f on an open subset U of T^*S^n we assign a symplectic vector field X_f on U in the usual way;

$$i_{X_f} d\alpha = -df .$$

It is easy to see that if a function f on $\mathring{T}^*S^n$ is (positively) homogeneous of degree one, then the vector field X_f is (positively) homogeneous, and $\alpha(X_f)=f$.

A riemannian metric g on S^n induces a bundle isomorphism $\sharp_g$ from the cotangent bundle T^*S^n to the tangent bundle TS^n such that

$$g\big(\sharp_g(\lambda), v\big) = \lambda(v), \qquad \lambda \in T^*_x S^n , \qquad v \in T_x S^n , \qquad x \in S^n .$$

Let $\widetilde{\mathscr{H}}^2$ be the vector space of functions on T^*S^n which are homogeneous polynomials of degree 2 on each fibre $T^*_x S^n$ $(x \in S^n)$. To each symmetric 2-form h we assign an element $h^{\sharp_g}$ of $\widetilde{\mathscr{H}}^2$ by

$$h^{\sharp_g}(\lambda) = h\big(\sharp_g(\lambda), \sharp_g(\lambda)\big) , \qquad \lambda \in T^*S^n .$$

In particular we call $E=\frac{1}{2} g^{\sharp_g}$ the energy function and X_E (or the one-parameter group of transformations generated by it) the geodesic flow associated with the riemannian metric g.

Now let $\{g_t\}$ be a $C_{2\pi}$-deformation of the standard metric g_0 on S^n. Put $h_t = \frac{d}{dt} g_t$.

LEMMA 1.1. *For any geodesic $\gamma(s)$ of (S^n, g_t) parametrized by arc-length, we have*

$$\int_0^{2\pi} h_t\big(\dot{\gamma}(s), \dot{\gamma}(s)\big)\, ds = 0 .$$

For the proof we refer to [1] Proposition 5.86.

For simplicity's sake we shall write $\sharp_t$ instead of $\sharp_{g_t}$. Put

$$E_t = \frac{1}{2} g_t^{\sharp_t} .$$

The following proposition is another representation of Lemma 1.1, which is essentially due to Weinstein [7] (see also [1] Proposition 4.46).

PROPOSITION 1.2. *There is a one-parameter family of homogeneous symplectic vector fields $\{X_t\}$ on $\mathring{T}^*S^n$ such that*

$$X_t E_t = \dot{E}_t ,$$

where the dot denotes the derivative in the parameter t.

PROOF. Let $\{\xi_s^t\}_{s\in\boldsymbol{R}}$ be the geodesic flow associated with the the riemannian metric g_t. Then $\{\xi_s^t\}_{s\in\boldsymbol{R}}$ induces a free S^1-action of period 2π on the unit cotangent bundle $S^*_{(t)}S^n=E_t^{-1}\left(\frac{1}{2}\right)$. It is easy to see that

$$\dot{E}_t=-\frac{1}{2}h_t^{\sharp_t}$$

and $\sharp_t(\xi_s^t(\lambda))=\dot{\gamma}(s)$, where $\lambda\in S^*_{(t)}S^n$ and $\gamma(s)$ denotes the geodesic $\pi(\xi_s^t(\lambda))$ of (S^n, g_t). Hence it follows from Lemma 1. 1 that

$$\int_0^{2\pi}\dot{E}_t\left(\xi_s^t(\lambda)\right)ds=0\,, \qquad \lambda\in S^*_{(t)}S^n\,.$$

Define a function H_t on $\mathring{T}^*S^n$ by the conditions

(i) $H_t(\lambda)=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\int_0^s\dot{E}_t\left(\xi_r^t(\lambda)\right)drds\,, \qquad \lambda\in S^*_{(t)}S^n\,,$

(ii) H_t is positively homogeneous of degree one.

From the condition (ii) it follows that $X_{E_t}H_t$ is positively homogeneous of degree two. For $\lambda\in S^*_{(t)}S^n$ we have

$$(X_{E_t}H_t)(\lambda)=\frac{d}{ds}H_t\left(\xi_s^t(\lambda)\right)\Big|_{s=0}=-\dot{E}_t(\lambda)\,.$$

Since $\dot{E}_t$ is also positively homogeneous of degree two, we see that

$$X_{E_t}H_t=-\dot{E}_t$$

on $\mathring{T}^*S^n$. Clearly we have

$$H_t(-\lambda)=-H_t(\lambda)\,, \qquad \lambda\in S^*_{(t)}S^n\,.$$

Hence H_t is homogeneous of degree one. Set

$$X_t=X_{H_t}\,.$$

Then X_t is homogeneous and we have

$$X_tE_t=\dot{E}_t$$

by the anti-commutativity of the Poisson bracket.

For simplicity's sake we shall write $\sharp$, $\{\xi_s\}$, and S^*S^n instead of $\sharp_0$, $\{\xi_s^0\}$, and $S^*_{(0)}S^n$ respectively.

Define a linear operator G on the vector space $C^\infty(S^*S^n)$ of C^∞-functions on S^*S^n by

$$G(f)(\lambda)=\frac{1}{2\pi}\int_0^{2\pi} f(\xi_s\lambda)\,ds\,, \qquad f\in C^\infty(S^*S^n)\,, \qquad \lambda\in S^*S^n\,.$$

We assign a homogeneous symplective vector field $X(h)$ to each $h\in\mathscr{K}^2$ as follows: Let H be the function on $\mathring{T}^*S^n$ which is positively homogeneous of degree one and satisfies

$$H(\lambda)=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{s} h^\sharp(\xi_r\lambda)\,drds\,, \qquad \lambda\in S^*S^n\,;$$

We set $X(h)=X_H$. By the proof of Proposition 1.2 we see that $X(h)$ is homogeneous and satisfies

$$X(h)\,E_0=h^\sharp\,.$$

It should be noticed that $X(h)$ is uniquely determined by the conditions $X(h)\,E_0=h^\sharp$ and $G(\alpha(X(h)))=0$.

We then define a bilinear map K: $\mathscr{K}^2\times\mathscr{K}^2\to C^\infty(S^*S^n)$ by

$$K(f,h)=G\big(X(f)\,h^\sharp\big)\,, \qquad f,\ h\in\mathscr{K}^2\,,$$

where $X(f)\,h^\sharp$ should be considered as a function on S^*S^n by restriction.

LEMMA 1.3. *K is symmetric.*

PROOF. Let f and h be elements of $\mathscr{K}^2$, and let F and H be the functions defined as above by f and h respectively. Then we have

$$X(f)\,h^\sharp=X_F h^\sharp=X_F X_H E_0\,.$$

Thus

$$\begin{aligned} X(f)\,h^\sharp-X(h)f^\sharp &= [X_F,\,X_H]\,E_0 \\ &= X_{X_F H}E_0=-X_{E_0}(X_F H)\,. \end{aligned}$$

Since $G\circ X_{E_0}=0$, it follows that

$$G\big(X(f)\,h^\sharp\big)=G\big(X(h)\,f^\sharp\big)\,.$$

Let $\mathscr{A}^2$ be the vector space of functions on S^*S^n which are the restrictions of elements of $\widetilde{\mathscr{A}}^2$. We shall say that an element h of $\mathscr{K}^2$ satisfies the second order condition if

$$K(h,h)\in G(\mathscr{A}^2)\,.$$

PROPOSITION 1.4. *Every infinitesimal $C_{2\pi}$-deformation of g_0 satisfy the second order condition.*

PROOF. Let $\{g_t\}$ be a $C_{2\pi}$-deformation of g_0 and put $\frac{d}{dt}g_t|_{t=0}=h$. Let

$\{E_t\}$ be the corresponding energy functions. Following the proof of Proposition 1.2 we construct the one-parameter family of homogeneous symplectic vector fields $\{X_t\}$ on $\overset{\circ}{T}{}^*S^n$. By differentiating the formula $X_tE_t=\dot{E}_t$ in the parameter t and putting $t=0$, we have

$$\dot{X}_0E_0+X_0\dot{E}_0=\ddot{E}_0\,.$$

Since $\dot{X}_0$ is homogeneous, it follows that $\dot{X}_0=X_f$, f being $\alpha(\dot{X}_0)$. Thus

$$\dot{X}_0E_0=-X_{E_0}f\,,$$

which implies $G(\dot{X}_0E_0)=0$. Since $\dot{E}_0=-\dfrac{1}{2}h^\sharp$ and $X_0=-\dfrac{1}{2}X(h)$ by the construction, we have

$$G\big(X(h)\,h^\sharp\big)=4G(\ddot{E}_0)\in G(\mathscr{H}^2)\,.$$

REMARK. In the case of S^2 it is known that

$$G(\mathscr{H}^2)=G\big(C^\infty(S^2)\,E_0\big)=\text{the image of } G$$

(cf. [3] Appendix). Therefore Proposition 1.4 turns out to be trivial in this case.

Next we shall give more explicit expression of the second order condition in the case where symmetric 2-forms are conformal to g_0. Let f be a function on S^n. Then it is known that fg_0 belongs to $\mathscr{K}^2$ if and only if f is an odd function, i.e., $\tau^*f=-f$, τ being the antipodal map of S^n (cf. [1] p. 123).

Let $\iota_0\colon S^n\to \boldsymbol{R}^{n+1}$ be the canonical embedding. Then we can embed T^*S^n into $\boldsymbol{R}^{2n+2}=T\boldsymbol{R}^{n+1}$ by the map $\iota_1=\iota_{0*}\circ\sharp$.

$$\begin{array}{ccc} T^*S^n & \xrightarrow{\ \sharp\ } & TS^n \\ & \underset{\iota_1}{\searrow} & \downarrow{\scriptstyle \iota_{0*}} \\ & & \boldsymbol{R}^{2n+2} \end{array}$$

Let $x=(x_1,\cdots,x_{n+1})$ be the canonical coordinate system of $\boldsymbol{R}^{n+1}$ and $(x,\zeta)=(x_1,\cdots,x_{n+1},\zeta_1,\cdots,\zeta_{n+1})$ be that of $\boldsymbol{R}^{2n+2}=T\boldsymbol{R}^{n+1}$. It is easy to see that

$$\iota_1(T^*S^n)=\Big\{(x,\zeta)\in\boldsymbol{R}^{2n+2}\Big|\sum_i x_i^2=1,\ \sum_i x_i\zeta_i=0\Big\}\,,$$

$$\iota_1(S^*S^n)=\Big\{(x,\zeta)\in\boldsymbol{R}^{2n+2}\Big|\sum_i x_i^2=\sum_i \zeta_i^2=1,\ \sum_i x_i\zeta_i=0\Big\}\,.$$

We shall denote by ι the restriction of ι_1 onto S^*S^n.

Define a one-parameter group of transformations $\{\xi_t\}$ of $\boldsymbol{R}^{2n+2}$ by

$$\xi_t(x,\zeta)=(x\cos t+\zeta\sin t,-x\sin t+\zeta\cos t)\,.$$

Let $\tilde{X}_{E_0}$ be its infinitesimal generator. Then

$$\tilde{X}_{E_0}=\sum_i\left(\zeta_i\frac{\partial}{\partial x_i}-x_i\frac{\partial}{\partial\zeta_i}\right).$$

Define a linear operator $\tilde{G}$: $C^\infty(\boldsymbol{R}^{2n+2})\to C^\infty(\boldsymbol{R}^{2n+2})$ by

$$\tilde{G}(f)(x,\zeta)=\frac{1}{2\pi}\int_0^{2\pi}f\left(\tilde{\xi}_t(x,\zeta)\right)dt\,.$$

It is easy to see that $\tilde{\xi}_t(\iota(\lambda))=\iota(\xi_t(\lambda))$ for $\lambda\in S^*S^n$. Hence it follows that $\tilde{X}_{E_0}=\iota_*X_{E_0}$ on $\iota(S^*S^n)$, and $\iota^*\tilde{G}(f)=G(\iota^*f)$ for any function f defined on a neighborhood of $\iota(S^*S^n)$.

Let α_0 be the 1-form on $\boldsymbol{R}^{2n+2}$ defined by $\alpha_0=\sum_i\zeta_i dx_i$. Then $d\alpha_0$ defines a symplectic structure on $\boldsymbol{R}^{2n+2}$. It is easy to see that $\iota_1{}^*\alpha_0=\alpha$. To each function u on an open subset U of $\boldsymbol{R}^{2n+2}$ we assign a symplectic vector field Y_u on U by

$$i_{Y_u}d\alpha_0=-du\,.$$

Let f be a function defined on a neighborhood U of $\iota_0(S^n)$ in $\boldsymbol{R}^{n+1}$. Put $\tilde{f}(x,\zeta)=f\left(\frac{x}{|x|}\right)$. Then $\tilde{f}$ is a function on $(\boldsymbol{R}^{n+1}-\{0\})\times\boldsymbol{R}^{n+1}$.

LEMMA 1.5. *$\iota_{1*}X_{\iota_1{}^*\tilde{f}}=Y_{\tilde{f}}$ on $\iota_1(T^*S^n)$.*

PROOF. Since $\iota_1{}^*d\alpha_0=d\alpha$, we only need to verify that $Y_{\tilde{f}}$ is tangent to $\iota_1(T^*S^n)$ at each point of $\iota_1(T^*S^n)$. We have

$$\begin{aligned}Y_{\tilde{f}}&=-\sum_i\frac{\partial\tilde{f}}{\partial x_i}\frac{\partial}{\partial\zeta_i}\\&=-\sum_{ij}\frac{\partial f}{\partial x_j}(\delta_{ij}-x_ix_j)\frac{\partial}{\partial\zeta_i}\end{aligned}$$

at $(x,\zeta)\in\iota_1(T^*S^n)$. Thus we have

$$Y_{\tilde{f}}(\sum_i x_i^2)=Y_{\tilde{f}}(\sum_i x_i\zeta_i)=0$$

on $\iota_1(T^*S^n)$, which implies that $Y_{\tilde{f}}$ is tangent to $\iota_1(T^*S^n)$.

Define a bilinear map F: $C^\infty(\boldsymbol{R}^{n+1})\times C^\infty(\boldsymbol{R}^{n+1})\to C^\infty(\boldsymbol{R}^{2n+2})$ by

$$F(f,h)(x,\zeta)=\sum_{i,j=1}^{n+1}(\delta_{ij}-x_ix_j-\zeta_i\zeta_j)\frac{\partial f}{\partial x_i}(x)\int_0^\pi\frac{\partial h}{\partial x_j}$$
$$(x\cos t+\zeta\sin t)\sin t\,dt,\ f,\ h\in C^\infty(\boldsymbol{R}^{n+1})\,.$$

Let f and h be odd functions on $\boldsymbol{R}^{n+1}$, i.e., $f(-x)=-f(x)$, $h(-x)=-h(x)$. Then $(\iota_0{}^*f)\,g_0$ and $(\iota_0{}^*h)\,g_0$ belong to $\mathscr{K}^2$, and we have

PROPOSITION 1.6. *$K((\iota_0{}^*f)\, g_0, (\iota_0{}^*h)\, g_0) \in G(\mathscr{H}^2)$ if and only if $G\iota^*F(f, h) \in G(\mathscr{H}^2)$.*

PROOF. For simplicity's sake we put $\hat{f}=(\iota_0{}^*f)\, g_0$ and $\hat{h}=(\iota_0{}^*h)\, g_0$. Define a function H on $(\boldsymbol{R}^{n+1}-\{0\})\times(\boldsymbol{R}^{n+1}-\{0\})$ by

$$H(x, \zeta)=\frac{1}{2}|x|\,|\zeta|\int_0^\pi h\left(\frac{x}{|x|}\cos t+\frac{\zeta}{|\zeta|}\sin t\right)dt\,.$$

Then the function $\iota_1{}^*H$ on $\mathring{T}^*S^n$ is positively homogeneous of degree one, and satisfies

$$(X_{E_0}\iota_1{}^*H)(\lambda)=-(\iota_0{}^*h)\big(\pi(\lambda)\big)\,, \qquad \lambda\in S^*S^n\,.$$

This shows that $X_{E_0}\iota_1{}^*H=-\hat{h}^\sharp$ on $\mathring{T}^*S^n$. Moreover, since $\iota_1{}^*H$ is odd with respect to τ^*, the differential of the antipodal map, it follows that $G(\iota_1{}^*H)=0$. Thus we have

$$X(\hat{h})=X_{\iota_1{}^*H}\,.$$

Then

$$\begin{aligned} X(\hat{h})\hat{f}^\sharp &= X_{\iota_1{}^*H}\big(2(\iota_0{}^*f)\, E_0\big) \\ &= 2(X_{\iota_1{}^*H}\iota_0{}^*f)\, E_0+4\,(\iota_0{}^*f)\,(\iota_0{}^*h)\, E_0\,. \end{aligned}$$

Since $X_{\iota_1{}^*H}\iota_0{}^*f=-X_{\iota_0{}^*f}\iota_1{}^*H=-\iota_1{}^*(Y_{\tilde{f}}H)$ by Lemma 1.5, we see that the condition $K(\hat{f}, \hat{h})\in G(\mathscr{H}^2)$ is equivalent to the condition $G\iota^*(Y_{\tilde{f}}H)\in G(\mathscr{H}^2)$. An explicit claculation shows that

$$Y_{\tilde{f}}H=-\frac{1}{2}F(f, h)-\frac{1}{2}\tilde{X}_{E_0}\left(f(x)\int_0^\pi h(x\cos t+\zeta\sin t)\, dt\right)-fh$$

on $\iota(S^*S^n)$, which proves the proposition.

PROPOSITION 1.7. *Let f and h be odd functions on $\boldsymbol{R}^{n+1}$. Then*

$$\tilde{G}F(f, h)=\tilde{G}F(h, f)\,.$$

PROOF. We have

$$\tilde{G}F(f, h)=\sum_{i,j}(\delta_{ij}-x_i x_j-\zeta_i\zeta_j)\times u_{ij}\,,$$

where

$$u_{ij}=\int_0^{2\pi}\frac{\partial f}{\partial x_i}(x\cos r+\zeta\sin r)\int_0^\pi\frac{\partial h}{\partial x_j}\big(x\cos(t+r)+\zeta\sin(t+r)\big)\sin t\, dt dr\,.$$

Then it is easy to see that

$$u_{ij}=\int_0^{2\pi}\frac{\partial h}{\partial x_j}(x\cos r+\zeta\sin r)\int_0^{\pi}\frac{\partial f}{\partial x_i}\big(x\cos(t+r)+\zeta\sin(t+r)\big)\sin t\,dt\,dr\,,$$

which proves the proposition.

§ 2. The laplacian on $C^\infty(S^*S^n)$

We first define a riemannian metric on S^*S^n. The riemannian metric g_0 on S^n induces the horizontal subspace H_v of T_vTS^n at each $v\in TS^n$. Let V_v be the vertical subspace of T_vTS^n. Then we have the decomposition $T_vTS^n=V_v+H_v$ (direct sum). Let $\pi: TS^n\to S^n$ be the projection and $\pi(v)=x$. Define a riemannian metric $\tilde{g}_1$ on TS^n by the conditions:

(i) The canonical identification $V_v\to T_xS^n$ is an isometry;

(ii) π_*; $H_v\to T_xS^n$ is an isometry;

(iii) V_v and H_v are orthogonal to each other.

Let g_1 be the riemannian metric on the unit tangent bundle SS^n which is the pull back of $\tilde{g}_1$ by the inclusion $SS^n\to TS^n$. Then the riemannian manifold (SS^n, g_1) has the following properties (cf. [1] Chapter 1, K):

(a) Each fibre S_xS^n is totally geodesic;

(b) The parallel translation of a unit vector along a geodesic of (S^n, g_0) is a geodesic.

Now we define a riemannian metric on S^*S^n in such a way that $\sharp$: $S^*S^n\to SS^n$ is an isometry. We shall also denote this metric by g_1.

Let Δ be the laplacian defined by the riemannian metric g_1 which operates on $C^\infty(S^*S^n)$. We need the explicit expression of Δ in terms of the euclidian coordinates. Let ι: $S^*S^n\to \boldsymbol{R}^{2n+2}=\{(x,\zeta)\}$ be the embedding defined in § 1.

LEMMA 2.1. *Let $f(x,\zeta)$ be a function defined on a neighborhood of $\iota(S^*S^n)$. Then*

$$\Delta(\iota^*f)=\iota^*\Big\{(n-1)\Big(\sum_i x_i\frac{\partial f}{\partial x_i}+\sum_i\zeta_i\frac{\partial f}{\partial\zeta_i}\Big)+\Big(\sum_i x_i\frac{\partial}{\partial x_i}\Big)^2 f+\Big(\sum_i\zeta_i\frac{\partial}{\partial\zeta_i}\Big)^2 f$$
$$+2\sum_{j,k}x_j\zeta_k\frac{\partial^2 f}{\partial\zeta_j\partial x_k}-\sum_i\frac{\partial^2 f}{\partial x_i^2}-\sum_i\frac{\partial^2 f}{\partial\zeta_i^2}\Big\}.$$

PROOF. We identify S^*S^n with $\iota(S^*S^n)$. Fix $(x,\zeta)\in S^*S^n$ and choose vectors $e_1,\cdots,e_{n-1}$ in $\boldsymbol{R}^{n+1}$ such that $(x,\zeta,e_1,\cdots,e_{n-1})$ is an orthonormal basis of $\boldsymbol{R}^{n+1}$. Consider the following curves on S^*S^n starting at (x,ζ);

$$\gamma_i(t)=(x,\zeta\cos t+e_i\sin t)\qquad(1\leqq i\leqq n-1)\,,$$
$$\eta_i(t)=(x\cos t+e_i\sin t;\zeta)\qquad(1\leqq i\leqq n-1)\,,$$
$$\xi(t)=(x\cos t+\zeta\sin t,\ -x\sin t+\zeta\cos t)=\xi_t(x,\zeta)\,.$$

It is easily seen from the properties (a) and (b) that these curves are geodesics of (S^*S^n, g_1). Moreover, the vectors $\dot{\gamma}_i(0)$, $\dot{\eta}_i(0)$ $(1 \leqq i \leqq n-1)$, and $\dot{\xi}(0)$ form an orthonormal basis of $T_{(x,\zeta)}S^*S^n$. Thus we have

$$\begin{aligned} &\Delta(\iota^*f)(x, \zeta) \\ &= -\sum_{i=1}^{n-1} \frac{d^2}{dt^2} f\big(\gamma_i(t)\big)\Big|_{t=0} - \sum_{i=1}^{n-1} \frac{d^2}{dt^2} f\big(\eta_i(t)\big)\Big|_{t=0} - \frac{d^2}{dt^2} f\big(\xi(t)\big)\Big|_{t=0}, \end{aligned}$$

from which the lemma immediately follows.

LEMMA 2.2. *X_{E_0} is an infinitesimal isometry of (S^*S^n, g_1).*

For the proof we refer to Besse [1] Proposition 1.104.

The following corollary is an immediate consequence of Lemma 2.2.

COROLLARY 2.3. *The operators ξ_t^*, X_{E_0}, and G on $C^\infty(S^*S^n)$ commute with Δ.*

The riemannian metric g_1 naturally induces an inner product on $C^\infty(S^*S^n)$;

$$(f, h) = \int_{S^*S^n} fh \, d\mu_1, \qquad f, h \in C^\infty(S^*S^n),$$

where $d\mu_1$ is the canonical measure defined by g_1.

LEMMA 2.4. *The operator G is self-adjoint with respect to the inner product (,).*

PROOF. Let $GeodS^n$ be the quotient manifold of S^*S^n by the S^1-action of $\{\xi_t\}_{t \in \boldsymbol{R}}$. Since $\{\xi_t\}$ are isometries, we can take a riemannian metric on $GeodS^n$ such that the projection $S^*S^n \to GeodS^n$ is a riemannian submersion. Let $d\rho$ be the measure on $GeodS^n$ defined by this riemannian structure. Then we have

$$(f, h) = 2\pi \int_{GeodS^n} G(fh) \, d\rho, \qquad f, h \in C^\infty(S^*S^n),$$

where the function $G(fh)$ should be considered as a function on $GeodS^n$. Since $G(G(f)h) = G(G(f)G(h)) = G(fG(h))$, it follows that

$$\big(G(f), h\big) = \big(G(f), G(h)\big) = \big(f, G(h)\big).$$

Let $\boldsymbol{R}[x, \zeta]$ be the polynomial algebra in the variables $(x, \zeta) = (x_1, \cdots, x_{n+1}, \zeta_1, \cdots, \zeta_{n+1})$ with real coefficients. We set

$$P = \iota^* \boldsymbol{R}[x, \zeta] \subset C^\infty(S^*S^n).$$

Let $\boldsymbol{R}[x, \zeta]_k$ be the subspace of $\boldsymbol{R}[x, \zeta]$ spanned by homogeneous polynomials of degree k. We say that a monomial $f(x, \zeta)$ is of bidegree (i, j) if $f(x, \zeta)$ is of degree i in x and of degree j in ζ. Let $\boldsymbol{R}[x, \zeta]_{i,j}$ be the vector space spanned by monomials of bidegree (i, j). We set

$$P_k = \iota^* \boldsymbol{R}[x, \zeta]_k , \qquad P_{i,j} = \iota^* \boldsymbol{R}[x, \zeta]_{i,j} ,$$

and $P^j = \sum_{i \geqq 0} P_{i,j}$. It is easy to see that $P_k \subset P_{k+2}$, $P = \sum_{k \geqq 0} P_k$, and $P_k = \sum_{i+j=k} P_{i,j}$.

Let Q_k be the orthogonal complement of P_{k-2} in P_k with respect to the inner product $(\ , \ )$; $P_k = P_{k-2} + Q_k$ (direct sum).

LEMMA 2.5. *The operators Δ and G preserves the vector spaces P_k and Q_k.*

PROOF. By Lemma 2.1 we immediately have $\Delta(P_k) \subset P_k$. Since $G \circ \iota^* = \iota^* \circ \tilde{G}$ and $\tilde{G}$ preserves $\boldsymbol{R}[x, \zeta]_k$, it follows that $G(P_k) \subset P_k$. Since P_{k-2} is also preserved by these operators, so is its orthogonal complement Q_k.

The following corollary is an immediate consequence of Lemma 2.5 and Corollary 2.3.

COROLLARY 2.6. (i) $G(P_k) = G(P_{k-2}) + G(Q_k)$ *(orthogonal direct sum).*

(ii) *The laplacian Δ preserves the vector spaces $G(P_k)$ and $G(Q_k)$.*

In general, for a function h on $S^* S^n$ we denote by $h_{G(Q_k)}$ the $G(Q_k)$-component of h. Let V be a subspace of $C^\infty(S^* S^n)$ which is invariant by the laplacian Δ. Then we denote by $Spec(\Delta, V)$ the set of spectra of Δ on V.

Put $k!! = \prod_{p=0}^{[(k-1)/2]} (k-2p)$ for a positive integer k, and put $0!! = (-1)!! = 1$. Define real numbers a_β^α $(\alpha \geqq 0, \beta \geqq 0)$ by

$$a_\beta^\alpha = (-1)^\beta \frac{(2\alpha - 1)!!}{(2\beta)!!(2\alpha + 2\beta - 1)!!} .$$

PROPOSITION 2.7. (i) $G(P_{2m+1}) = G(Q_{2m+1}) = 0$ $(m \geqq 0)$.

(ii) *$Spec(\Delta, G(Q_{2m})) \subset \{N_{i,j} | i+j=2m, \ i \geqq j \geqq 0\}$, where $N_{i,j} = i(i+n-1) + j(j+n-1) - 2j$.*

PROOF. (i) Since $\tilde{\xi}_\pi(x, \zeta) = (-x, -\zeta)$, it follows that $\tilde{\xi}_\pi^* = (-1) \cdot$identity on $\boldsymbol{R}[x, \zeta]_{2m+1}$. Thus $\xi_\pi^* = (-1) \cdot$identity on P_{2m+1}. Since $G \circ \xi_\pi^* = G$, we have $G(P_{2m+1}) = 0$.

(ii) Since $\tilde{\xi}_{\pi/2}(x, \zeta) = (\zeta, -x)$, we have $\tilde{\xi}_{\pi/2}^* \boldsymbol{R}[x, \zeta]_{i,j} = \boldsymbol{R}[x, \zeta]_{j,i}$. This shows that $G(P_{i,j}) = G(P_{j,i})$, and we have

$$G(P_{2m}) = \sum_{\substack{i+j=2m \\ i \geqq j}} G(P_{i,j}) .$$

Let $f \in \boldsymbol{R}[x, \zeta]_{i,j}$ $(i \geqq j, \ i+j=2m)$. By Lemma 2.1 we see that

$$\varDelta G_\iota^* f = G_\iota^* \Big\{ \big(i(i+n-1) + j(j+n-1) \big) f + 2 \sum_{k,l} x_k \zeta_l \frac{\partial^2 f}{\partial \zeta_k \partial x_l} - \sum_k \frac{\partial^2 f}{\partial x_k^2} - \sum_k \frac{\partial^2 f}{\partial \zeta_k^2} \Big\} .$$

Since

$$\sum_{k,l} x_k \zeta_l \frac{\partial^2 f}{\partial \zeta_k \partial x_l} = \tilde{X}_{E_0} \Big(\sum_k x_k \frac{\partial f}{\partial \zeta_k} \Big) - \sum_k \zeta_k \frac{\partial f}{\partial \zeta_k} + \sum_{k,l} x_k x_l \frac{\partial^2 f}{\partial \zeta_k \partial \zeta_l} ,$$

we can rewrite it as

$$\varDelta G_\iota^* f = N_{i,j} G_\iota^* f + 2 G_\iota^* \Big(\Big(\sum_k x_k \frac{\partial}{\partial \zeta_k} \Big)^2 f \Big) - G_\iota^* \Big(\sum_k \frac{\partial^2 f}{\partial x_k^2} + \sum_k \frac{\partial^2 f}{\partial \zeta_k^2} \Big) .$$

By applying this formula to $\Big(\sum_k x_k \frac{\partial}{\partial \zeta_k} \Big)^{2p} f \in \boldsymbol{R}[x, \zeta]_{i+2p, j-2p}$ $(0 \leqq p \leqq [j/2])$, we also have

$$\varDelta G_\iota^* \Big(\Big(\sum_k x_k \frac{\partial}{\partial \zeta_k} \Big)^{2p} f \Big) = N_{i+2p, j-2p} G_\iota^* \Big(\Big(\sum_k x_k \frac{\partial}{\partial \zeta_k} \Big)^{2p} f \Big) + 2 G_\iota^* \Big(\Big(\sum_k x_k \frac{\partial}{\partial \zeta_k} \Big)^{2p+2} f \Big) - G_\iota^* \Big(\sum_l \Big(\frac{\partial^2}{\partial x_l^2} + \frac{\partial^2}{\partial \zeta_l^2} \Big) \Big(\sum_k x_k \frac{\partial}{\partial \zeta_k} \Big)^{2p} f \Big) .$$

Put

$$f_p = \sum_{q=p}^{[j/2]} a_{q-p}^{m-j+1+2p} \Big(\sum_k x_k \frac{\partial}{\partial \zeta_k} \Big)^{2q} f \qquad (0 \leqq p \leqq [j/2]) .$$

Then the above formulas imply that

$$\varDelta G_\iota^* f_p = N_{i+2p, j-2p} G_\iota^* f_p - G_\iota^* \Big(\sum_l \Big(\frac{\partial^2}{\partial x_l^2} + \frac{\partial^2}{\partial \zeta_l^2} \Big) f_p \Big) .$$

By taking the $G(Q_{2m})$-components of both sides, we obtain

$$\varDelta (G_\iota^* f_p)_{G(Q_{2m})} = N_{i+2p, j-2p} (G_\iota^* f_p)_{G(Q_{2m})} \qquad (0 \leqq p \leqq [j/2]) .$$

This shows that $(G_\iota^* f_p)_{G(Q_{2m})}$ is an eigenfunction of $\varDelta$ corresponding to the eigenvalue $N_{i+2p, j-2p}$ if it does not vanish.

Put $c_0^{i,j} = 1$, and define real numbers $c_q^{i,j}$ $(1 \leqq q \leqq [j/2])$ inductively by

$$c_q^{i,j} = - \sum_{p=0}^{q-1} c_p^{i,j} a_{q-p}^{m-j+1+2p} .$$

Then it follows that

$$(G_\iota^* f)_{G(Q_{2m})} = \sum_{p=0}^{[j/2]} c_p^{i,j} (G_\iota^* f_p)_{G(Q_{2m})} ,$$

which proves the proposition.

We set

$$Q_{i,j}=\left\{f\in G(Q_{2m})\middle|\Delta f=N_{i,j}f\right\}$$

for each (i,j) such that $i\geqq j\geqq 0$ and $i+j=2m$. Since

$$N_{m,m}<N_{m+1,m-1}<\cdots<N_{2m,0},$$

the subspaces $Q_{i,j}$ $(i+j=2m,\ i\geqq j\geqq 0)$ of $G(Q_{2m})$ are mutually orthogonal. We note that some $Q_{i,j}$ may be $\{0\}$.

Set

$$I_m=\left\{(i,j)\in \boldsymbol{Z}\times\boldsymbol{Z}\middle| i\geqq j\geqq 0,\ i+j=2m\right\},$$

and $I=\bigcup_{m\geqq 0} I_m$. Let $c_q^{i,j}$ be the constants defined in the proof of Proposition 2.7. For $h\in C^\infty(S^*S^n)$ we denote by $h_{Q_{i,j}}$ the $Q_{i,j}$-component of h. The following corollary is immediately obtained from the proof of Proposition 2.7.

COROLLARY 2.8. (i) $G(Q_{2m})=\sum_{(i,j)\in I_m} Q_{i,j}$, $G(P)=\sum_{(i,j)\in I} Q_{i,j}$ *(orthogonal direct sum).*

(ii) *For* $f\in\boldsymbol{R}[x,\zeta]_{i,j}$ $((i,j)\in I_m)$

$$(G_t{}^*f)_{Q_{i+2p,j-2p}}=c_p^{i,j}\,G_t{}^*\left(\sum_{q=p}^{[j/2]} a_{q-p}^{m-j+1+2p}\left(\sum_k x_k\frac{\partial}{\partial\zeta_k}\right)^{2q}f\right)_{G(Q_{2m})}$$

$(0\leqq p\leqq[j/2])$.

REMARK. It possively occurs that $N_{i,j}=N_{k,l}$ with $i+j\neq k+l$. For example, $N_{2m-2,4}=N_{2m,0}$ if $n=4m-5$. Thus the decomposition $G(P)=\sum_{(i,j)\in I} Q_{i,j}$ is in general finer than the simple eigenspace decomposition.

We define a partial ordering on the set of indices I as follows: $(k,l)\leqq(i,j)$ if $k+l\leqq i+j$, $l\leqq j$, and $j-l$ is even.

PROPOSITION 2.9. (i) $G(P_{i,j})\subset\sum_{(k,l)\leqq(i,j)} Q_{k,l}$, $(i,j)\in I$.

(ii) $Q_{i,j}\subset\sum_{(k,l)\leqq(i,j)} G(P_{k,l})$, $(i,j)\in I$.

(iii) $G(P^j)=\sum Q_{k,l}$, *where the sum is taken over all* $(k,l)\in I$ *such that* $l\leqq j$ *and* $j-l$ *is even.*

PROOF. We first prove (i) and (ii) at the same time by induction on the integer $i+j$. It is clear that $G(P_{0,0})=Q_{0,0}=\{$constant functions$\}$. Fix an integer $m>0$ and assume that (i) and (ii) hold for every $(i,j)\in I$ with $i+j<2m$.

Take $(i,j)\in I_m$ and $f\in\boldsymbol{R}[x,\zeta]_{i,j}$. By the proof of Proposition 2.7 we can write

$$f=\sum_{p=0}^{[j/2]} c_p^{i,j} f_p,$$

where $f_p=\sum_{q=p}^{[j/2]} a_{q-p}^{m-j+1+2p}\left(\sum_k x_k \frac{\partial}{\partial \zeta_k}\right)^{2q} f$. Each f_p satisfies

$$\Delta G_t^* f_p = N_{i+2p,j-2p} G_t^* f_p - G_t^*\left(\sum_k\left(\frac{\partial^2}{\partial x_k^2}+\frac{\partial^2}{\partial \zeta_k^2}\right)f_p\right),$$

and

$$G_t^*\left(\sum_k\left(\frac{\partial^2}{\partial x_k^2}+\frac{\partial^2}{\partial \zeta_k^2}\right)f_p\right)\in \sum_{q=p}^{[j/2]} G(P_{i+2q-2,j-2q}).$$

If $i>j$, then $(i-2,j)\in I$. Since $(i+2q-2, j-2q)\leqq(i-2,j)$, it follows that

$$G(P_{i+2q-2,j-2q})\subset \sum_{(k,l)\leqq(i-2,j)} Q_{k,l} \qquad (0\leqq q\leqq [j/2])$$

by the assumption. If $i=j$, then $G(P_{i-2,j})=G(P_{i,i-2})$ and $(i, i-2)\in I$. Since $(i+2q-2, j-2q)\leqq(i, i-2)$ $(1\leqq q\leqq[j/2])$, we have in this case

$$G(P_{i+2q-2,j-2q})\subset \sum_{(k,l)\leqq(i,i-2)} Q_{k,l} \qquad (0\leqq q\leqq [j/2]).$$

Hence we see that $\Delta G_t^* f_p - N_{i+2p,j-2p} G_t^* f_p$ lies in $\sum_{(k,l)\leqq(i-2,j)} Q_{k,l}$ if $i>j$, and in $\sum_{(k,l)\leqq(i,i-2)} Q_{k,l}$ if $i=j$.

Let

$$\Delta G_t^* f_p - N_{i+2p,j-2p} G_t^* f_p = \sum_{(k,l)} f_p^{k,l} \qquad (f_p^{k,l}\in Q_{k,l})$$

be the corresponding decomposition. We notice here that $N_{k,l}\neq N_{i+2p,j-2p}$ if $f_p^{k,l}\neq 0$, because

$$\begin{aligned} N_{k,l}(G_t^* f_p, f_p^{k,l}) &= (G_t^* f_p, \Delta f_p^{k,l}) = (\Delta G_t^* f_p, f_p^{k,l}) \\ &= N_{i+2p,j-2p}(G_t^* f_p, f_p^{k,l}) + (f_p^{k,l}, f_p^{k,l}). \end{aligned}$$

Then it is easily seen that the function

$$h_p = G_t^* f_p + \sum_{(k,l)} (N_{i+2p,j-2p} - N_{k,l})^{-1} f_p^{k,l}$$

is an eigenfunction corresponding to the eigenvalue $N_{i+2p,j-2p}$ if it does not vanish.

We must show that $h_p\in \sum_{(k,l)\leqq(i,j)} Q_{k,l}$. Since $h_p\in G(P_{2m})$, we have the decomposition

$$h_p = \sum_{\substack{(r,s)\in I \\ r+s\leqq 2m}} h_p^{r,s}, \qquad h_p^{r,s}\in Q_{r,s}.$$

The eigenvalue condition implies that $h_p^{r,s}=0$ if $N_{r,s}\neq N_{i+2p,j-2p}$. If $r+s=2m$ and $N_{r,s}=N_{i+2p,j-2p}$, then we have $(r, s)=(i+2p, j-2p)$. Suppose that $r+s<$

$2m$ and $N_{r,s}=N_{i+2p,j-2p}$. Since $N_{i+2p,j-2p}=N_{r,s}<N_{2m-s,s}$ and $N_{2m-s,s}$ is monotonously decreasing in s, it follows that $s<j-2p$. Let σ be the isometry of (S^*S^n, g_1) defined by $\sigma(x, \zeta)=(x, -\zeta)$. Since $\sigma\circ\xi_t=\xi_{-t}\circ\sigma$, it follows that $\sigma^*\circ G=G\circ\sigma^*$. This implies $\sigma^*=(-1)^s\cdot$identity on $G(P_{r,s})$. Hence by the induction assumption we have $\sigma^*=(-1)^s\cdot$identity on $Q_{r,s}$ if $r+s<2m$. Thus we have $\sigma^* h_p=(-1)^j h_p$ by the definition of h_p. Since $(\sigma^* h_p, \sigma^* h_p^{r,s})=(h_p, h_p^{r,s})$, it follows that $h_p^{r,s}=0$ if $r+s<2m$ and $j-s$ is odd. Hence we have $h_p\in \sum_{(k,l)\leqq(i,j)} Q_{k,l}$, and therefore

$$G\iota^* f_p = h_p - \sum_{(k,l)} (N_{i+2p,j-2p}-N_{k,l})^{-1} f_p^{k,l} \in \sum_{(k,l)\leqq(i,j)} Q_{k,l}\,.$$

Furthermore, considering the case $p=0$ we have

$$\begin{aligned}(G\iota^* f_0)_{Q_{i,j}} &= h_0^{i,j}\\ &= G\iota^* f_0 - \sum_{(r,s)\neq(i,j)} h_0^{r,s} + \sum_{(k,l)} (N_{i,j}-N_{k,l})^{-1} f_0^{k,l}\,.\end{aligned}$$

The second and the third term of the right-hand side belong to $\sum_{\substack{(k,l)\leqq(i,j)\\ k+l<2m}} Q_{k,l}$, which is contained in $\sum_{(k,l)\leqq(i,j)} G(P_{k,l})$ by the induction assumption. Since the linear map $G(P_{i,j})\to Q_{i,j}$ defined by $G\iota^* f\to(G\iota^* f_0)_{Q_{i,j}}$ is surjective, it follows that

$$Q_{i,j}\subset \sum_{(k,l)\leqq(i,j)} G(P_{k,l})\,.$$

Hence (i) and (ii) have been proved.

For (iii) we observe that $G(P^j)=\sum_{p\geqq0} G(P_{j+2p,j})$. Then (iii) immediately follows from (i) and (ii).

Let $\mathscr{A}^k$ $(k\geqq0)$ be the vector space of functions f on S^*S^n such that $f|_{S_x^*S^n}$ are the restrictions of homogeneous polynomials of degree k on $T_x^*S^n$ to $S_x^*S^n$ for all $x\in S^n$.

PROPOSITION 2.10. *$G(P^k)$ is C^0-dense, and hence L^2-dense, in $G(\mathscr{A}^k)$.*

PROOF. First remark that $\mathscr{A}^k$ is generated by

$$\left\{\iota^*(\zeta_{i_1}\cdots\zeta_{i_k})\middle| 1\leqq i_1\leqq\cdots\leqq i_k\leqq n+1\right\}$$

as a $C^\infty(S^n)$-module. By the Stone-Weierstrass approximation theorem (cf. [2] 7.3.1) we see that $\iota_0^*\boldsymbol{R}[x_1, \cdots, x_{n+1}]$ is dense in $C^\infty(S^n)$ in the C^0-topology. Hence P^k is C^0-dense in $\mathscr{A}^k$. Since the operator G is C^0-continuous, the proposition follows.

REMARK. By applying the Stone-Weierstrass theorem we can also see that $G(P)$ is C^0-dense in $G(C^\infty(S^*S^n))$. But this fact is not used in this paper.

§ 3. A result for homogeneous polynomials

In the rest of the paper we shall assume that n, the dimension of the sphere under consideration, is equal to or greater than 3. The main purpose of this section is to prove

PROPOSITION 3.1. *Let $f\in \boldsymbol{R}[x_1, \cdots, x_{n+1}]_{2j+1}$ $(j\geqq 2)$, and suppose that $G_t{}^*F(f,f)\in G(\mathscr{H}^2)$. Then there are constants a_i, $b_i\in \boldsymbol{R}$ $(1\leqq i\leqq n+1)$ such that*

$$f\equiv (\sum_i a_i x_i)^{2j}(\sum_i b_i x_i) \qquad \mathrm{mod}\,(\sum_i x_i^2)\,\boldsymbol{R}[x]_{2j-1}.$$

We first give another representation of $\tilde{G}F(f, h)$ defined in § 1. Set

$$\boldsymbol{R}[x]_{od}=\sum_{k\geqslant 0}\boldsymbol{R}[x]_{2k+1}.$$

LEMMA 3.2. *Let $f\in \boldsymbol{R}[x]_{2j+1}$ $(j\geqq 0)$ and $h\in \boldsymbol{R}[x]_{od}$. Then*

$$\begin{aligned}\tilde{G}F(f, h) &= \tilde{G}\Big(\sum_i \frac{\partial f}{\partial x_i}(x)\int_0^\pi \frac{\partial h}{\partial x_i}(x\cos t+\zeta \sin t)\sin t\, dt\Big)\\ &\quad -(2j+3)\,\tilde{G}\Big(f(x)\int_0^\pi \sum_i \frac{\partial h}{\partial x_i}(x\cos t+\zeta\sin t)\, x_i \sin t\, dt\Big)\\ &\quad +2\tilde{G}\big(f(x)\,h(x)\big).\end{aligned}$$

PROOF. By the homogeneity of f we have

$$\begin{aligned}&\sum_{i,j} x_i x_j \frac{\partial f}{\partial x_i}(x)\int_0^\pi \frac{\partial h}{\partial x_j}(x\cos t+\zeta\sin t)\sin t\, dt\\ &\quad =(2j+1)f(x)\int_0^\pi \sum_j \frac{\partial h}{\partial x_j}(x\cos t+\zeta \sin t)\, x_j \sin t\, dt.\end{aligned}$$

Since $\sum_i \zeta_i \dfrac{\partial f}{\partial x_i}=\tilde{X}_{E_0}f$ and $\tilde{G}\circ\tilde{X}_{E_0}=0$, it follows that

$$\begin{aligned}&\tilde{G}\Big(\sum_{i,j}\zeta_i\zeta_j\frac{\partial f}{\partial x_i}(x)\int_0^\pi \frac{\partial h}{\partial x_j}(x\cos t+\zeta\sin t)\sin t\, dt\Big)\\ &\quad = -\tilde{G}\Big(f(x)\int_0^\pi \sum_j \frac{d}{dt}\Big\{\frac{\partial h}{\partial x_j}(x\cos t+\zeta\sin t)\Big\}\zeta_j \sin t\, dt\Big)\\ &\quad\quad +\tilde{G}\Big(f(x)\int_0^\pi \sum_j \frac{\partial h}{\partial x_j}(x\cos t+\zeta\sin t)\, x_j\sin t\, dt\Big).\end{aligned}$$

The first term of the right-hand side is

$$\tilde{G}\Big(f(x)\int_0^\pi \sum_j \frac{\partial h}{\partial x_j}(x\cos t+\zeta\sin t)\,\zeta_j\cos t\, dt\Big)$$

$$=\tilde{G}\left(f(x)\int_0^\pi \sum_j \frac{\partial h}{\partial x_j}(x\cos t+\zeta\sin t)\, x_j \sin t\, dt\right)$$
$$+\tilde{G}\left(f(x)\int_0^\pi \frac{d}{dt} h(x\cos t+\zeta\sin t)\, dt\right).$$

The lemma easily follows from these formulas.

The following corollary is an immediate consequence of Lemma 3. 2.

COROLLARY 3. 3. *Let* $f\in \boldsymbol{R}[x]_{od}$ *and* $h\in\boldsymbol{R}[x]_1+\boldsymbol{R}[x]_3$. *Then*

$$G_t^* F(f,h)\in G(\mathscr{K}^2).$$

In particular, if $G_t^*F(f,f)\in G(\mathscr{K}^2)$, *then*

$$G_t^* F(f+h,\, f+h)\in G(\mathscr{K}^2).$$

Define bilinear maps $F_l:\ \boldsymbol{R}[x]_{2i+1}\times\boldsymbol{R}[x]_{2j+1}\to\boldsymbol{R}[x,\zeta]_{2i+2l+2,2j-2l}$ $(0\leqq l\leqq j)$ by

$$F_l(f,h)=f(x)\left(\sum_k x_k\frac{\partial}{\partial\zeta_k}\right)^{2l+1} h(\zeta),\ f\in\boldsymbol{R}[x]_{2i+1},\ h\in\boldsymbol{R}[x]_{2j+1}.$$

Put

$$I_l^j=\int_0^\pi(\cos t)^{2l}(\sin t)^{2j+1-2l}dt=\frac{2(2l-1)!!\,(2j-2l)!!}{(2j+1)!!}\qquad(0\leqq l\leqq j).$$

Then we have

$$f(x)\int_0^\pi\sum_k\frac{\partial h}{\partial x_k}(x\cos t+\zeta\sin t)\,x_k\sin t\,dt$$
$$=\sum_{l=0}^j\frac{1}{(2l)!}I_l^j F_l(f,h).$$

We now assume $i\geqq j\geqq 0$. Define real constants $d_l^{i,j}$ $(0\leqq l\leqq j)$ inductively by $d_0^{i,j}=I_0^j$ and

$$d_l^{i,j}=\frac{1}{(2l)!}I_l^j-\sum_{p=0}^{l-1}d_p^{i,j}\,a_{l-p}^{i-j+2+2p}\qquad(l\geqq 1).$$

Then we have

$$f(x)\int_0^\pi\sum_k\frac{\partial h}{\partial x_k}(x\cos t+\zeta\sin t)\,x_k\sin t\,dt$$
$$=\sum_{p=0}^j d_p^{i,j}\sum_{q=p}^j a_{q-p}^{i-j+2+2p}F_q(f,h).$$

Put

$$J_\beta^\alpha=(-1)^\beta a_\beta^\alpha=\frac{(2\alpha-1)!!}{(2\beta)!!\,(2\alpha+2\beta-1)!!}\qquad(\alpha\geqq 0,\ \beta\geqq 0).$$

LEMMA 3.4. (i) $\sum_{p=0}^{\beta} a_p^{\alpha} J_{\beta-p}^{\alpha+\beta-1+p}=\delta_{\beta,0}$ $(\alpha\geqq 1,\ \beta\geqq 0)$.

(ii) $d_l^{i,j}>0$ $\quad(0\leqq l\leqq j\leqq i)$.

PROOF. (i) In case $\beta=0$ the formula is obvious. Assume $\beta\geqq 1$, and consider the identity

$$t^{2\alpha+2\beta-3}(1+t^2)^{\beta}=\sum_{p=0}^{\beta}\binom{\beta}{p}t^{2\alpha+2\beta+2p-3}.$$

By applying $\left(t^{-1}\frac{d}{dt}\right)^{\beta-1}$ to both sides and putting $t=\sqrt{-1}$, we have

$$\sum_{p=0}^{\beta}(-1)^p\binom{\beta}{p}\frac{(2\alpha+2\beta+2p-3)!!}{(2\alpha+2p-1)!!}=0.$$

This proves (i).

(ii) Since $\sum_{p=0}^{q} d_p^{i,j} a_{q-p}^{i-j+2+2p}=\frac{1}{(2q)!}I_q^j$, it follows that

$$\begin{aligned}&\sum_{q=0}^{l}\frac{1}{(2q)!}I_q^j J_{l-q}^{i-j+1+l+q}\\&=\sum_{p=0}^{l} d_p^{i,j}\sum_{q=p}^{l} a_{q-p}^{i-j+2+2p}J_{l-q}^{i-j+1+l+q}\\&=\sum_{p=0}^{l} d_p^{i,j}\delta_{l,p}=d_l^{i,j}.\end{aligned}$$

Hence we have the lemma.

PROPOSITION 3.5. *Let $f\in \boldsymbol{R}[x]_{2i+1}$ and $h\in\boldsymbol{R}[x]_{2j+1}$ $(i\geqq j\geqq 2)$. Suppose that $G_t{}^*F(f,h)\in G(\mathscr{K}^2)$. Then*

$$\left(G_t{}^*F_p(f,h)\right)_{Q_{2i+2p+2,2j-2p}}=0$$

for all p such that $0\leqq p\leqq j-2$.

PROOF. We have

$$G(P^2)=\sum_{r\geqq 0}Q_{2r,0}+\sum_{r\geqq 1}Q_{2r,2}$$

by Proposition 2.9 (iii). Since $G(P^2)$ is L^2-dense in $G(\mathscr{K}^2)$, it thus follows that

$$\left(G_t{}^*F(f,h)\right)_{Q_{2i+2p+2,2j-2p}}=0\qquad(0\leqq p\leqq j-2).$$

Now observe the formula stated in Lemma 3.2. Since $G_t{}^*(f(x)h(x))\in G(P^0)=\sum_{r\geqq 0}Q_{2r,0}$ and

$$G_t^*\left(\sum_k \frac{\partial f}{\partial x_k}(x)\int_0^\pi \frac{\partial h}{\partial x_k}(x\cos t+\zeta\sin t)\sin t\,dt\right)\in G(P_{2i+2j}),$$

it follows that

$$\begin{aligned}&\left(G_t^* F(f,h)\right)_{Q_{2i+2p+2,2j-2p}}\\ &\quad=-(2i+3)\left(G_t^*\left(f(x)\int_0^\pi \sum_k \frac{\partial h}{\partial x_k}(x\cos t+\zeta\sin t)\,x_k\sin t\,dt\right)\right)_{Q_{2i+2p+2,2j-2p}}.\end{aligned}$$

We have seen above that

$$\begin{aligned}&G_t^*\left(f(x)\int_0^\pi \sum_k \frac{\partial h}{\partial x_k}(x\cos t+\zeta\sin t)\,x_k\sin t\,dt\right)\\ &\quad=\sum_{r=0}^{j} d_r^{i,j}\,G_t^*\left(\sum_{q=r}^{j} a_{q-r}^{i-j+2+2r}F_q(f,h)\right).\end{aligned}$$

But Corollary 2.8 (ii) implies that

$$\begin{aligned}&G_t^*\left(\sum_{q=r}^{j} a_{q-r}^{i-j+2+2r}F_q(f,h)\right)_{G(Q_{2i+2j+2})}\\ &\quad=\left(G_t^* F_r(f,h)\right)_{Q_{2i+2r+2,2j-2r}}\qquad (0\leqq r\leqq j).\end{aligned}$$

Therefore we have

$$\left(G_t^* F(f,h)\right)_{Q_{2i+2p+2,2j-2p}}=-(2i+3)\,d_p^{i,j}\left(G_t^* F_p(f,h)\right)_{Q_{2i+2p+2,2j-2p}}.$$

Since $d_p^{i,j}>0$, the proposition follows.

We denote by $C_C^\infty(M)$ the vector space of complex-valued functions on a manifold M. The operators G, $\tilde{G}$, X_{E_0}, $\tilde{X}_{E_0}$, ι^*, and Δ can be naturally extended to $\boldsymbol{C}$-linear operators (the complexifications) on the spaces $C_C^\infty(S^*S^n)$, $C_C^\infty(\boldsymbol{R}^{2n+2})$, etc., which will be deboted by the same symbols.

Let $\boldsymbol{C}[x]$ (resp. $\boldsymbol{C}[x,\zeta]$) be the polynomial algebra in the variables $x=(x_1,\cdots,x_{n+1})$ (resp. $(x,\zeta)=(x_1,\cdots,x_{n+1},\zeta_1,\cdots,\zeta_{n+1})$) with complex coefficients. We denote by $\boldsymbol{C}[x]_k$ and $\boldsymbol{C}[x,\zeta]_k$ (resp. $\boldsymbol{C}[x,\zeta]_{i,j}$) the vector spaces spanned by homogeneous polynomials of degree k (resp. bihomogeneous polynomials of bidegree (i,j)) as in the real polynomials. In general, for a commutative ring R we denote by $(f_1,\cdots,f_r)$ the ideal in R generated by $f_1,\cdots,f_r\in R$.

Considering $\boldsymbol{C}[x,\zeta]$ as a subalgebra of $C_C^\infty(\boldsymbol{R}^{2n+2})$, we have

LEMMA 3.6. *The kernel of* $\iota^*|_{\boldsymbol{C}[x,\zeta]_{2k}}$ $(k\geqq 1)$ *is*

$$\left(\sum_i (x_i^2-\zeta_i^2)\right)\boldsymbol{C}[x,\zeta]_{2k-2}+\left(\sum_i x_i\zeta_i\right)\boldsymbol{C}[x,\zeta]_{2k-2}.$$

PROOF. Let σ_1 and σ_2 be the linear transformations of $\boldsymbol{R}^{2n+2}$ defined by

$$\sigma_1(x, \zeta) = (x, -\zeta), \qquad \sigma_2(x, \zeta) = (\zeta, x).$$

Then $\sigma_1^* \sigma_2^* = \sigma_2^* \sigma_1^*$ on $\boldsymbol{C}[x, \zeta]_{2k}$, and we have the decomposition

$$\boldsymbol{C}[x, \zeta]_{2k} = V_{0,0} + V_{1,0} + V_{0,1} + V_{1,1},$$

where $V_{i,j} = \{f \in \boldsymbol{C}[x, \zeta]_{2k} \mid \sigma_1^* f = (-1)^i f,\ \sigma_2^* f = (-1)^j f\}$ $(i, j = 0, 1)$. Identify $S^* S^n$ with $\iota(S^* S^n) \subset \boldsymbol{R}^{2n+2}$. Then we see that σ_1 and σ_2 preserve $S^* S^n$, and $\iota^* \sigma_i^* = \sigma_i^* \iota^*$ $(i = 1, 2)$. Hence

$$\text{Kernel of } \iota^*|_{\boldsymbol{C}[x,\zeta]_{2k}} = \sum_{i,j=0}^{1} \text{Kernel of } \iota^*|_{V_{i,j}}.$$

Take $f \in V_{0,0}$ such that $\iota^* f = 0$. Since $\sigma_1^* f = f$, we can write

$$f = \sum_{p=0}^{k} f_p, \qquad f_p \in \boldsymbol{C}[x, \zeta]_{2p, 2k-2p}.$$

Let $W_1 = \{(x, \zeta) \in \boldsymbol{R}^{2n+2} \mid \sum_i (x_i^2 - \zeta_i^2) = \sum_i x_i \zeta_i = 0\}$. Since $f = 0$ on $S^* S^n$ and f is homogeneous, it follows that $f = 0$ on W_1. Let $W_2 = \{(x, \zeta) \in \boldsymbol{R}^{2n+2} \mid \sum_i x_i \zeta_i = 0\}$. Define $f' \in \boldsymbol{C}[x, \zeta]_{2k,2k}$ by

$$f' = \sum_{p=0}^{k} (\sum_i x_i^2)^{k-p} (\sum_i \zeta_i^2)^p f_p.$$

Then $f' = 0$ on W_1. Since f' is bihomogeneous, it follows that $f' = 0$ on W_2. It is clear that

$$(\sum_i x_i^2)^k f - f' \in \Big(\sum_i (x_i^2 - \zeta_i^2)\Big) \boldsymbol{C}[x, \zeta]_{4k-2}.$$

Since $\sigma_2^* f = f$, we see that $f_p(x, \zeta) = f_{k-p}(\zeta, x)$ $(0 \leqq p \leqq k)$. Hence $\sigma_2^* f' = f'$.

Put $y_i = x_i + \zeta_i$, $\xi_i = x_i - \zeta_i$ $(1 \leqq i \leqq n+1)$, and define $f'_p \in \boldsymbol{C}[y, \xi]_{p, 4k-p}$ $(0 \leqq p \leqq 4k)$ by

$$f'\left(\frac{y+\xi}{2}, \frac{y-\xi}{2}\right) = \sum_{p=0}^{4k} f'_p(y, \xi).$$

Since $\sigma_2(y, \xi) = (y, -\xi)$, it follows that $f'_p = 0$ if p is odd. Define $f'' \in \boldsymbol{C}[y, \xi]_{4k,4k}$ by

$$f''(y, \xi) = \sum_{p=0}^{2k} (\sum_i y_i^2)^{2k-p} (\sum_i \xi_i^2)^p f'_{2p}(y, \xi).$$

Since $f''(y, \xi) = (\sum_i y_i^2)^{2k} f'\left(\dfrac{y+\xi}{2}, \dfrac{y-\xi}{2}\right) = 0$ on

$$W_2 = \left\{(y, \xi) \in \boldsymbol{R}^{2n+2} \,\Big|\, \sum_i y_i^2 = \sum_i \xi_i^2\right\},$$

it follows that f'' is identically zero. We have

$$(\sum_i y_i^2)^{2k} f'\left(\frac{y+\xi}{2}, \frac{y-\xi}{2}\right) - f''(y, \xi) \in \left(\sum_i (y_i^2 - \xi_i^2)\right) \boldsymbol{C}[y, \xi]_{8k-2}.$$

This implies

$$\left(\sum_i (x_i^2 + \zeta_i^2)\right)^{2k} f'(x, \zeta) \in (\sum_i x_i \zeta_i) \boldsymbol{C}[x, \zeta]_{8k-2}.$$

Hence we have

$$\left(\sum_i (x_i^2 + \zeta_i^2)\right)^{3k} f(x, \zeta) \in \left(\sum_i (x_i^2 - \zeta_i^2)\right) \boldsymbol{C}[x, \zeta]_{8k-2} + (\sum_i x_i \zeta_i) \boldsymbol{C}[x, \zeta]_{8k-2}.$$

In general, it is known that the ring $\boldsymbol{C}[X_1, \cdots, X_m]/\left(\sum_{i=1}^m X_i^2\right)$ is a UFD (a unique factorization domain) if $m \geqq 5$. Hence the ring $\boldsymbol{C}[x, \zeta]/(\sum_i (x_i^2 - \zeta_i^2))$ is a UFD. It is easy to see that the image of $\sum_i x_i \zeta_i$ by the homomorphism $\boldsymbol{C}[x, \zeta] \rightarrow \boldsymbol{C}[x, \zeta]/(\sum_i (x_i^2 - \zeta_i^2))$ is irreducible, and hence is a prime element. Therefore the ideal $(\sum_i (x_i^2 - \zeta_i^2), \sum_i x_i \zeta_i)$ in $\boldsymbol{C}[x, \zeta]$ is prime. Since $\sum_i (x_i^2 + \zeta_i^2)$ does not belong to this ideal, it follows that

$$f \in \left(\sum_i (x_i^2 - \zeta_i^2), \sum_i x_i \zeta_i\right).$$

In case that f belongs to $V_{1,0}$ or $V_{0,1}$ or $V_{1,1}$, we define $h \in V_{0,0}$ by $h = x_1 \zeta_1 f$ if $f \in V_{1,0}$, $h = (x_i^2 - \zeta_i^2) f$ if $f \in V_{0,1}$, and $h = x_1 \zeta_1 (x_1^2 - \zeta_1^2) f$ if $f \in V_{1,1}$. If $\iota^* f = 0$, then $\iota^* h = 0$, and we have $h \in (\sum_i (x_i^2 - \zeta_i^2), \sum_i x_i \zeta_i)$. Since $x_1 \zeta_1$ and $x_1^2 - \zeta_1^2$ do not belong to this prime ideal, we also have $f \in (\sum_i (x_i^2 - \zeta_i^2), \sum_i x_i \zeta_i)$ in these cases. By considering the homogeneity we have the lemma.

We complexify the vector spaces $G(P_k)$, $G(P_{i,j})$, $G(P^j)$, $G(Q_k)$, and $Q_{i,j}$, and denote them by the same symbols.

COROLLARY 3.7. *Let* $f \in \boldsymbol{C}[x, \zeta]_{2k}$ $(k \geqq 1)$.

(i) $(G\iota^* f)_{G(Q_{2k})} = 0$ *if and only if* $\tilde{G} f$ *belong to the ideal* $(\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i \zeta_i)$.

(ii) *Suppose that* f *is a polynomial in only* 4 *variables* $(x_1, x_2, \zeta_1, \zeta_2)$. *Then* $(G\iota^* f)_{G(Q_{2k})} = 0$ *if and only if* $\tilde{G} f = 0$.

PROOF. (i) First assume that $\tilde{G} f \in (\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i \zeta_i)$. Then there are homogeneous polynomials $R_i \in \boldsymbol{C}[x, \zeta]_{2k-2}$ $(i = 1, 2, 3)$ such that

$$\tilde{G} f = \sum_i x_i^2 R_1 + \sum_i \zeta_i^2 R_2 + \sum_i x_i \zeta_i R_3.$$

By applying ι^* to this formula, we have

$$G\iota^* f = \iota^*(R_1+R_2) = G\iota^*(R_1+R_2) \in G(P_{2k-2}) .$$

Hence $(G\iota^* f)_{G(Q_{2k})}=0$.

Next assume that $(G\iota^* f)_{G(Q_{2k})}=0$. Then there is $h_0 \in \boldsymbol{C}[x, \zeta]_{2k-2}$ such that $G\iota^* f = G\iota^* h_0$. This shows that

$$\iota^*\left(\tilde{G}f - \frac{1}{2}\sum_i (x_i^2+\zeta_i^2)\,\tilde{G}h_0\right) = 0 .$$

Thus by Lemma 3.6 there are polynomials $h_1, h_2 \in \boldsymbol{C}[x, \zeta]_{2k-2}$ such that

$$\tilde{G}f = \frac{1}{2}\sum_i (x_i^2+\zeta_i^2)\,\tilde{G}h_0 + \sum_i (x_i^2-\zeta_i^2)\,h_1 + \sum_i x_i\zeta_i h_2 .$$

(ii) Since f is a polynomial in the variables $(x_1, x_2, \zeta_1, \zeta_2)$, so is $\tilde{G}f$. Assume that $\tilde{G}f$ is in the ideal $(\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i\zeta_i)$. Fix $(x_1, x_2, \zeta_1, \zeta_2) \in \boldsymbol{R}^4$ and put

$$x_3 = \sqrt{-1}\,\sqrt{x_1^2+x_2^2}\,\cos\theta , \qquad \zeta_3 = \sqrt{-1}\,\sqrt{\zeta_1^2+\zeta_2^2}\,\cos\eta ,$$

$$x_4 = \sqrt{-1}\,\sqrt{x_1^2+x_2^2}\,\sin\theta , \qquad \zeta_4 = \sqrt{-1}\,\sqrt{\zeta_1^2+\zeta_2^2}\,\sin\eta ,$$

and $x_i=\zeta_i=0$ $(5 \leqq i \leqq n+1)$, where θ and η are real numbers such that

$$\sqrt{(x_1^2+x_2^2)(\zeta_1^2+\zeta_2^2)}\,\cos(\theta-\eta) = x_1\zeta_1 + x_2\zeta_2 .$$

Then we have $\sum_i x_i^2 = \sum_i \zeta_i^2 = \sum_i x_i\zeta_i = 0$, and hence

$$(\tilde{G}f)(x_1, x_2, \zeta_1, \zeta_2) = 0 .$$

Since $(x_1, x_2, \zeta_1, \zeta_2) \in \boldsymbol{R}^4$ is arbitrary, it follows that $\tilde{G}f=0$. This completes the proof.

PROPOSITION 3.8. *Let $f \in \boldsymbol{C}[x]_{2i+1}$ and $h \in \boldsymbol{C}[x]_{2j+1}$ $(i \geqq j \geqq 2)$. Suppose that f and h are polynomials in two variables (x_1, x_2). Then*

(i) $(G\iota^* F_p(f, h))_{Q_{2i+2p+2,2j-2p}}=0$ *if and only if*

$$\tilde{G}\left(\sum_{q=p}^{j} a_{q-p}^{i-j+2+2p} F_q(f, h)\right) = 0 \qquad (0 \leqq p \leqq j) .$$

(ii) *If $(G\iota^* F_{i-2}(f, f))_{Q_{4i-2,4}}=0$, then f must be of the form*

$$f = (a_1 x_1 + a_2 x_2)^{2i}(b_1 x_1 + b_2 x_2) ,$$

where a_1, a_2 and b_1, b_2 are complex constants.

PROOF. (i) Since

$$\left(G\iota^* F_p(f, h)\right)_{Q_{2i+2p+2, 2j-2p}} = \left(G\iota^* \sum_{q=p}^{j} a_{q-p}^{i-j+2+2p} F_q(f, h)\right)_{G(Q_{2i+2j+2})},$$

(i) follows from Corollary 3. 7 (ii).

(ii) The general linear group $GL(2, \boldsymbol{C})$ naturally acts on the polynomial algebras $\boldsymbol{C}[x_1, x_2]$ and $\boldsymbol{C}[x_1, x_2, \zeta_1, \zeta_2]$. It is easy to see that these actions commute with the operators $\tilde{G}$ and F_p. Hence we see from (i) that $(G\iota^* F_{i-2}(f,f))_{Q_{4i-2,4}}=0$ if and only if

$$\left(G\iota^* F_{i-2}(A^*f, A^*f)\right)_{Q_{4i-2,4}} = 0 \text{ for } f\in\boldsymbol{C}[x_1, x_2]_{2i+1} \text{ and } A\in GL(2, \boldsymbol{C}).$$

Put

$$f=\sum_{k=0}^{2i+1} c_k x_1^k x_2^{2i+1-k};$$

By considering A^*f instead of f for a suitable $A\in GL(2, \boldsymbol{C})$ if necessary, we may assume that $c_0=0$. A direct computation shows that

$$\begin{aligned} &F_{i-2}(x_1^k x_2^{2i+1-k},\ x_1^l x_2^{2i+1-l}) \\ &= \frac{(2i-3)!}{4!}\{l(l-1)(l-2)(l-3)\, x_1^{k+l-4} x_2^{4i+2-(k+l)} \zeta_1^4 \\ &+4l(l-1)(l-2)(2i+1-l)\, x_1^{k+l-3} x_2^{4i+1-(k+l)} \zeta_1^3\zeta_2 \\ &+6l(l-1)(2i+1-l)(2i-l)\, x_1^{k+l-2} x_2^{4i-(k+l)} \zeta_1^2\zeta_2^2 \\ &+4l(2i+1-l)(2i-l)(2i-1-l)\, x_1^{k+l-1} x_2^{4i-1-(k+l)} \zeta_1\zeta_2^3 \\ &+(2i+1-l)(2i-l)(2i-1-l)(2i-2-l)\, x_1^{k+l} x_2^{4i-2-(k+l)} \zeta_2^4\} \end{aligned}$$

$(0\leqq k,\ l\leqq 2i+1)$. For each p, q such that $0\leqq p-q\leqq 4i-1$, $0\leqq q\leqq 3$, we have

$$\begin{aligned} \tilde{X}_{E_0}(x_1^{p-q} x_2^{4i-1-p+q} \zeta_1^q \zeta_2^{3-q}) &= (p-q)\, x_1^{p-q-1} x_2^{4i-1-p+q} \zeta_1^{q+1} \zeta_2^{3-q} \\ &+(4i-1-p+q)\, x_1^{p-q} x_2^{4i-2-p+q} \zeta_1^q \zeta_2^{4-q} + h, \end{aligned}$$

where $h\in\boldsymbol{C}[x, \zeta]_{4i,2}$. Since $(G\iota^* h)_{Q_{4i-2,4}}=0$ by Proposition 2. 9 (i), it follows that

$$\begin{aligned} &(p-q)\left(G\iota^*(x_1^{p-q-1} x_2^{4i-1-p+q} \zeta_1^{q+1} \zeta_2^{3-q})\right)_{Q_{4i-2,4}} \\ &= -(4i-1-p+q)\left(G\iota^*(x_1^{p-q} x_2^{4i-2-p+q} \zeta_1^q \zeta_2^{4-q})\right)_{Q_{4i-2,4}} \end{aligned}$$

for p, q with $0\leqq p-q\leqq 4i-1$, $0\leqq q\leqq 3$. Using this formula successively, we have

$$\begin{aligned} &\left(G\iota^* F_{i-2}(x_1^k x_2^{2i+1-k},\ x_1^l x_2^{2i+1-l})\right)_{Q_{4i-2,4}} \\ &= A_{k,l}^i\left(G\iota^*(x_1^{k+l} x_2^{4i-2-(k+l)} \zeta_2^4)\right)_{Q_{4i-2,4}} \qquad (0\leqq k,\ l\leqq 2i+1), \end{aligned}$$

where $A^i_{k,l}=0$ if $0\leqq k+l\leqq 3$ or $4i-1\leqq k+l\leqq 4i+2$, and

$$A^i_{k,l}=\frac{(2i-1)!}{4!}\frac{1}{(k+l)(k+l-1)(k+l-2)(k+l-3)}$$
$$\times\Big[2i(2i+1)\big\{k(k-1)(k-2)(k-3)+l(l-1)(l-2)(l-3)\big\}$$
$$-4(i-1)\,lk\big\{i(4k^2+4l^2-6lk-6k-6l+10)+3kl-3k-3l+3\big\}\Big]$$

if $4\leqq k+l\leqq 4i-2$. Especially we have

$$A^i_{k,1}=A^i_{1,k}=\frac{(2i)!}{4!}\frac{(2i+1)(k-7)+12}{k+1}\qquad(3\leqq k\leqq 2i+1)$$

and

$$A^i_{k,k}=\frac{3\cdot(2i-1)!}{4!}\frac{(2i+1-k)(2i-k)}{(2k-1)(2k-3)}\qquad(2\leqq k\leqq 2i-1),$$

which are not zero.

Now we can write

$$\left(G_t{}^*F_{i-2}(f,f)\right)_{Q_{4i-2,4}}$$
$$=\sum_{p=4}^{4i-2}\sum_{k+l=p}c_kc_lA^i_{k,l}\left(G_t{}^*(x_1{}^px_2{}^{4i-2-p}\zeta_2{}^4)\right)_{Q_{4i-2,4}}.$$

In Lemma 3.9 stated later we shall prove that

$$\left(G_t{}^*(x_1{}^px_2{}^{4i-2-p}\zeta_2{}^4)\right)_{Q_{4i-2,4}}\qquad(4\leqq p\leqq 4i-2)$$

are linearly independent. By using this fact we have

$$(\sharp)_p\qquad\sum_{k+l=p}c_kc_lA^i_{k,l}=0\qquad(4\leqq p\leqq 4i-2).$$

If $c_1=0$, then by the formulas $(\sharp)_{2p}$ $(2\leqq p\leqq 2i-1)$ and the fact $A^i_{p,p}\neq 0$ $(2\leqq p\leqq 2i-1)$ we have $c_k=0$ $(1\leqq k\leqq 2i-1)$. In this case

$$f=x_1{}^{2i}(c_{2i}x_2+c_{2i+1}x_1).$$

If $c_1\neq 0$, then by the formulas $(\sharp)_p$ $(4\leqq p\leqq 2i+2)$ and the fact $A^i_{p,1}\neq 0$ $(3\leqq p\leqq 2i+1)$ we see that c_k $(3\leqq k\leqq 2i+1)$ are uniquely determined by c_1 and c_2. In this case we consider the following polynomial;

$$g(x)=c_1x_1\left(x_2+\frac{c_2}{2ic_1}x_1\right)^{2i}.$$

Put $g(x)=\sum_{k=0}^{2i+1}b_kx_1{}^kx_2{}^{2i+1-k}$. Then $b_0=0$, $b_1=c_1$, and $b_2=c_2$. Moreover, for a suitable $A\in GL(2,\boldsymbol{C})$, A^*g becomes

$$\alpha x_1^{2i} x_2 + \beta x_1^{2i+1} \qquad (\alpha, \beta \in \boldsymbol{C}).$$

Then it follows from the above formula that

$$\left(G_\iota^* F_{i-2}(A^* g, A^* g)\right)_{Q_{4i-2,4}} = 0,$$

and hence

$$\left(G_\iota^* F_{i-2}(g, g)\right)_{Q_{4i-2,4}} = 0.$$

Thus we have

$$\sum_{k+l=p} b_k b_l A_{k,l}^i = 0 \qquad (4 \leqq p \leqq 4i-2).$$

Since $b_1 = c_1$ and $b_2 = c_2$, we can conclude that $b_k = c_k$ $(0 \leqq k \leqq 2i+1)$. Therefore $f = g$, and the proposition has been proved.

LEMMA 3.9. *$2k-3$ elements $(G_\iota^*(x_1^p x_2^{2k-p} \zeta_2^4))_{Q_{2k,4}}$ $(4 \leqq p \leqq 2k)$ of $Q_{2k,4}$ are linearly independent, where $k \geqq 2$.*

PROOF. By Corollary 2.8 (ii) we see that

$$\begin{aligned}
&\left(G_\iota^*(x_1^p x_2^{2k-p} \zeta_2^4)\right)_{Q_{2k,4}} \\
&= \left(G_\iota^*\left(x_1^p x_2^{2k-p} \zeta_2^4 - \frac{6}{2k-1} x_1^p x_2^{2k-p+2} \zeta_2^2\right.\right. \\
&\qquad \left.\left. + \frac{3}{(2k+1)(2k-1)} x_1^p x_2^{2k-p+4}\right)\right)_{G(Q_{2k+4})}.
\end{aligned}$$

Thus in view of Corollary 3.7 (ii) it is enough to show that $2k-3$ polynomials

$$h_p = \tilde{G}\left(x_1^p x_2^{2k-p} \zeta_2^4 - \frac{6}{2k-1} x_1^p x_2^{2k-p+2} \zeta_2^2 + \frac{3}{(2k+1)(2k-1)} x_1^p x_2^{2k-p+4}\right)$$

$(4 \leqq p \leqq 2k)$ are linearly independent. An explicit computation shows that the coefficient of $x_1^p \zeta_2^{2k-p+4}$ in h_p is

$$\frac{(p-1)!!\,(2k-p-1)!!}{(2k)!!} \; \frac{p(p-2)}{(2k+1)(2k-1)}$$

if p is even, and the coefficient of $x_1^{p-1} \zeta_1 \zeta_2^{2k-p+4}$ in h_p is

$$\frac{(p-2)!!\,(2k-2p)!!}{(2k)!!} \; \frac{(p-1)^2(p-3)}{(2k+1)(2k-1)}$$

if p is odd. Thus we have $h_p \neq 0$ $(4 \leqq p \leqq 2k)$. Since h_p $(4 \leqq p \leqq 2k)$ have mutually different degrees in the variables (x_1, ζ_1), it follows that they are linearly independent.

We set

$$\tilde{S}=\left\{x\in \boldsymbol{C}^{n+1}\middle|\sum_i x_i^2=0\right\},$$

$$\tilde{S}_1=\left\{(x,\zeta)\in \boldsymbol{C}^{2n+2}\middle|\sum_i x_i^2=\sum_i \zeta_i^2=\sum_i x_i\zeta_i=0\right\}.$$

Let V be a 2-dimensional subspace of $\boldsymbol{C}^{n+1}$ which is contained in $\tilde{S}$, and let $\kappa: \boldsymbol{C}^2=\{(x_1, x_2)\}\rightarrow V$ be a linear isomorphism. Then the isomorphism κ induces the isomorphism

$$\kappa\times\kappa:\ \boldsymbol{C}^4=\left\{(x_1, x_2, \zeta_1, \zeta_2)\right\}\longrightarrow V\times V\subset \boldsymbol{C}^{2n+2}=\left\{(x,\zeta)\right\},$$

which will also be denoted by κ. In this case it is easy to see that $V\times V$ is contained in $\tilde{S}_1$.

By identifying $\boldsymbol{C}[x_1, x_2]$ (resp. $\boldsymbol{C}[x_1, x_2, \zeta_1, \zeta_2]$) with a subalgebra of $\boldsymbol{C}[x]$ (resp. $\boldsymbol{C}[x,\zeta]$) naturally, the operators $\tilde{G}$ and F_p are defined on $\boldsymbol{C}[x_1, x_2, \zeta_1, \zeta_2]$ and on $\boldsymbol{C}[x_1, x_2]\times \boldsymbol{C}[x_1, x_2]$ respectively. Then we have $\tilde{G}\circ\kappa^*=\kappa^*\circ\tilde{G}$ and $F_p(\kappa^*f, \kappa^*h)=\kappa^*F_p(f, h)$, $f,\ h\in \boldsymbol{C}[x]$.

PROPOSITION 3.10. *Let V and $\kappa: \boldsymbol{C}^2=\{(x_1, x_2)\}\rightarrow V$ be as above. Suppose that $f\in \boldsymbol{C}[x]_{2j+1}$ $(j\geqq 2)$ satisfies $G_t^*F(f,f)\in G(\mathscr{H}^2)$. Then*

$$(\kappa^*f)(x_1, x_2)=(a_1x_1+a_2x_2)^{2j}(b_1x_1+b_2x_2)$$

for some constants a_k, $b_k\in \boldsymbol{C}$ $(k=1, 2)$.

PROOF. By Proposition 3.5 we have

$$\left(G_t^*F_{j-2}(f,f)\right)_{Q_{4j-2,4}}=0\,.$$

By Corollary 2.8 (ii) this implies

$$\left(G_t^*\sum_{q=j-2}^{j}a_{q-(j-2)}^{2+2(j-2)}F_q(f,f)\right)_{G(Q_{4j+2})}=0\,.$$

Hence we have

$$\tilde{G}\left(\sum_{q=j-2}^{j}a_{q-(j-2)}^{2+2(j-2)}F_q(f,f)\right)\in\left(\sum_i x_i^2,\ \sum_i\zeta_i^2,\ \sum x_i\zeta_i\right)$$

by Corollary 3.7 (i). By applying κ^* we have

$$\tilde{G}\left(\sum_{q=j-2}^{j}a_{q-(j-2)}^{2+2(j-2)}F_q(\kappa^*f, \kappa^*f)\right)=0\,.$$

The proposition now follows from Proposition 3.8 (i) (ii).

In view of Proposition 3.10, it is enough to prove the following proposition in order to show Proposition 3.1.

PROPOSITION 3.11. *Let $f\in \boldsymbol{R}[x_1, \cdots, x_{n+1}]_{2j+1}$ $(j\geqq 2)$. Suppose that for* *each 2-dimensional subspace V of $C^{n+1}=\{(x)\}$ contained in $\tilde{S}$, $f|_V$ has the form $(\alpha_1 z_1+\alpha_2 z_2)^{2j}(\beta_1 z_1+\beta_2 z_2)$, α_i, $\beta_i \in C$ $(i=1,2)$, where (z_1, z_2) is a linear coordinate system of V. Then there are constants a_i, $b_i \in R$ $(1 \leqq i \leqq n+1)$ such that*

$$f \equiv (\sum_i a_i x_i)^{2j} (\sum_i b_i x_i) \qquad \mathrm{mod}\,(\sum_i x_i^2)\, R[x]_{2j-1}\,.$$

PROOF. We need different considerations according as $n=3$ or $n \geqq 4$.

I. *The case* $n=3$. Define a bilinear map ϕ: $C^2 \times C^2 \to C^4$ by

$$\phi(y, z)=$$
$$\left(\frac{1}{2}(y_1 z_1+y_2 z_2), \frac{1}{2\sqrt{-1}}(y_1 z_1-y_2 z_2), \frac{1}{2}(y_2 z_1-y_1 z_2), \frac{1}{2\sqrt{-1}}(y_2 z_1+y_1 z_2)\right),$$

where $(y, z)=((y_1, y_2), (z_1, z_2)) \in C^2 \times C^2$. (The induced map $P^1 \times P^1 \to P^3$ is known as Segre embedding, where P^k denotes the k-dimensional complex projective space.) It is easy to see that the image of ϕ is $\tilde{S}$. Moreover any 2-dimensional subspace V of C^4 which is contained in $\tilde{S}$ is of the form $\phi(C^2 \times \{a\})$ or $\phi(\{a\} \times C^2)$, $a \in C^2-\{0\}$. Thus $\phi^* f \in C[y, z]$ is homogeneous of degree $2j+1$ in both variables y and z, and for each $a \in C^2-\{0\}$ there are constants α_1, α_2, β_1, $\beta_2 \in C$ (resp. α_1', α_2', β_1', $\beta_2' \in C$) such that $\phi^* f(y, a)=(\alpha_1 y_1+\alpha_2 y_2)^{2j}(\beta_1 y_1+\beta_2 y_2)$ (resp. $\phi^* f(a, z)=(\alpha_1' z_1+\alpha_2' z_2)^{2j}(\beta_1' z_1+\beta_2' z_2)$).

We denote by $C[y, z]_{k,l}$ the vector space of polynomials which are homogeneous of degree k in the variables y and homogeneous of degree l in the variables z.

LEMMA 3.12. *Let $h \in C[y, z]_{k,l}$ $(l \geqq 3)$. Assume that h satisfies the following condition: For each $a \in C^2-\{0\}$ there are constants α_1, α_2, β_1, $\beta_2 \in C$ such that*

$$h(a, z)=(\alpha_1 z_1+\alpha_2 z_2)^{l-1}(\beta_1 z_1+\beta_2 z_2)\,.$$

Then there are homogeneous polynomials h_0, γ_1, γ_2, δ_1, $\delta_2 \in C[y]$ with $\deg \gamma_1 = \deg \gamma_2$, $\deg \delta_1 = \deg \delta_2$ *such that*

$$h(y, z)=h_0(\gamma_1 z_1+\gamma_2 z_2)^{l-1}(\delta_1 z_1+\delta_2 z_2)\,,$$

and $\gamma_1 z_1+\gamma_2 z_2$ and $\delta_1 z_1+\delta_2 z_2$ *are irreducible in* $C[y, z]$.

PROOF OF LEMMA 3.12. Let P_y^1 be the complex projective line with homogeneous coordinates $[y_1, y_2]$, and let $C(P_y^1)$ its function field, i.e.,

$$C(P_y^1)=\left\{\frac{v}{u} \,|\, v,\ u \in C[y_1, y_2], \text{ homogeneous of the same degree, } u \neq 0\right\}.$$

For a homogeneous polynomial $u \in C[y]$ we set

$$V(u)=\left\{[a_1, a_2]\in P_y^1 \middle| u(a_1, a_2)=0\right\},$$

and for $v\in \boldsymbol{C}(P_y^1)$

$$W(v)=\left\{p\in P_y^1 \middle| v(p)=0 \quad \text{or} \quad v(p)=\infty\right\}.$$

Moreover for a polynomial $v=\sum_i v_i t^i \in \boldsymbol{C}(P_y^1)[t]$ ($v_i\in \boldsymbol{C}(P_y^1)$) we set $W(v)=\bigcup W(v_i)$, where the union is taken over all i such that $v_i\neq 0$.

We assume that h is not identically zero. By changing the coordinates (z_1, z_2) linearly if necessary, we may also assume that the coefficient $h_1(y)\in \boldsymbol{C}[y]_k$ of z_1^l in $h(y, z)$ is not zero. Define the monic polynomial $g=g(y_1, y_2, t)$ $\in \boldsymbol{C}(P_y^1)[t]$ of degree l by the formula

$$h(y, z)=h_1(y)\, z_2^l\, g\left(y_1, y_2, \frac{z_1}{z_2}\right).$$

Let

$$g=g_1^{r_1}\cdots g_m^{r_m},\quad g_i=g_i(y_1, y_2, t)\in \boldsymbol{C}(P_y^1)[t] \qquad (1\leqq i\leqq m)$$

be its irreducible decomposition in $\boldsymbol{C}(P_y^1)[t]$. We assume that each g_i is monic and $\deg g_1\leqq\cdots\leqq \deg g_m$. Let $D(g_i)\in \boldsymbol{C}(P_y^1)$ be the discriminant of g_i and $D(g_i, g_j)\in \boldsymbol{C}(P_y^1)$ the resultant of g_i and g_j $(i\neq j)$. Since g_i $(1\leqq i\leqq m)$ are irreducible and mutually prime, they do not vanish. Set

$$W=V(h_1)\cup \bigcup_i W(g_i)\cup \bigcup_i W\big(D(g_i)\big)\cup \bigcup_{i\neq j} W\big(D(g_i, g_j)\big).$$

Then W is a finite subset of P_y^1.

Take $[a_1, a_2]\in P_y^1-W$. Then $g_i(a_1, a_2, t)\in \boldsymbol{C}[t]$ $(1\leqq i\leqq m)$. Moreover we see that the polynomials $g_i(a_1, a_2, t)$ are mutually prime and each algebraic equiation $g_i(a_1, a_2, t)=0$ has only simple roots. Hence by the assumption we easily have $\deg g_i=1$ $(1\leqq i\leqq m)$, $m=1$ or 2, and $r_1=1$ or $r_2=1$ in case $m=2$. In any case we can write

$$\begin{aligned} h(y, z) &= h_1\left(z_1+\frac{\gamma_2}{\gamma_1}z_2\right)^{l-1}\left(z_1+\frac{\delta_2}{\delta_1}z_2\right) \\ &= \frac{h_1}{\gamma_1^{l-1}\delta_1}(\gamma_1 z_1+\gamma_2 z_2)^{l-1}(\delta_1 z_1+\delta_2 z_2), \end{aligned}$$

where γ_1 and γ_2 (resp. δ_1 and δ_2) are homogeneous polynomials in the variables y of the same degree and mutually prime. Since $\gamma_1 z_1+\gamma_2 z_2$ and $\delta_1 z_1+\delta_2 z_2$ are irreducible in $\boldsymbol{C}[y, z]$, it follows that $\dfrac{h_1}{\gamma_1^{l-1}\delta_1}\in \boldsymbol{C}[y_1, y_2]$. This finishes the proof of the lemma.

We now continue the proof of the case $n=3$. By applying Lemma 3.12 to $\psi^*f(y, z)$ in both variables y and z, we see that the irreducible decomposition of ψ^*f in $C[y, z]$ must be one of the following:

(i) $f_1(y, z)^{2j}f_2(y, z)$, $f_1, f_2\in C[y, z]_{1,1}$,

(ii) $f_1(y, z)^{2j}f_2(y)f_3(z)$, $f_1\in C[y, z]_{1,1}$, $f_2\in C[y]_1$, $f_3\in C[z]_1$,

(iii) $f_1(y)^{2j}f_2(z)^{2j}f_3(y, z)$, $f_1\in C[y]_1$, $f_2\in C[z]_1$, $f_3\in C[y, z]_{1,1}$,

(iv) $f_1(y)^{2j}f_2(z)^{2j}f_3(y)f_4(z)$, $f_1, f_3\in C[y]_1$, $f_2, f_4\in C[z]_1$.

Here it is not assumed that f_1 and f_2 are mutually prime in the case (i), and so are not the pairs (f_1, f_3) and (f_2, f_4) in the case (iv). In any case ψ^*f is of the form

$$\left(\sum_{i,k=1}^{2}\alpha_{i,k}y_i z_k\right)^{2j}\left(\sum_{i,k=1}^{2}\beta_{i,k}y_i z_k\right), \qquad \alpha_{i,k}, \beta_{i,k}\in C.$$

Thus there are constants a_i, $b_i\in C$ $(1\leqq i\leqq 4)$ such that

$$f(x)=(\sum_i a_i x_i)^{2j}(\sum_i b_i x_i) \qquad \text{on } \tilde{S}.$$

Since the ideal $\left(\sum_{i=1}^{4}x_i^2\right)C[x]$ is prime, it follows that

$$f\equiv(\sum_i a_i x_i)^{2j}(\sum_i b_i x_i) \qquad \mathrm{mod}\,(\sum_i x_i^2)\,C[x]_{2j-1}.$$

Next we must show that the coefficients a_i, b_i $(1\leqq i\leqq 4)$ can be taken from the real numbers. We may assume that the vectors $a=(a_1, a_2, a_3, a_4)$ and $b=(b_1, b_2, b_3, b_4)$ are not zero. Since f is real, we have

$$(\sum_i \bar{a}_i x_i)^{2j}(\sum_i \bar{b}_i x_i)\equiv(\sum_i a_i x_i)^{2j}(\sum_i b_i x_i) \qquad \mathrm{mod}\,(\sum_i x_i^2)\,C[x]_{2j-1},$$

where the bars denote the complex conjugates. Then there are two cases; the ideal $J=(\sum_i a_i x_i, \sum_i x_i^2)$ in $C[x]$ is prime or not.

Case 1. *J is prime.* In this case

$$(\sum_i \bar{a}_i x_i)^{2j}(\sum_i \bar{b}_i x_i)\equiv 0 \qquad \mathrm{mod}\, J.$$

If $\sum_i \bar{b}_i x_i\notin J$, then $\sum_i \bar{a}_i x_i\in J$. If $\sum_i \bar{b}_i x_i\in J$, then we have $\sum_i \bar{b}_i x_i=c\sum_i a_i x_i$ $(c\in C-\{0\})$ by comparing the degrees. Then

$$c(\sum_i \bar{a}_i x_i)^{2j}\equiv(\sum_i a_i x_i)^{2j-1}(\sum_i b_i x_i) \qquad \mathrm{mod}\,(\sum_i x_i^2),$$

and we also have $\sum_i \bar{a}_i x_i\in J$. By comparing the degrees we see that

$$\sum_i \bar{a}_i x_i=d\sum_i a_i x_i, \qquad d\in C,\ |d|=1.$$

Take $e\in C$ such that $e^2=d$. Then $e\sum_i a_i x_i\in R[x]$. Moreover, since

$$e^{2j}\sum_i \bar{b}_i x_i \equiv \bar{e}^{2j}\sum_i b_i x_i \qquad \mathrm{mod}\,(\sum_i x_i^2)\,,$$

it follows that $e^{2j}\sum_i \bar{b}_i x_i = \bar{e}^{2j}\sum_i b_i x_i$. Thus $\bar{e}^{2j}\sum_i b_i x_i \in \boldsymbol{R}[x]$, and

$$f\equiv (e\sum a_i x_i)^{2j}(\bar{e}^{2j}\sum b_i x_i) \qquad \mathrm{mod}\,(\sum x_i^2)\,\boldsymbol{R}[x]_{2j-1}\,.$$

Case 2. *J is not prime.* In this case the image of $\sum_i x_i^2$ by the homomorphism $\boldsymbol{C}[x]\to\boldsymbol{C}[x]/(\sum_i a_i x_i)$ is not irreducible, because the ring $\boldsymbol{C}[x]/(\sum_i a_i x_i)$ is a UFD. Thus we can write

$$\sum_i x_i^2 \equiv (\sum_i c_i x_i)(\sum_i d_i x_i) \qquad \mathrm{mod}\,(\sum_i a_i x_i)\,.$$

Then the 2-dimensional subspace V defined by $\sum_i a_i x_i = \sum_i c_i x_i = 0$ is contained in $\tilde{S}$. On the other hand, it is easy to see that the real orthogonal group $O(4, \boldsymbol{R})$ acts transitively on the set of 2-dimensional subspaces contained in $\tilde{S}$. Since the 2-dimensional subspace defined by $x_1+\sqrt{-1}\,x_2 = x_3+\sqrt{-1}\,x_4 = 0$ is contained in $\tilde{S}$, it follows that

$$\sum_i a_i x_i = A^*\left(\alpha(x_1+\sqrt{-1}\,x_2)+\beta(x_3+\sqrt{-1}\,x_4)\right)$$

for a suitable $A\in O(4,\boldsymbol{R})$ and $\alpha,\ \beta\in\boldsymbol{C}$. This implies that $a=(a_1,\cdots,a_4)$ is in $\tilde{S}$. Moreover, we see that $O(4,\boldsymbol{R})$ acts transitively on $\tilde{S}-\{0\}$ up to positive constant factors. Hence there is $B\in O(4,\boldsymbol{R})$ such that

$$B^*\sum_i a_i x_i = e(x_1+\sqrt{-1}\,x_2)\,, \qquad e>0\,.$$

By applying B^* to the congruence

$$(\sum_i \bar{a}_i x_i)^{2j}(\sum_i \bar{b}_i x_i) \equiv (\sum_i a_i x_i)^{2j}(\sum_i b_i x_i) \qquad \mathrm{mod}\,(\sum_i x_i^2)\,,$$

we have

$$(x_1-\sqrt{-1}\,x_2)^{2j}(\sum_i \bar{b}'_i x_i) \equiv (x_1+\sqrt{-1}\,x_2)^{2j}(\sum_i b'_i x_i) \qquad \mathrm{mod}\,(\sum_i x_i^2)\,,$$

where $\sum_i b'_i x_i = B^*\sum_i b_i x_i$. Since $x_1-\sqrt{-1}\,x_2$ does not belong to the prime ideals $(x_1+\sqrt{-1}\,x_2,\ x_3+\sqrt{-1}\,x_4)$ and $(x_1+\sqrt{-1}\,x_2,\ x_3-\sqrt{-1}\,x_4)$, and since $(x_1+\sqrt{-1}\,x_2,\ x_3+\sqrt{-1}\,x_4)\cap(x_1+\sqrt{-1}\,x_2,\ x_3-\sqrt{-1}\,x_4) = (x_1+\sqrt{-1}\,x_2,\ \sum_i x_i^2)$, it follows that $\sum_i \bar{b}'_i x_i \in (x_1+\sqrt{-1}\,x_2,\ \sum_i x_i^2)$. Thus

$$\sum_i \bar{b}'_i x_i = d(x_1+\sqrt{-1}\,x_2)\,, \qquad d\in\boldsymbol{C}-\{0\}\,.$$

Then the above congruence shows that

$$x_1-\sqrt{-1}\,x_2\in(x_1+\sqrt{-1}\,x_2,\ \sum_i x_i^2),$$

which is a contradiction. Therefore we see that the ideal J must be prime.

This completes the proof of the case $n=3$.

II. *The case* $n\geqq 4$. We may assume that $f\not\equiv 0 \mod(\sum_i x_i^2)$. We first decompose f into irreducible components in the ring $\boldsymbol{C}[x]/(\sum_i x_i^2)$;

$$f\equiv f_1^{r_1}\cdots f_m^{r_m} \qquad \mod(\sum_i x_i^2),$$

where $f_i\in\boldsymbol{C}[x]$ are homogeneous polynomials, $\deg f_1\leqq\cdots\leqq\deg f_m$, and the images of f_i by the homomorphism $\boldsymbol{C}[x]\to\boldsymbol{C}[x]/(\sum x_i^2)$ are irreducible and mutually prime.

Let P^n be the complex projective space of dimension n with the homogeneous coordinates $[x]=[x_1,\cdots,x_{n+1}]$. In general, for homogeneous polynomials $h_1,\cdots,h_k\in\boldsymbol{C}[x]$ we denote by $V(h_1,\cdots,h_k)$ the algebraic subset of P^n defined by $h_1=\cdots=h_k=0$. We set

$$S=V(\sum_i x_i^2).$$

Let $h\in\boldsymbol{C}[x]$ be a homogeneous polynomial whose image by the homomorphism $\boldsymbol{C}[x]\to\boldsymbol{C}[x]/(\sum_i x_i^2)$ is not zero, and is irreducible. Since the ring $\boldsymbol{C}[x]/(\sum_i x_i^2)$ is a UFD, it follows that the ideal $(h,\sum_i x_i^2)$ in $\boldsymbol{C}[x]$ is prime. Set

$$\mathrm{Sing}\ V(h)=\left\{[x]\in V(h)\,\middle|\,\frac{\partial h}{\partial x_1}(x)=\cdots=\frac{\partial h}{\partial x_{n+1}}(x)=0\right\}$$

and $\mathrm{Reg}\,V(h)=V(h)-\mathrm{Sing}\,V(h)$. We denote by $V(h)_p$ (resp. S_p) the tangent hyperplane to $V(h)$ (resp. S) at $p\in\mathrm{Reg}\,V(h)$ (resp. $p\in S$). Set

$$U=\left\{p\in\mathrm{Reg}\ V(h)\cap S\,\middle|\,V(h)_p\neq S_p\right\}$$

and $U(p)=S_p\cap S-S_p\cap S\cap V(h)_p$ for $p\in U$. Then we set

$$T=\left\{(p,q)\in P^n\times P^n\,\middle|\,p\in S\cap V(h),\ q\in S_p\cap S\right\},$$

$$U_1=\left\{(p,q)\in P^n\times P^n\,\middle|\,p\in U,\ q\in U(p)\right\}.$$

LEMMA 3.13. *The subset* U_1 *is open and dense in the set* T

PROOF OF LEMMA 3.13. We shall first show that U is open and dense in $S\cap V(h)$. Since $[a_1,\cdots,a_{n+1}]\in S\cap V(h)$ is in U if and only if $a=(a_1,\cdots,a_{n+1})$ and $\left(\frac{\partial h}{\partial x_1}(a),\cdots,\frac{\partial h}{\partial x_{n+1}}(a)\right)$ are linearly independent, it follows that U is open in $S\cap V(h)$ in the Zariski topology. If $(a_1, \cdots, a_{n+1})$ and $\left(\frac{\partial h}{\partial x_1}(a), \cdots, \frac{\partial h}{\partial x_{n+1}}(a)\right)$ are linearly dependent for all $[a]\in S\cap V(h)$, then there are polynomials $\alpha_{i,j}$, $\beta_{i,j}$ $(1\leqq i, j\leqq n+1)$ such that

$$x_i\frac{\partial h}{\partial x_j}-x_j\frac{\partial h}{\partial x_i}=\alpha_{i,j}\sum_k x_k^2+\beta_{i,j}h\,,$$

$\alpha_{j,i}=-\alpha_{i,j}$, $\beta_{j,i}=-\beta_{i,j}$. By the homogeneity we may assume that $\beta_{i,j}\in \boldsymbol{C}$. Then we have

$$(\deg h)\,x_i h-\sum_j x_j^2\frac{\partial h}{\partial x_i}=\sum_j \alpha_{i,j}x_j\sum_k x_k^2+\sum_j \beta_{i,j}x_j h \quad (1\leqq i\leqq n+1)\,,$$

and hence

$$(\deg h)\,x_i\equiv \sum_j \beta_{i,j}x_j \qquad \mathrm{mod}\,(\sum_i x_i^2)\,.$$

Since $\beta_{i,i}=0$, this is a contradiction. Therefore we see that $U\neq\phi$. Since $S\cap V(h)$ is an algebraic variety, it follows that U is open and dense in $S\cap V(h)$ in the classical topology.

Next we shall show that $U(p)$ is open and dense in $S_p\cap S$ for each $p\in U$. As is easily seen, $S_p\cap S$ is a variety and $U(p)$ is its Zariski-open subset. If $U([a])=\phi$ for an $[a]\in U$, then $V(h)_{[a]}\supset S_{[a]}\cap S$. This implies that

$$\sum_i x_i\frac{\partial h}{\partial x_i}(a)\in(\sum_i x_i^2,\ \sum_i a_i x_i)\,.$$

Then we have by the homogeneity $\sum_i x_i\frac{\partial h}{\partial x_i}(a)\in(\sum_i a_i x_i)$, which is impossible when $[a]\in U$. Hence $U(p)\neq\phi$ for each $p\in U$, and we see that $U(p)$ is open and dense in $S_p\cap S$ for each $p\in U$.

Let

$$\mathrm{pr}_1:\ T\longrightarrow S\cap V(h)$$

be the projection to the first term. Then we see that $\mathrm{pr}_1: T\rightarrow S\cap V(h)$ is locally trivial, i. e., for each $p\in S\cap V(h)$ there is a neighborhood W of p in $S\cap V(h)$ and a fibre-preserving homeomorphism

$$\mathrm{pr}_1^{-1}(W)\rightarrow W\times(S_p\cap S)\,.$$

This, together with the above facts, implies that U_1 is dense in T. Since $U(p)$ depends continuously on $p\in U$, we also see that U_1 is open in T.

COROLLARY 3.14. *Let K be a Zariski-closed subset of S such that $V(h)\cap S\not\subset K$. Then there is a projective line L contained in S such that*

$V(h)\cap L\cap K=\phi$ and $V(h)\cap L$ consists of k distinct points, where $k=\deg h$.

PROOF OF COROLLARY 3.14. First take $(p, q)\in U_1$ such that $p\notin K$. Then the projective line L_0 through p and q lies in S, and L_0 and $V(h)\cap S$ intersect transversally at p. Assume that there is a projective line L in S such that L and $U-K$ intersect transversally at least at l distinct points, say $p_1, \cdots, p_l$ $(1\leqq l<k)$. Since $l<k$, there is another point p_{l+1} in $V(h)\cap L$. Take any point $r\in L-\{p_{l+1}\}$. Then $(p_{l+1}, r)\in T$. If $p_{l+1}\in K$ or L and $S\cap V(h)$ does not intersect transversally at p_{l+1}, then we can take $(p'_{l+1}, r')\in U_1$ near (p_{l+1}, r) such that $p'_{l+1}\notin K$. Let L' be the projective line through p'_{l+1} and r'. If we take (p'_{l+1}, r') sufficiently close to (p_{l+1}, r), then L' and $V(h)\cap S$ intersect transversally at points $p'_i\notin K$ near p_i $(1\leqq i\leqq l)$. Therefore we have found a line L' such that L' and $V(h)\cap S-K$ intersect transversally at least at $l+1$ distinct points. The corollary now follows by industion.

We now continue the proof of the case $n\geqq 4$. As usual $\sharp(K)$ will denote the number of elements in a set K. Corollary 3.14 shows that there is a projective line L_1 in S such that $L_1\not\subset V(f)$ and $\sharp(V(f_m)\cap L_1)=\deg f_m$. Then we have

$$\deg f_m \leqq \sharp\left(V(f)\cap L_1\right)<\infty .$$

Since $\sharp(V(f)\cap L_1)=1$ or 2 or ∞ by the assumption on f, it follows that $\deg f_m\leqq 2$. Assume that $\deg f_m=2$. In this case we have $\deg f_1=1$, since $\deg f$ is odd. Since $V(f_m)\cap S\not\subset V(f_1)\cap S$, we see by Corollary 3.14 that there is a projective line L_2 in S such that $L_2\not\subset V(f)$ and $(V(f_1)\cup V(f_m))\cap L_2$ consists of three points. This being a contradiction, we have $\deg f_m=1$.

Now we may assume that $r_1\geqq \cdots \geqq r_m$. Since $V(f_i)\cap S\not\subset V(f_j)\cap S$ $(i\neq j)$, an induction argument as in the proof of Corollary 3.14 implies that there is a projective line L_3 in S such that $\sharp(V(f)\cap L_3)=m$. Hence $m=1$ or 2. Let V be the 2-dimensional subspace of $\boldsymbol{C}^{n+1}$ whose image by the quotient map $\boldsymbol{C}^{n+1}-\{0\}\to P^n$ is L_3. Then by considering $f|_V$ we see that $r_1=2j$ and $r_2=1$ if $m=2$. Thus we have

$$f\equiv(\sum_i a_i x_i^{2j})(\sum_i b_i x_i) \mod (\sum_i x_i^2),\ a_i,\ b_i\in\boldsymbol{C} \qquad (1\leqq i\leqq n+1)\,.$$

Since f is real and $\boldsymbol{C}[x]/(\sum_i x_i^2)$ is a UFD, it easily follows that the coefficients a_i and b_i $(1\leqq i\leqq n+1)$ can be taken from the real numbers.

This completes the proof of Proposition 3.11.

§ 4. The main result

Let $O(n+1, \boldsymbol{R})$ be the orthogonal group of degree $n+1$, which naturally acts on $\boldsymbol{R}^{n+1}=\{(x)\}$ and hence on $\boldsymbol{R}[x]$. In this and the next sections we shall prove the following

THEOREM 4.1. *Let $f\in\boldsymbol{R}[x_1,\cdots,x_{n+1}]_{od}$. Then f satisfies the condition $G\iota^*F(f,f)\in G(\mathscr{H}^2)$ if and only if f has one of the following forms* (i) *and* (ii):

(i) $f\equiv h_1+h_3+\sum_{i=2}^{m}(\sum_k a_kx_k)^{2i}(\sum_j b_{i,j}x_j) \qquad \mod(1-\sum_i x_i^2),$

where a_k, $b_{i,j}\in\boldsymbol{R}$, $h_1\in\boldsymbol{R}[x]_1$, and $h_3\in\boldsymbol{R}[x]_3$;

(ii) $f\equiv h_1+h_3+cA^*h \qquad \mod(1-\sum_i x_i^2),$

where $h_1\in\boldsymbol{R}[x_1]$, $h_3\in\boldsymbol{R}[x]_3$, $c\in\boldsymbol{R}$, $A\in O(n+1,\boldsymbol{R})$, and h is a polynomial of degree 21 *in the variables (x_1,x_2) of the following form*

$$h=\sum_{i=2}^{10}\alpha_{2i+1}x_1^{2i+1}+\sum_{i=2}^{6}\beta_{2i+1}x_1^{2i}x_2+\sum_{i=2}^{6}\gamma_{2i+1}x_1^{2i-1}x_2^2$$
$$+\delta_5x_1^2x_2^3+\varepsilon_5x_1x_2^4,$$

with $\beta_{13}\in\boldsymbol{R}$, $\gamma_{13}\in\boldsymbol{R}-\{0\}$, and

$$\alpha_5=\frac{10}{13}\gamma_{13}-\frac{25}{13^2\cdot192}\frac{\beta_{13}^4}{\gamma_{13}^2}+\frac{15}{4}\frac{\beta_{13}^2}{\gamma_{13}}+45,$$

$$\alpha_7=-\frac{10}{13}\gamma_{13}-5\frac{\beta_{13}^2}{\gamma_{13}}-120,\qquad \alpha_9=\frac{5}{13}\gamma_{13}+\frac{15}{4}\frac{\beta_{13}^2}{\gamma_{13}}+210,$$

$$\alpha_{11}=-\frac{1}{13}\gamma_{13}-\frac{3}{2}\frac{\beta_{13}^2}{\gamma_{13}}-252,\qquad \alpha_{13}=\frac{1}{4}\frac{\beta_{13}^2}{\gamma_{13}}+210,$$

$$\alpha_{15}=-120,\qquad \alpha_{17}=45,\qquad \alpha_{19}=-10,\qquad \alpha_{21}=1,$$

$$\beta_5=\frac{75}{13}\beta_{13}-\frac{25}{13^2\cdot24}\frac{\beta_{13}^3}{\gamma_{13}},\qquad \beta_7=-\frac{140}{13}\beta_{13},\qquad \beta_9=\frac{135}{13}\beta_{13},$$

$$\beta_{11}=-\frac{66}{13}\beta_{13},\qquad \gamma_5=\frac{25}{13}\gamma_{13}-\frac{25}{13^2\cdot8}\beta_{13}^2,$$

$$\gamma_7=-\frac{70}{13}\gamma_{13},\qquad \gamma_9=\frac{90}{13}\gamma_{13},\qquad \gamma_{11}=-\frac{55}{13}\gamma_{13},$$

$$\delta_5=-\frac{25}{13^2\cdot6}\beta_{13}\gamma_{13},\qquad \varepsilon_5=-\frac{25}{13^2\cdot12}\gamma_{13}^2.$$

We first remark various actions of the orthogonal group. The orthogonal group $O(n+1,\boldsymbol{R})$ acts on $\boldsymbol{R}^{2n+2}$ by the map $(x,\zeta)\to(Ax,A\zeta)$ $(A\in O(n+1,\boldsymbol{R}))$, and via the inclusions $\iota_0: S^n\to\boldsymbol{R}^{n+1}$ and $\iota: S^*S^n\to\boldsymbol{R}^{2n+2}$, it also acts on (S^n,g_0) and (S^*S^n,g_1) as isometries. Clearly the induced actions of $O(n+1,\boldsymbol{R})$ on $C^\infty(\boldsymbol{R}^{n+1})$, $C^\infty(\boldsymbol{R}^{2n+2})$, $C^\infty(S^n)$, and $C^\infty(S^*S^n)$ commute with the operators $\tilde{G}$, G,

ι^*, Δ, F_p, and F. It follows that the subspaces $G(P_k)$, $G(Q_k)$, and $Q_{i,j}$ of $C^\infty(S^*S^n)$ are preserved by this action. In particular we see that $G\iota^*F(f,f) \in G(\mathscr{A}^2)$ if and only if $G\iota^*F(A^*f, A^*f)\in G(\mathscr{A}^2)$, where $f\in \boldsymbol{R}[x]_{od}$ and $A\in O(n+1,\boldsymbol{R})$.

PROPOSITION 4. 2. *Let $f\in\boldsymbol{R}[x]_{od}$. Suppose that f is of the form* (i) *in Theorem* 4. 1. *Then f satisfies the condition $G\iota^*F(f,f)\in G(\mathscr{A}^2)$.*

PROOF. Let u, $v\in\boldsymbol{R}[x]_{od}$. Proposition 1. 6 shows that if u or v belongs to the ideal $(1-\sum x_i^2)$, then $G\iota^*F(u, v)\in G(\mathscr{A}^2)$. Thus we may assume that

$$f=h_1+h_3+\sum_{i=2}^{m}(\sum_k a_k x_k)^{2i}(\sum_j b_{i,j}x_j)\,.$$

In view of Corollary 3. 3 we may further assume that

$$f=\sum_{i=2}^{m}(\sum_k a_k x_k)^{2i}(\sum_j b_{i,j}x_j)\,.$$

By considering the action of the orthogonal group, we may consequently assume that f is of the form

$$\sum_{i=2}^{m} x_1^{2i}(\sum_j b_{i,j}x_j)\,, \qquad b_{i,j}\in\boldsymbol{R}\,.$$

Then the proposition follows from the next lemma.

LEMMA 4. 3. $G\iota^*F(x_1^{2i}x_k,\ x_1^{2j}x_l)\in G(\mathscr{A}^2) \qquad (1\leq k, l\leqq n+1)$.

PROOF. Let U_m $(m\geq 3)$ be the real vector space spanned by

$$\Big\{G\iota^*(x_1^{2p}x_kx_l\zeta_1^{2q}),\ G\iota^*(x_1^{2p+1}x_k\zeta_1^{2q-1}\zeta_l),\ G\iota^*(x_1^{2p+1}x_l\zeta_1^{2q-1}\zeta_k),$$
$$G\iota^*(x_1^{2p+2}\zeta_1^{2q-2}\zeta_k\zeta_l)\Big|p+q=m\Big\}\,.$$

Then from the definition of F we have

$$G\iota^*F(x_1^{2i}x_k,\ x_1^{2j}x_l)\in U_{i+j}+U_{i+j-1}\,.$$

Consider the following identities:

$$\tilde{X}_{E_0}(x_1^{2p+1}x_kx_l\zeta_1^{2q-1})=(2p+1)x_1^{2p}x_kx_l\zeta_1^{2q}+x_1^{2p+1}x_l\zeta_1^{2q-1}\zeta_k$$
$$+x_1^{2p+1}x_k\zeta_1^{2q-1}\zeta_l-(2q-1)x_1^{2p+2}x_kx_l\zeta_1^{2q-2}\,,$$

$$\tilde{X}_{E_0}(x_1^{2p+2}x_l\zeta_1^{2q-2}\zeta_k)=(2p+2)x_1^{2p+1}x_l\zeta_1^{2q-1}\zeta_k+x_1^{2p+2}\zeta_1^{2q-2}\zeta_k\zeta_l$$
$$-(2q-2)x_1^{2p+3}x_l\zeta_1^{2q-3}\zeta_k-x_1^{2p+2}x_kx_l\zeta_1^{2q-2}\,,$$

$$\tilde{X}_{E_0}(x_1^{2p+2}x_k\zeta_1^{2q-2}\zeta_l)=(2p+2)x_1^{2p+1}x_k\zeta_1^{2q-1}\zeta_l+x_1^{2p+2}\zeta_1^{2q-2}\zeta_k\zeta_l$$
$$-(2q-2)x_1^{2p+3}x_k\zeta_1^{2q-3}\zeta_l-x_1^{2p+2}x_kx_l\zeta_1^{2q-2}\,,$$

$$\tilde{X}_{E_0}(x_1^{2p+3}\zeta_1^{2q-3}\zeta_k\zeta_l)=(2p+3)\,x_1^{2p+2}\zeta_1^{2q-2}\zeta_k\zeta_l-(2q-3)\,x_1^{2p+4}\zeta_1^{2q-4}\zeta_k\zeta_l$$
$$-x_1^{2p+3}x_k\zeta_1^{2q-3}\zeta_l-x_1^{2p+3}x_l\zeta_1^{2q-3}\zeta_k\,.$$

These identities imply that $G\iota^*(x_1^{2p}x_kx_l\zeta_1^{2q})$, $G\iota^*(x_1^{2p+1}x_k\zeta_1^{2q-1})$, $G\iota^*(x_1^{2p+1}x_l\zeta_1^{2q-1}\zeta_k)$, and $G\iota^*(x_1^{2p+2}\zeta_1^{2q-2}\zeta_k\zeta_l)$ are linear combinations of $G\iota^*(x_1^{2p+2}x_kx_l\zeta_1^{2q-2})$, $G\iota^*(x_1^{2p+3}x_k\zeta_1^{2q-3}\zeta_l)$, $G\iota^*(x_1^{2p+3}x_l\zeta_1^{2q-3}\zeta_k)$, and $G\iota^*(x_1^{2p+4}\zeta_1^{2q-4}\zeta_k\zeta_l)$, provided $q\geq 2$ By using this fact successively, we have $U_m\subset G(\mathscr{H}^2)$. This proves the lemma.

Hereafter we make the convention that the degree of the polynomial 0 is $-\infty$. Let $f\in \boldsymbol{R}[x]_{od}$ with $\deg f=2m+1$ $(m\geq 2)$. Consider the following conditions for f:

(i) The homogeneous part of f of degree $2m+1$ is of the form $x_1^{2m}(ax_1+bx_2)$, $(a, b)\in \boldsymbol{R}^2-\{0\}$;

(ii) The homogeneous parts of f of degrees 1 and 3 are zero;

(iii) The degree of f in the variable x_{n+1} is at most 1.

LEMMA 4.4. *Let $h\in\boldsymbol{R}[x]_{od}$ with $\deg h=2m+1$ $(m\geq 2)$. Suppose that $G\iota^*F(h, h)\in G(\mathscr{H}^2)$. Then there are $A\in O(n+1, \boldsymbol{R})$, $h_1\in\boldsymbol{R}[x]_1$, $h_3\in\boldsymbol{R}[x]_3$, and $f\in\boldsymbol{R}[x]_{od}$ such that* (a) *either $f=0$ or $5\leq\deg f\leq 2m+1$ and f satisfies the above conditions* (i) (ii) (iii), *and*

(b) $A^*h\equiv h_1+h_3+f \qquad \mathrm{mod}\,(1-\sum_i x_i^2)\,.$

PROOF. We may assume that $h\not\equiv 0$ $\mathrm{mod}\,(1-\sum_i x_i^2)$. Then there are m' $(0\leq m'\leq m)$ and $h'\in\boldsymbol{R}[x]_{2m'+1}$ such that

$$h\equiv h' \qquad \mathrm{mod}\,(1-\sum_i x_i^2)$$

and $h'\not\equiv 0$ mod $\sum_i x_i^2$. If $m'\leq 1$, then there is nothing to prove. We now assume $m'\geq 2$. Since

$$G\iota^*F(h', h')\in G(\mathscr{H}^2)\,,$$

it follows from Proposition 3.1 that there are constants a_i, $b_i\in\boldsymbol{R}$ $(1\leq i\leq n+1)$ such that

$$h'\equiv(\sum_i a_ix_i)^{2m'}(\sum_i b_ix_i) \qquad \mathrm{mod}\,(\sum_i x_i^2)\,\boldsymbol{R}[x]_{2m'-1}\,.$$

Hence there are $A\in O(n+1, \boldsymbol{R})$, $(a, b)\in\boldsymbol{R}^2-\{0\}$, and $h''\in\boldsymbol{R}[x]_{2m'-1}$ such that

$$A^*h'\equiv x_1^{2m'}(ax_1+bx_2)+h'' \qquad \mathrm{mod}\,(1-\sum_i x_i^2)\,.$$

It is easy to see that there are homogeneous polynomials $h_{2k+1}\in\boldsymbol{R}[x]_{2k+1}$ $(0\leq k\leq m'-1)$ such that the degrees of h_{2k+1} in the variable x_{n+1} are at most 1 and

$$h'' \equiv \sum_{k=0}^{m'-1} h_{2k+1} \qquad \bmod (1-\sum_i x_i^2)\,.$$

By putting $f=x_1^{2m'}(ax_1+bx_2)+\sum_{k=2}^{m'-1} h_{2k+1}$, we have the lemma.

In view of this lemma we may restrict our attention to the polynomials satisfying the above conditions (i) (ii) (iii). Now we fix $f\in \boldsymbol{R}[x]_{od}$ with $\deg f=2m+1$ $(m\geq 2)$ which satisfies the above conditions (i) (ii) (iii) and the condition

$$G\iota^* F(f,f)\in G(\mathscr{H}^2)\,.$$

Let f_{2i+1} $(2\leq i\leq m)$ be the homogeneous part of f of degree $2i+1$, $f=\sum_{i=2}^{m} f_{2i+1}$, and let $f_{2m+1}=x_1^{2m}(ax_1+bx_2)$, $(a, b)\in \boldsymbol{R}^2-\{0\}$.

For a polynomial $h\in \boldsymbol{C}[x, \zeta]$ we denote by $\deg_1 h$ (resp. $\deg_2 h$) the degree of h in the variables (x_1, ζ_1) (resp. in the variables $(x_2, \cdots, x_{n+1}, \zeta_2, \cdots, \zeta_{n+1})$). For elements of $\boldsymbol{C}[x]$ we shall also apply this notation by considering $\boldsymbol{C}[x]$ as a subalgebra of $\boldsymbol{C}[x, \zeta]$. Put

$$d_i=\deg_2 f_{2i+1} \qquad (2\leq i\leq m)\,.$$

Let i_0 be the index such that $d_i\leq d_{i_0}$ if $2\leq i\leq i_0$ and $d_i<d_{i_0}$ if $i_0<i\leq m$. In particular $d_m=0$ or 1, and $d_{i_0}=\max_i d_i$. Consider f_{2i+1} $(2\leq i\leq m)$ as a polynomial in the variables $(x_2, \cdots, x_{n+1})$ with coefficients in $\boldsymbol{R}[x_1]$, and let h_{2i+1} be its homogeneous part of degree d_i. Then $f_{2i+1}=h_{2i+1}$ if $d_i\leq 0$, and $\deg_2 (f_{2i+1}-h_{2i+1})\leq d_i-1$ if $d_i\geq 1$.

If $d_{i_0}\leq 1$, then it is clear that f is of the form (i) in Theorem 4. 1. Now we assume that $d_{i_0}\geq 2$. In this case we have $i_0<m$. Let i_1 $(>i_0)$ be the index such that $d_i\leq d_{i_1}$ if $i_0<i\leq i_1$ and $d_i<d_{i_1}$ if $i_1<i\leq m$. In the rest of this section we shall prove the following

PROPOSITION 4. 5. *Under the assumption $d_{i_0}\geq 2$, we have $m=10$ (*$\deg f=21$*), $i_0=2$, $i_1=6$, $d_{i_0}=4$, $d_{i_1}=2$, and $d_i\leq 0$ $(7\leq i\leq 10)$.*

We shall prepare some lemmas. For any positive integer N, let V_N be the direct sum of vector spaces $Q_{2p,2q}$ in $G(P_{4m+2})$ such that $N_{2p,2q}=N$.

LEMMA 4. 6. *Let $u\in \boldsymbol{R}[x, \zeta]_{2k,2l}$ $(k\geq l,\ k+l\leq 2m+1)$ with $\deg_2 u=d$. Then there is a polynomial $v=\sum_{j=0}^{k+l} v_{2j}$ $(v_{2j}\in \boldsymbol{R}[x, \zeta]_{2j})$ such that $\deg_2 v\leq d$ and*

$$(G\iota^* u)_{V_N}=G\iota^* v\,,$$

where $(G\iota^* u)_{V_N}$ stands for the V_N-component of $G\iota^* u$.

PROOF. We shall prove this by induction on the integer $2k+2l=\deg u$. If $\deg u\leq 0$, then it is obvious. Assume that for each $u'\in \boldsymbol{R}[x, \zeta]_{2k',2l'}$ with

$k'+l'<k+l$ and for each N there is a polynomial $v'=\sum_{j=0}^{k'+l'} v'_{2j}$ $(v'_{2j}\in \boldsymbol{R}[x,\zeta]_{2j})$ such that $\deg_2 v'\le \deg_2 u'$ and $(G\iota^* u')_{V_N}=G\iota^* v'$. By the proof of Proposition 2.7 we can write

$$u=\sum_{p=0}^{l} u_p\,,\qquad u_p\in \sum_{q=p}^{l} \boldsymbol{R}[x,\zeta]_{2k+2q,2l-2q}$$

such that

$$\Delta G\iota^* u_p=N_{2k+2p,2l-2p}G\iota^* u_p+G\iota^* w_p\,,\qquad w_p=-\sum_i\left(\frac{\partial^2 u_p}{\partial x_i^2}+\frac{\partial^2 u_p}{\partial \zeta_i^2}\right),$$

and $\deg_2 u_p\le d$. Let $G\iota^* w_p=\sum_N (G\iota^* w_p)_{V_N}$ be the eigenspace decomposition. As was seen in the proof of Proposition 2.9, $(G\iota^* w_p)_{V_N}=0$ if $N=N_{2k+2p,2l-2p}$. Therefore if we put $N_p=N_{2k+2p,2l-2p}$, we see that

$$G\iota^* u_p+\sum_{N\neq N_p}(N_p-N)^{-1}(G\iota^* w_p)_{V_N}$$

is an eigenfunction corresponding to the eigenvalue N_p, and we have the decomposition of $G\iota^* u_p$ into eigenfunctions;

$$G\iota^* u_p=\Big(G\iota^* u_p+\sum_{N\neq N_p}(N_p-N)^{-1}(G\iota^* w_p)_{V_N}\Big)-\sum_{N\neq N_p}(N_p-N)^{-1}(G\iota^* w_p)_{V_N}\,.$$

Since $\deg_2 w_p\le d$, the assumption implies that for each N there is a polynomial $v=\sum_{j=0}^{k+l} v_{2j}$ such that $\deg_2 v\le d$ and $(G\iota^* u_p)_{V_N}=G\iota^* v$. This proves the lemma.

Fix an index $(2p,2q)$ such that $q\le p$ and $p+q\le 2m+1$. Let V(resp. V') be the direct sum of vector spaces $Q_{2p',2q'}$ in $G(P_{4m+2})$ such that $N_{2p',2q'}=N_{2p,2q}$ (resp. $N_{2p',2q'}=N_{2p,2q}$ and $p'+q'\ge p+q$).

COROLLARY 4.7. *Let* $u_1\in \boldsymbol{R}[x]_{2k+1}$ *and* $u_2\in \boldsymbol{R}[x]_{2l+1}$ $(k,l\le m)$ *with* $\deg_2 u_1+\deg_2 u_2=d$. *Then there are polynomials* $v=\sum_{j=0}^{k+l+1} v_{2j}$ $(v_{2j}\in \boldsymbol{R}[x,\zeta]_{2j})$ *and* $w\in \boldsymbol{R}[x,\zeta]_{2p+2q-2}$ *such that* $\deg_2 v\le d$ *and*

$$\Big(G\iota^* F(u_1,u_2)\Big)_{V'}=G\iota^* v+G\iota^* w\,.$$

PROOF. By the definition of V and V' we have

$$\Big(G\iota^* F(u_1,u_2)\Big)_{V}-\Big(G\iota^* F(u_1,u_2)\Big)_{V'}\in G(P_{2p+2q-2})\,.$$

Moreover we see from the definition of F that $\deg_2 F(u_1,u_2)\le d$. Hence the corollary follows from Lemma 4.6.

LEMMA 4.8. *Let* $w\in \boldsymbol{R}[x,\zeta]_{2k}$ $(k\ge 2)$. *Suppose that* w *belongs to the* *ideal* $(\sum_i x_i^2,\ \sum_i \zeta_i^2,\ \sum_i x_i\zeta_i)$ *in* $\boldsymbol{R}[x,\zeta]$. *Then there are polynomials* $w_i\in \boldsymbol{R}[x,\zeta]_{2k-2}$ $(i=1,2,3)$ *with* $\deg_2 w_i\leq \deg_2 w-2$ *such that*

$$w=\sum_i(x_i^2+\zeta_i^2)\,w_1+\sum_i(x_i^2-\zeta_i^2)\,w_2+\sum_i x_i\zeta_i w_3\,.$$

PROOF. We first put

$$w=\sum_i(x_i^2+\zeta_i^2)\,w_1+\sum_i(x_i^2+\zeta_i^2)\,w_2+\sum_i x_i\zeta_i w_3$$

for some $w_i\in\boldsymbol{R}[x,\zeta]_{2k-2}$ $(i=1,2,3)$. Let $w_i=\sum_{j=0}^{2k-2}w_{i,j}$ $(i=1,2,3)$ be the decomposition of w_i into its homogeneous parts in the variables $(x_2,\cdots,x_{n+1},\zeta_2,\cdots,\zeta_{n+1})$, $\deg_2 w_{i,j}=j$ if $w_{i,j}\neq 0$. Assume that $\deg_2 w=d<2k$, and that there is d_1 with $d-2<d_1\leq 2k-2$ such that $w_{i,j}=0$ for all $j>d_1$ and i. By comparing the homogeneous parts of degree d_1+2 in the variables $(x_2,\cdots,x_{n+1},\zeta_2,\cdots,\zeta_{n+1})$, we have

$$0=\sum_{i\geq 2}(x_i^2+\zeta_i^2)\,w_{1,d_1}+\sum_{i\geq 2}(x_i^2-\zeta_i^2)\,w_{2,d_1}+\sum_{i\geq 2}x_i\zeta_i w_{3,d_1}\,.$$

Since the ideal $(\sum_{i\geq 2}(x_i^2-\zeta_i^2),\ \sum_{i\geq 2}x_i\zeta_i)$ is prime (cf. the proof of Lemma 3.6), we can find polynomials $v_i\in\boldsymbol{R}[x,\zeta]_{2k-4}$ $(i=1,2,3)$ such that they are homogeneous of degree d_1-2 in the variables $(x_2,\cdots,x_{n+1},\zeta_2,\cdots,\zeta_{n+1})$ and

$$w_{1,d_1}=\sum_{i\geq 1}(x_i^2-\zeta_i^2)\,v_2+\sum_{i\geq 2}x_i\zeta_i v_3\,,$$

$$w_{2,d_1}=-\sum_{i\geq 2}(x_i^2+\zeta_i^2)\,v_2+\sum_{i\geq 2}x_i\zeta_i v_1\,,$$

$$w_{3,d_1}=-\sum_{i\geq 2}(x_i^2+\zeta_i^2)\,v_3-\sum_{i\geq 2}(x_i^2-\zeta_i^2)\,v_1\,.$$

Define $w_i'\in\boldsymbol{R}[x,\zeta]_{2k-2}$ $(i=1,2,3)$ by the conditions;

$$w_i'=\sum_{j=0}^{2k-2}w_{i,j}',\quad w_{i,j}'=w_{i,j}\ (j\neq d_1,d_1-2)\,,\quad w_{i,d_1}'=0\,,$$

$$w_{1,d_1-2}'=w_{1,d_1-2}-(x_1^2-\zeta_1^2)\,v_2-x_1\zeta_1 v_3\,,$$

$$w_{2,d_1-2}'=w_{2,d_1-2}-x_1\zeta_1 v_1+(x_1^2+\zeta_1^2)\,v_2\,,$$

$$w_{3,d_1-2}'=w_{3,d_1-2}+(x_1^2-\zeta_1^2)\,v_1+(x_1^2+\zeta_1^2)\,v_3\,.$$

Then we have

$$w=\sum_i(x_i^2+\zeta_i^2)\,w_i'+\sum_i(x_i^2-\zeta_i^2)\,w_2'+\sum_i x_i\zeta_i w_3'$$

and $\deg_2 w_i'\leq d_1-1$ $(i=1,2,3)$. Therefore the lemma can be proved by induction on the integer d_1.

COROLLARY 4.9. *Let* $w\in\boldsymbol{R}[x,\zeta]_{2k}-\{0\}$ $(k\geq 2)$ *be also homogeneous in* *the variables* $(x_2, \cdots, x_{n+1}, \zeta_2, \cdots, \zeta_{n+1})$. *Suppose that there are polynomials* $v_{2i} \in \boldsymbol{R}[x, \zeta]_{2i}$ $(0 \leq i \leq l,\ k \leq l)$ *and* $w' \in \boldsymbol{R}[x, \zeta]_{2k-2}$ *such that* $\deg_2 v_{2i} < \deg_2 w$ $(k \leq i \leq l)$ *and*

$$\iota^* w = \iota^* \sum_{i=0}^{l} v_{2i} + \iota^* w' .$$

Then w *belongs to the ideal* $(\sum_{i \geq 2} x_i^2, \sum_{i \geq 2} \zeta_i^2, \sum_{i \geq 2} x_i \zeta_i)$.

PROOF. We can easily deduce from Lemma 3.6 that there are polynomials $u_i \in \boldsymbol{R}[x, \zeta]$ with $\deg u_i \leq 2l-2$ $(i=1, 2, 3)$ such that

$$w - \sum_{i=0}^{l} v_{2i} - w' = \Big(\sum_i (x_i^2 + \zeta_i^2) - 2\Big) u_1 + \sum_i (x_i^2 - \zeta_i^2) u_2 + \sum_i x_i \zeta_i u_3 .$$

Put

$$u_i = \sum_{j=0}^{l-1} u_{i,2j} , \qquad u_{i,2j} \in \boldsymbol{R}[x, \zeta]_{2j} \ (i = 1, 2, 3) .$$

Assume that $k < l$, and there is j_0 $(k-1 < j_0 \leq l-1)$ such that

$$\deg_2 u_{i,2j} < \deg_2 w - 2$$

for all $j > j_0$ and i. Take the homogeneous parts of degree $2j_0+2$ in the above formula;

$$\begin{aligned} -v_{2j_0+2} + 2u_{1,2j_0+2} = & \sum_i (x_i^2 + \zeta_i^2) u_{1,2j_0} + \sum_i (x_i^2 - \zeta_i^2) u_{2,2j_0} \\ & + \sum_i x_i \zeta_i u_{3,2j_0} . \end{aligned}$$

Since $\deg_2(-v_{2j_0+2} + 2u_{1,2j_0+2}) < \deg_2 w$, we see by Lemma 4.8 that there are polynomials $u'_{i,2j_0} \in \boldsymbol{R}[x, \zeta]_{2j_0}$ $(i=1, 2, 3)$ such that $\deg_2 u'_{i,2j_0} < \deg_2 w - 2$ and

$$\begin{aligned} -v_{2j_0+2} + 2u_{1,2j_0+2} = & \sum_i (x_i^2 + \zeta_i^2) u'_{1,2j_0} + \sum_i (x_i^2 - \zeta_i^2) u'_{2,2j_0} \\ & + \sum_i x_i \zeta_i u'_{3,2j_0} . \end{aligned}$$

Since

$$\begin{aligned} & \sum_i (x_i^2 + \zeta_i^2)(u'_{1,2j_0} - u_{1,2j_0}) + \sum_i (x_i^2 - \zeta_i^2)(u'_{2,2j_0} - u_{2,2j_0}) \\ & \quad + \sum_i x_i \zeta_i (u'_{3,2j_0} - u_{3,2j_0}) = 0 , \end{aligned}$$

there are polynomials $\alpha_i \in \boldsymbol{R}[x, \zeta]_{2j_0-2}$ $(i=1, 2, 3)$ such that

$$u'_{1,2j_0} = u_{1,2j_0} + \sum_i (x_i^2 - \zeta_i^2) \alpha_2 + \sum_i x_i \zeta_i \alpha_3 ,$$

$$u'_{2,2j_0} = u_{2,2j_0} - \sum_i (x_i^2 + \zeta_i^2) \alpha_2 + \sum_i x_i \zeta_i \alpha_1 ,$$

$$u'_{3,2j_0}=u_{3,2j_0}-\sum_i(x_i^2+\zeta_i^2)\,\alpha_3-\sum_i(x_i^2-\zeta_i^2)\,\alpha_1\,.$$

Set $u'_{1,2j_0-2}=u_{1,2j_0-2}$, $u'_{2,2j_0-2}=u_{2,2j_0-2}+2\alpha_2$, $u'_{3,2j_0-2}=u_{3,2j_0-2}+2\alpha_3$, and $u'_{i,2j}=u_{i,2j}$ if $j\neq j_0$, j_0-1 $(i=1, 2, 3)$. Then setting $u'_i=\sum_{j=0}^{l-1}u'_{i,2j}$ we have

$$w-\sum_{j=0}^{l}v_{2j}-w'=\Big(\sum_i(x_i^2+\zeta_i^2)-2\Big)\,u'_1+\sum_i(x_i^2-\zeta_i^2)\,u'_2+\sum_i x_i\zeta_i u'_3\,,$$

and $\deg_2 u'_{i,2j}<\deg_2 w-2$ for all $j\geq j_0$ and i. Thus by induction on j_0 we see that there are polynomials $u''_i=\sum_{j=0}^{l-1}u''_{i,2j}$, $u''_{i,2j}\in\boldsymbol{R}[x, \zeta]_{2j}$ $(i=1, 2, 3)$ such that

$$w-\sum_{j=0}^{l}v_{2j}-w'=\Big(\sum_i(x_i^2+\zeta_i^2)-2\Big)\,u''_i+\sum_i(x_i^2-\zeta_i^2)\,u''_2+\sum_i x_i\zeta_i u''_3$$

and

$$\deg_2 u''_{i,2j}<\deg_2 w-2$$

for all $j\geq k$ and i.

In case $k=l$ we put $u''_i=u_i$ $(i=1, 2, 3)$. Consider the homogeneous parts of degree $2k$;

$$w-v_{2k}+2u''_{1,2k}=\sum_i(x_i^2+\zeta_i^2)\,u''_{1,2k-2}+\sum_i(x_i^2-\zeta_i^2)\,u''_{2,2k-2}+\sum_i x_i\zeta_i u''_{3,2k-2}$$

$(u''_{1,2k}=0$ if $k=l)$. Since $\deg_2(w-v_{2k}+2u''_{1,2k})\leq\deg_2 w$, we may assume that $\deg_2 u''_{i,2k-2}\leq\deg_2 w-2$ $(i=1, 2, 3)$ by Lemma 4.8. Then, by taking the homogeneous parts of degree $\deg_2 w$ in the variables $(x_2, \cdots, x_{n+1}, \zeta_2, \cdots, \zeta_{n+1})$ in the above formula, we have

$$w\in\Big(\sum_{i\geq2}(x_i^2+\zeta_i^2),\ \sum_{i\geq2}(x_i^2-\zeta_i^2),\ \sum_{i\geq2}x_i\zeta_i\Big)\,.$$

Let V and V' be as before Corollary 4.7. We remark that V' is also defined as the direct sum of vector spaces $Q_{2p',2q'}$ in $G(P_{4m+2})$ such that $N_{2p',2q'}=N_{2p,2q}$ and $q'\geq q$. This can be easily seen from the fact that $k\rightarrow N_{2k,2l}$, $l\rightarrow N_{2k,2l}$, and $l\rightarrow N_{2k+2l\;2k-2l}$ are monotonously increasing.

COROLLARY 4.10. *Take $u_1\in\boldsymbol{R}[x]_{2k+1}$ and $u_2\in\boldsymbol{R}[x]_{2l+1}$ such that $l\leq k\leq m$, $q\leq l$, and $k+l+1=p+q$. Suppose that u_1 and u_2 are also homogeneous in the variables $(x_2, \cdots, x_{n+1})$ and* $\deg_2 u_1+\deg_2 u_2=d$. *Furthermore suppose that there are polynomials v_i, $w_i\in\boldsymbol{R}[x]_{od}$ $(i=1, \cdots, r)$ such that* $\deg v_i\leq 2m+1$, $\deg w_i\leq 2m+1$, $\deg_2 v_i+\deg_2 w_i<d$ $(i=1, \cdots, r)$, *and*

$$\Big(G_t{}^*F(u_1, u_2)\Big)_{V'}=\sum_{i=1}^{r}\Big(G_t{}^*F(v_i, w_i)\Big)_{V'}\,.$$

Then the polynomial

$$\sum_{j=l-q}^{l} a_{j-(l-q)}^{k-l+2+2(l-q)} \tilde{G}F_j(u_1, u_2)$$

belongs to the ideal $(\sum_{i\geq 2} x_i^2, \sum_{i\geq 2} \zeta_i^2, \sum_{i\geq 2} x_i\zeta_i)$

PROOF. Since $\deg \tilde{G}F(u_1, u_2) \leq 2p+2q$, and since $V' \cap G(P_{2p+2q}) = Q_{2p,2q}$, it follows that

$$\left(G\iota^* F(u_1, u_2)\right)_{V'} = \left(G\iota^* F(u_1, u_2)\right)_{Q_{2p,2q}}.$$

As we have seen in the proof of Lemma 3. 5,

$$\begin{aligned}&\left(G\iota^* F(u_1, u_2)\right)_{Q_{2p,2q}} \\ &\quad = -(2k+3)\, d_{l-q}^{k,l} \left(G\iota^* \sum_{j=l-q}^{l} a_{j-(l-q)}^{k-l+2+2(l-q)} F_j(u_1, u_2)\right)_{G(Q_{2p+2q})}.\end{aligned}$$

Hence there is $\alpha_1 \in \boldsymbol{R}[x, \zeta]_{2p+2q-2}$ such that

$$\begin{aligned}&\left(G\iota^* F(u_1, u_2)\right)_{V'} \\ &\quad = \iota^* \left\{-(2k+3)\, d_{l-q}^{k,l} \sum_{j=l-q}^{l} a_{j-(l-q)}^{k-l+2+2(l-q)} \tilde{G}F_j(u_1, u_2) + \alpha_1\right\}.\end{aligned}$$

On the other hand, it follows from Corollary 4. 7 that there are polynomials $\beta \in \boldsymbol{R}[x, \zeta]$ and $\alpha_2 \in \boldsymbol{R}[x, \zeta]_{2p+2q-2}$ such that $\deg \beta \leq 4m+2$, $\deg_2 \beta < d$, and

$$\sum_{i=1}^{r} \left(G\iota^* F(v_i, w_i)\right)_{V'} = \iota^* (\beta + \alpha_2).$$

Therefore the corollary follows from Corollary 4. 9.

Let $\nu = (\nu_2, \cdots, \nu_{n+1}) \in \boldsymbol{C}^n - \{0\}$ satisfy $\sum_{i=2}^{n+1} \nu_i^2 = 0$. Define a homomorphism $\boldsymbol{C}[x, \zeta] \to \boldsymbol{C}[x_1, x_2, \zeta_1, \zeta_2]$ $(u \to u^\nu)$ by

$$u^\nu(x_1, x_2, \zeta_1, \zeta_2) = u(x_1, \nu_2 x_2, \cdots, \nu_{n+1} x_2, \zeta_1, \nu_2\zeta_2, \cdots, \nu_{n+1}\zeta_2).$$

The following formulas are easily verified:

$$\sum_{i=1}^{2} x_i \frac{\partial u^\nu}{\partial \zeta_i} = \left(\sum_{i=1}^{n+1} x_i \frac{\partial u}{\partial \zeta_i}\right)^\nu, \qquad F_j(u_1^\nu, u_2^\nu) = F_j(u_1, u_2)^\nu,$$

$$\tilde{G}(u^\nu) = (\tilde{G}u)^\nu, \qquad u \in \boldsymbol{C}[x, \zeta], \qquad u_1, u_2 \in \boldsymbol{C}[x].$$

LEMMA 4. 11. *Under the same assumptions and terminologies as in Corollary* 4. 10, *we have*

$$\left(G\iota^* F_{l-q}(u_1^\nu, u_2^\nu)\right)_{Q_{2p,2q}} = 0.$$

PROOF. Since the ideal $(\sum_{i\geq 2} x_i^2, \sum_{i\geq 2} \zeta_i^2, \sum_{i\geq 2} x_i\zeta_i)$ in $\boldsymbol{C}[x, \zeta]$ is contained in the kernel of the homomorphism

$$\boldsymbol{C}[x, \zeta] \longrightarrow \boldsymbol{C}[x_1, x_2, \zeta_1, \zeta_2] \qquad (u \to u^\nu),$$

we have

$$\sum_{j=l-q}^{l} a_{j-(l-q)}^{k-l+2+2(l-q)} \tilde{G}F_j(u_1^\nu, u_2^\nu) = 0$$

by Corollary 4.10. Since

$$\left(Gt^* F_{l-q}(u_1^\nu, u_2^\nu)\right)_{Q_{2p,2q}}$$
$$= \left(Gt^* \sum_{j=l-q}^{l} a_{j-(l-q)}^{k-l+2+2(l-q)} F_j(u_1^\nu, u_2^\nu)\right)_{G(Q_{2p+2q})},$$

the lemma follows.

PROOF of Proposition 4.5. Let V_1 (resp. V_1') be the direct sum of vector spaces $Q_{2p,2q} \cap G(P_{4m+2})$ such that $N_{2p,2q} = N_{4i_0-2,4}$ (resp. $N_{2p,2q} = N_{4i_0-2,4}$ and $q \geq 2$). Since V_1' is orthogonal to $G(P^2)$ and $G(P_{4i_0})$, we have

$$\begin{aligned} 0 &= \left(Gt^* F(f, f)\right)_{V_1'} \\ &= \left(Gt^* F(h_{2i_0+1}, h_{2i_0+1})\right)_{V_1'} + 2\left(Gt^* F(h_{2i_0+1}, f_{2i_0+1} - h_{2i_0+1})\right)_{V_1'} \\ &+ \left(Gt^* F(f_{2i_0+1} - h_{2i_0+1}, f_{2i_0+1} - h_{2i_0+1})\right)_{V_1'} \\ &+ \sum \left(Gt^* F(f_{2i+1}, f_{2j+1})\right)_{V_1'}, \end{aligned}$$

where the sum in the last term is taken over all 2-tuples of integers $(i. j)$ such that $2 \leq i, j \leq m$, $i+j \geq 2i_0$, $(i, j) \neq (i_0, i_0)$. For such (i, j) we have $\deg_2 f_{2i+1} + \deg_2 f_{2j+1} < 2d_{i_0}$ by the definition of i_0. Hence it follows from Lemma 4.11 that

$$\left(Gt^* F_{i_0-2}(h_{2i_0+1}^\nu, h_{2i_0+1}^\nu)\right)_{Q_{4i_0-2,4}} = 0.$$

Since $h_{2i_0+1} \neq 0$ and the degree of h_{2i_0+1} in the variable x_{n+1} is at most 1, it follows that h_{2i_0+1} does not belong to the ideal $(\sum_{i\geq 2} x_i^2)$ in $\boldsymbol{C}[x]$. This ideal being prime, we can choose $\nu = (\nu_2, \cdots, \nu_{n+1}) \in \boldsymbol{C}^n - \{0\}$ with $\sum_{i\geq 2} \nu_i^2 = 0$ such that $h_{2i_0+1}^\nu \neq 0$. Then by Proposition 3.8 (ii)

$$h_{2i_0+1}^\nu = (\alpha_1 x_1 + \alpha_2 x_2)^{2i_0} (\beta_1 x_1 + \beta_2 x_2)$$

for some $\alpha_1, \alpha_2. \beta_1. \beta_2 \in \boldsymbol{C}$. On the other hand, $h_{2i_0+1}^\nu$ must be of the form

$$c x_1^{2i_0+1-d_{i_0}} x_2^{d_{i_0}}, \qquad c \in \boldsymbol{C} - \{0\}$$

by the definition of h_{2i_0+1}. Since $i_0 \geq 2$ and $d_{i_0} \geq 2$, we can conclude that

$$d_{i_0} = 2i_0 \text{ or } 2i_0+1 .$$

Let V_2 (resp. V_2') be the direct sum of vector spaces $Q_{2p,2q}$ in $G(P_{4m+2})$ such that $N_{2p,2q} = N_{2i_0+2i_1-2,4}$ (resp. $N_{2p,2q} = N_{2i_0+2i_1-2,4}$ and $q \geq 2$). Then we have

$$\begin{aligned} 0 &= \left(G\iota^* F(f,f)\right)_{V_2'} \\ &= 2\left(G\iota^* F(h_{2i_1+1}, h_{2i_0+1})\right)_{V_2'} + 2\left(G\iota^* F(h_{2i_1+1}, f_{2i_0+1} - h_{2i_0+1})\right)_{V_2'} \\ &+ 2\left(G\iota^* F(f_{2i_1+1} - h_{2i_1+1}, h_{2i_0+1})\right)_{V_2'} + 2\left(G\iota^* F(f_{2i_1+1} - h_{2i_1+1}, f_{2i_0+1} - h_{2i_0+1})\right)_{V_2'} \\ &+ \sum \left(G\iota^* F(f_{2i+1}, f_{2j+1})\right)_{V_2'} , \end{aligned}$$

where the sum in the last term is taken over all 2-tuples of integers (i, j) such that $2 \leq i,\ j \leq m,\ i_0 + i_1 \leq i + j,\ \{i, j\} \neq \{i_0, i_1\}$. For such (i, j) we easily see that

$$\deg_2 f_{2i+1} + \deg_2 f_{2j+1} < d_{i_0} + d_{i_1} .$$

Hence it follows from Lemma 4. 11 that

$$\left(G\iota^* F_{i_0-2}(h^{\nu}_{2i_1+1}, h^{\nu}_{2i_0+1})\right)_{Q_{2i_0+2i_1-2,4}} = 0$$

for each $\nu = (\nu_2, \cdots, \nu_{n+1}) \in \boldsymbol{C}^{n+1} - \{0\}$ with $\sum_{i \geq 2} \nu_i^2 = 0$. We can choose ν such that $h^{\nu}_{2i_0+1}$ and $h^{\nu}_{2i_1+1}$ do not vanish. We shall consider the two cases separately; I) $d_{i_0} = 2i_0 + 1$, II) $d_{i_0} = 2i_0$.

I. $d_{i_0} = 2i_0 + 1$. In this case, $h^{\nu}_{2i_0+1} = c_0 x_2^{2i_0+1}$ and $h^{\nu}_{2i_1+1} = c_1 x_1^{2i_1+1-d_{i_1}} x_2^{d_{i_1}}$ $(c_0, c_1 \in \boldsymbol{C} - \{0\})$. Then we have

$$\begin{aligned} 0 &= \left(G\iota^* F_{i_0-2}(x_1^{2i_1+1-d_{i_1}} x_2^{d_{i_1}}, x_2^{2i_0+1})\right)_{Q_{2i_0+2i_1-2,4}} \\ &= \frac{(2i_0+1)!}{4!} \left(G\iota^* (x_1^{2i_1+1-d_{i_1}} x_2^{2i_0-3+d_{i_1}} \zeta_2^4)\right)_{Q_{2i_0+2i_1-2,4}} . \end{aligned}$$

In view of Lemma 3. 9 the last term does not vanish if $2i_1 + 1 - d_{i_1} \geq 4$. Thus we have $d_{i_1} \geq 2i_1 - 2$. On the other hand, the definition of d_{i_1} implies that $d_{i_1} \leq d_{i_0} - 1 = 2i_0 \leq 2i_1 - 2$. Therefore it follows that

$$d_{i_1} = 2i_1 - 2 , \qquad i_1 = i_0 + 1 .$$

Since $d_{i_1} \geq 4$ in this case, it follows that $i_1 \neq m$. Let i_2 $(i_1 < i_2 \leq m)$ be the index such that $d_i \leq d_{i_2}$ if $i_1 < i \leq i_2$ and $d_i < d_{i_2}$ if $i_2 < i \leq m$. Then there are three cases; I-1) $d_{i_2} \leq d_{i_1} - 2$, I-2) $d_{i_2} = d_{i_1} - 1$, $i_2 \geq i_1 + 2$, I-3) $d_{i_2} = d_{i_1} - 1$, $i_2 = i_1 + 1$.

I-1. $d_{i_2} \leq d_{i_1} - 2$. Let V_3 (resp. V_3') be the direct sum of vector sapces

$Q_{2p,2q}$ in $G(P_{4m+2})$ such that $N_{2p,2q}=N_{4i_1-2,4}$ (resp. $N_{2p,2q}=N_{4i_1-2,4}$ and $q\geq 2$). Then we have

$$\begin{aligned} 0 &= \left(Gt^*F(f,f)\right)_{V_3'} \\ &= \left(Gt^*F(h_{2i_1+1}, h_{2i_1+1})\right)_{V_3'} + 2\left(Gt^*F(f_{2i_1+1}-h_{2i_1+1}, h_{2i_1+1})\right)_{V_3'} \\ &+ \left(Gt^*F(f_{2i_1+1}-h_{2i_1+1}, f_{2i_1+1}-h_{2i_1+1})\right)_{V_3'} + \sum \left(Gt^*F(f_{2i+1}, f_{2j+1})\right)_{V_3'}, \end{aligned}$$

where the sum in the last term is taken over all 2-tuples of integers (i, j) such that $2\leq i,\ j\leq m,\ i+j\geq 2i_1,\ (i, j)\neq(i_1, i_1)$. If (i, j) $(i\geq j)$ satisfies these conditions, then $i>i_1$, and hence

$$\deg_2 f_{2i+1}\leq d_{i_2}\leq d_{i_1}-2\,.$$

Thus it follows that

$$\deg_2 f_{2i+1}+\deg_2 f_{2j+1}\leq d_{i_1}-2+d_{i_0}=2d_{i_1}-1\,.$$

Then we have by Lemma 4. 11

$$\left(Gt^*F_{i_1-2}(h^{\nu}_{2i_1+1}, h^{\nu}_{2i_1+1})\right)_{Q_{4i_1-2,4}}=0\,.$$

We can choose ν such that $h^{\nu}_{2i_1+1}\neq 0$. Since $2i_1-2\geq 4$ and

$$h^{\nu}_{2i_1+1}=cx_1^{2i_1+1-d_{i_1}}x_2^{d_{i_1}}=cx_1^3x_2^{2i_1-2} \qquad (c\in \boldsymbol{C}-\{0\})\,,$$

the above formula contradicts Proposition 3. 8 (ii).

I-2. $d_{i_2}=d_{i_1}-1$, $i_2\geq i_1+2$. Let V_4 (resp. V_4') be the direct sum of $Q_{2p,2q}$ in $G(P_{4m+2})$ such that $N_{2p,2q}=N_{2i_0+2i_2-2,4}$ (resp. $N_{2p,2q}=N_{2i_0+2i_2-2,4}$ and $q\geq 2$). Then we have

$$\begin{aligned} 0 &= \left(Gt^*F(f,f)\right)_{V_4'} \\ &= 2\left(Gt^*F(h_{2i_2+1}, h_{2i_0+1})\right)_{V_4'} + 2\left(Gt^*F(f_{2i_2+1}-h_{2i_2+1}, h_{2i_0+1})\right)_{V_4'} \\ &+ 2\left(Gt^*F(h_{2i_2+1}, f_{2i_0+1}-h_{2i_0+1})\right)_{V_4'} \\ &+ 2\left(Gt^*F(f_{2i_2+1}-h_{2i_2+1}, f_{2i_0+1}-h_{2i_0+1})\right)_{V_4'} \\ &+ \sum\left(Gt^*F(f_{2i+1}, f_{2j+1})\right)_{V_4'}, \end{aligned}$$

where the sum in the last term is taken over all 2-tuples of integers (i, j) such that $2\leq i,\ j\leq m,\ i+j\geq i_0+i_2,\ \{i, j\}\neq\{i_0, i_2\}$. Let (i, j) be such 2-tuple and assume that $i\geq j$. Since $i_2\geq i_1+2$, it follows that $i>i_1$. Hence we have either $i_1<i\leq i_2$ and $i_0<j$ or $i_2<i$. In any case we have

$$\deg_2 f_{2i+1} + \deg_2 f_{2j+1} \leq d_{i_0} + d_{i_2} - 1 .$$

Thus by applying Lemma 4. 11 we have

$$\left(G\iota^* F_{i_0-2}(h^{\nu}_{2i_2+1},\ h^{\nu}_{2i_0+1})\right)_{Q2i_0+2i_2-2,4} = 0 .$$

We can choose ν such that $h^{\nu}_{2i_0+1} \neq 0$ and $h^{\nu}_{2i_2+1} \neq 0$. Since $h^{\nu}_{2i_0+1} = c_0 x_2^{2i_0+1}$ and $h^{\nu}_{2i_2+1} = c_2 x_1^{2i_2+1-d_{i_2}} x_2^{d_{i_2}}$ $(c_0, c_2 \in \boldsymbol{C} - \{0\})$, it follows that

$$\left(G\iota^* (x_1^{2i_2+1-d_{i_2}} x_2^{2i_0-3+d_{i_2}} \zeta_2^4)\right)_{Q2i_0+2i_2-2,4} = 0 .$$

But, since $2i_2 + 1 - d_{i_2} \geq 2i_1 + 5 - (d_{i_1} - 1) = 8$, this contradicts Lemma 3. 9.

I-3. $d_{i_2} = d_{i_1} - 1$, $i_2 = i_1 + 1$. Let V_3 and V_3' be as before. In this case we have

$$\begin{aligned} 0 &= \left(G\iota^* F(f, f)\right)_{V_3'} \\ &= \left(G\iota^* F(h_{i_1+1},\ h_{2i_1+1})\right)_{V_3'} + 2\left(G\iota^* F(h_{2i_2+1},\ h_{2i_0+1})\right)_{V_3'} \\ &\quad + 2\left(G\iota^* F(f_{2i_1+1} - h_{2i_1+1},\ h_{2i_1+1})\right)_{V_3'} + \left(G\iota^* F(f_{2i_1+1} - h_{2i_1+1},\ f_{2i_1+1} - h_{2i_1+1})\right)_{V_3'} \\ &\quad + 2\left(G\iota^* F(f_{2i_2+1} - h_{2i_2+1},\ h_{2i_0+1})\right)_{V_3'} + 2\left(G\iota^* F(h_{2i_2+1},\ f_{2i_0+1} - h_{2i_0+1})\right)_{V_3'} \\ &\quad + 2\left(G\iota^* F\,(f_{2i_2+1} - h_{2i_2+1},\ f_{2i_0+1} - h_{2i_0+1})\right)_{V_3'} + \sum \left(G\iota^* F(f_{2i+1},\ f_{2j+1})\right)_{V_3'} , \end{aligned}$$

where the sum in the last term is taken over all 2-tuples of integers (i, j) such that $2 \leq i,\ j \leq m,\ i+j \geq 2i_1,\ (i, j) \neq (i_1,\ i_1),\ (i_0, i_2),\ (i_2, i_0)$. For such (i, j) we have

$$\deg_2 f_{2i+1} + \deg_2 f_{2j+1} \leq 2d_{i_1} - 1 .$$

Since

$$\begin{aligned} &\left(G\iota^* F(h_{2i_1+1},\ h_{i_1+1})\right)_{V_3'} \\ &\qquad = -(2i_1+3)\, d^{i_1, i_1}_{i_1-2} \left(G\iota^* F_{i_1-2}(h_{2i_1+1},\ h_{2i_1+1})\right)_{Q4i_1-2,4} , \\ &\left(G\iota^* F(h_{2i_2+1},\ h_{2i_0+1})\right)_{V_3'} \\ &\qquad = -(2i_2+3)\, d^{i_2, i_0}_{i_0-2} \left(G\iota^* F_{i_0-2}(h_{2i_2+1},\ h_{2i_0+1})\right)_{Q4i_1-2,4} , \end{aligned}$$

it is easily seen from the proof of Lemma 4. 11 that

$$\begin{aligned} &(2i_1+3)\, d^{i_1, i_1}_{i_1-2} \left(G\iota^* F_{i_1-2}(h^{\nu}_{2i_1+1},\ h^{\nu}_{2i_1+1})\right)_{Q4i_1-2,4} \\ &\qquad + 2(2i_2+3)\, d^{i_2, i_0}_{i_0-2} \left(G\iota^* F_{i_0-2}(h^{\nu}_{2i_2+1},\ h^{\nu}_{2i_0+1})\right)_{Q4i_1-2,4} = 0 . \end{aligned}$$

Fix $\nu=(\nu_2, \cdots, \nu_{n+1})\in \boldsymbol{C}^n-\{0\}$ with $\sum_{i\geq 2}\nu_i^2=0$ such that $h^{\nu}_{2i_0+1}$, $h^{\nu}_{2i_1+1}$, $h^{\nu}_{2i_2+1}$ do not vanish. Then

$$h^{\nu}_{2i_0+1}=c_0 x_2^{2i_0+1}, \qquad h^{\nu}_{2i_1+1}=c_1 x_1^3 x_2^{2i_1-2},$$
$$h^{\nu}_{2i_2+1}=c_2 x_1^6 x_2^{2i_2-5} \qquad (c_0, c_1, c_2\in \boldsymbol{C}-\{0\}).$$

We have

$$\Big(Gt^* F_{i_0-2}(h^{\nu}_{2i_2+1},\ h^{\nu}_{2i_0+1})\Big)_{Q_{4i_1-2,4}}=\frac{(2i_0+1)\,!}{4\,!}c_0c_2\Big(Gt^*(x_1^6 x_2^{4i_1-8}\zeta_2^4)\Big)_{Q_{4i_1-2,4}},$$
$$\Big(Gt^* F_{i_1-2}(h^{\nu}_{2i_1-1},\ h^{\nu}_{2i_1+1})\Big)_{Q_{4i_1-2,4}}=A^{i_1}_{3,3}c_1^2\Big(Gt^*(x_1^6 x_2^{4i_1-8}\zeta_2^4)\Big)_{Q_{4i_1-2,4}}$$

by the proof of Proposition 3. 8, where $A^{i_1}_{3,3}=\dfrac{(2i_1-1)\,!}{5\,!}(2i_1-2)(2i_1-3)$. Since $(Gt^*(x_1^6x_2^{4i_1-8}\zeta_2^4))_{Q_{4i_1-2,4}}\neq 0$ by Lemma 3. 9, it follows that

$$\frac{1}{5}(2i_1+3)(2i_1-2)(2i_1-3)\,d^{i_1,i_1}_{i_1-2}c_1^2+2(2i_2+3)\,d^{i_2,i_0}_{i_0-2}c_0c_2=0\,.$$

In particular we have $c_0c_2c_1^{-2}<0$.

Let V_5 (resp. V_5') be the direct sum of vector spaces $Q_{2p,2q}$ in $G(P_{4m+2})$ such that $N_{2p,2q}=N_{4i_1-4,6}$ (resp. $N_{2p,2q}=N_{4i_1-4,6}$ and $q\geq 3$). Since V_5' is orthogonal to $G(P^2)$ and $G(P_{4i_1})$, we have

$$\begin{aligned}0&=\Big(Gt^* F(f,f)\Big)_{V_5'}\\&=\Big(Gt^* F(h_{2i_1+1},\ h_{2i_1+1})\Big)_{V_5'}+2\Big(Gt^* F(h_{2i_2+1},\ h_{2i_0+1})\Big)_{V_5'}\\&+2\Big(Gt^* F(f_{2i_1+1}-h_{2i_1+1},\ h_{2i_1+1})\Big)_{V_5'}+2\Big(Gt^* F(f_{2i_1+1}-h_{2i_1+1},\ f_{2i_1+1}-h_{2i_1+1})\Big)_{V_5'}\\&+2\Big(Gt^* F(f_{2i_2+1}-h_{2i_2+1},\ h_{2i_0+1})\Big)_{V_5'}+2\Big(Gt^* F(h_{2i_2+1},\ f_{2i_0+1}-h_{2i_0+1})\Big)_{V_5'}\\&+2\Big(Gt^* F(f_{2i_2+1}-h_{2i_2+1},\ f_{2i_0+1}-h_{2i_0+1})\Big)_{V_5'}+\sum\Big(Gt^* F(f_{2i+1},\ f_{2j+1})\Big)_{V_5'},\end{aligned}$$

where the sum in the last term is taken over all 2-tuples of integers (i,j) such that $2\leq i,\ j\leq m,\ i+j\geq 2i_1,\ (i,j)\neq(i_1, i_1),\ (i_0, i_2),\ (i_2, i_0)$. For such (i,j) we have

$$\deg_2 f_{2i+1}+\deg_2 f_{2j+1}\leq 2d_{i_1}-1\,.$$

Since

$$\Big(Gt^* F(h_{2i_1+1},\ h_{2i_1+1})\Big)_{V_5'}=-(2i_1+3)\,d^{i_1,i_1}_{i_1-3}\Big(Gt^* F_{i_1-3}(h_{2i_1+1},\ h_{2i_1+1})\Big)_{Q_{4i_1-4,6}},$$
$$\Big(Gt^* F(h_{2i_2+1},\ h_{2i_0+1})\Big)_{V_5'}=-(2i_2+3)\,d^{i_2,i_0}_{i_0-3}\Big(Gt^* F_{i_0-3}(h_{2i_2+1},\ h_{2i_0+1})\Big)_{Q_{4i_1-4,6}},$$

it follows from the proof of Lemma 4.11 that

$$(2i_1+3)\, d_{i_1-3}^{i_1,i_1}\Big(G\iota^* F_{i_1-3}(h_{2i_1+1}^{\nu},\ h_{2i_1+1}^{\nu})\Big)_{Q4i_1-4,6}$$
$$+2(2i_2+3)\, d_{i_0-3}^{i_2,i_0}\Big(G\iota^* F_{i_0-3}(h_{2i_2+1}^{\nu},\ h_{2i_0+1}^{\nu})\Big)_{Q4i_1-4,6}=0\,,$$

where ν is the same one as above, and $d_{i_0-3}^{i_2,i_0}$ is considered to be zero in case $i_0=2$. We have

$$\Big(G\iota^* F_{i_0-3}(h_{2i_2+1}^{\nu},\ h_{2i_0+1}^{\nu})\Big)_{Q4i_1-4,6}=\frac{(2i_0+1)\,!}{6\,!}\,c_0c_2\Big(G\iota^*(x_1^6x_2^{4i_1-10}\zeta_2^6)\Big)_{Q4i_1-4,6}$$
$$(i_0\neq 2)\,,$$

$$\Big(G\iota^* F_{i_1-3}(h_{2i_1+1}^{\nu},\ h_{2i_1+1}^{\nu})\Big)_{Q4i_1-4,6}=c_1^2\Big(G\iota^* F_{i_1-3}(x_1^3x_2^{2i_1-2},\ x_1^3x_2^{2i_1-2})\Big)_{Q4i_1-4,6}\,.$$

LEMMA 4.12. (i)

$$\Big(G\iota^* F_{i_1-3}(x_1^3x_2^{2i_1-2},\ x_1^3x_2^{2i-2})\Big)_{Q4i_1-4,6}$$
$$=-\frac{(2i_1-2)\,!}{6\,!}(2i_1-2)\,(2i_1-3)\,(2i_1-4)\,\Big(G\iota^*(x_1^6x_2^{4i_1-10}\zeta_2^6)\Big)_{Q4i_1-4,6}\,.$$

(ii) $(G\iota^*(x_1^6x_2^{4i_1-10}\zeta_2^6))_{Q4i_1-4,6}\neq 0.$

PROOF. (i) We have

$$F_{i_1-3}(x_1^3x_2^{2i_1-2},\ x_1^3x_2^{2i_1-2})$$
$$=\frac{(2i_1-2)\,!}{3\,!}\,x_1^3x_2^{4i_1-7}\zeta_1^3\zeta_2^3+3(2i_1-5)\,\frac{(2i_1-2)\,!}{4\,!}\,x_1^4x_2^{4i-8}\zeta_1^2\zeta_2^4$$
$$+6\binom{2i_1-5}{2}\,\frac{(2i_1-2)\,!}{5\,!}\,x_1^5x_2^{4i_1-9}\zeta_1\zeta_2^5+6\binom{2i_1-5}{3}\,\frac{(2i_1-2)\,!}{6\,!}\,x_1^6x_2^{4i_1-10}\zeta_2^6\,.$$

Since

$$\tilde{X}_{E_0}(x_1^kx_2^{4i_1-3-k}\zeta_1^{6-k}\zeta_2^{k-1})=kx_1^{k-1}x_2^{4i_1-3-k}\zeta_1^{7-k}\zeta_2^{k-1}$$
$$+(4i_1-3-k)\,x_1^kx_2^{4i_1-4-k}\zeta_1^{6-k}\zeta_2^k-(6-k)\,x_1^{k+1}x_2^{4i_1-3-k}\zeta_1^{5-k}\zeta_2^{k-1}$$
$$-(k-1)\,x_1^kx_2^{4i_1-2-k}\zeta_1^{6-k}\zeta_2^{k-2}\qquad(4\leq k\leq 6)\,,$$

it follows that

$$\Big(G\iota^*(x_1^5x_2^{4i_1-9}\zeta_1\zeta_2^5)\Big)_{Q4i_1-4,6}=-\frac{4i_1-9}{6}\Big(G\iota^*(x_1^6x_2^{4i_1-10}\zeta_2^6)\Big)_{Q4i_1-4,6}\,,$$

$$\Big(G\iota^*(x_1^4x_2^{4i_1-8}\zeta_1^2\zeta_2^4)\Big)_{Q4i_1-4,6}=\frac{(4i_1-8)\,(4i_1-9)}{5\cdot 6}\Big(G\iota^*(x_1^6x_2^{4i_1-10}\zeta_2^6)\Big)_{Q4i_1-4,6}\,,$$

$$\left(G\iota^*(x_1^3x_2^{4i_1-7}\zeta_1^3\zeta_2^3)\right)_{Q_{4i_1-4,6}}$$
$$=-\frac{(4i_1-7)(4i_1-8)(4i_1-9)}{4\cdot5\cdot6}\left(G\iota^*(x_1^6x_2^{4i_1-10}\zeta_2^6)\right)_{Q_{4i_1-4,6}}.$$

Then (i) is easily obtained from these formulas.

(ii) By Corollary 2. 8 (ii) we have

$$\left(G\iota^*(x_1^6x_2^{4i_1-10}\zeta_2^6)\right)_{Q_{4i_1-4,6}}=\Bigl(G\iota^*\Bigl(x_1^6x_2^{4i_1-10}\zeta_2^6-\frac{15}{4i_1-7}x_1^6x_2^{4i_1-8}\zeta_2^4$$
$$+\frac{45}{(4i_1-5)(4i_1-7)}x_1^6x_2^{4i_1-6}\zeta_2^2$$
$$-\frac{15}{(4i_1-3)(4i_1-5)(4i_1-7)}x_1^6x_2^{4i_1-4}\Bigr)\Bigr)_{G(Q_{4i_1+2})}.$$

Then, in view of Corollary 3. 7 it is enough to prove that

$$J=\tilde{G}\Bigl(x_1^6x_2^{4i_1-10}\zeta_2^6-\frac{15}{4i_1-7}x_1^6x_2^{4i_1-8}\zeta_2^4+\frac{45}{(4i_1-5)(4i_1-7)}x_1^6x_2^{4i_1-6}\zeta_2^2$$
$$-\frac{15}{(4i_1-3)(4i_1-5)(4i_1-7)}x_1^6x_2^{4i_1-4}\Bigr)$$

does not vanish. A direct computation shows that the coefficient of $x_1^6\zeta_2^{4i_1-4}$ in the polynomial J is

$$I'_{6,2i_1-5}-\frac{15}{4i_1-7}I'_{5,2i_1-4}+\frac{45}{(4i_1-5)(4i_1-7)}I'_{4,2i_1-3}$$
$$-\frac{15}{(4i_1-3)(4i_1-5)(4i_1-7)}I'_{3,2i_1-2}$$
$$=\frac{1}{(4i_1-5)(4i_1-7)}\left\{\frac{11\cdot9\cdot7}{4i_1-9}-\frac{15\cdot9\cdot7}{4i_1-7}+\frac{45\cdot7}{4i_1-5}-\frac{15}{4i_1-3}\right\}I'_{3,2i_1-2}>0,$$

where $I'_{a,b}=\frac{1}{2\pi}\int_0^{2\pi}(\cos t)^{2a}(\sin t)^{2b}dt$. Thus we have $J\neq0$, proving the lemma.

As a consequence of this lemma, we have

$$-(2i_1+3)(2i_1-2)(2i_1-3)(2i_1-4)\,d_{i_1-3}^{i_1,i_1}c_1^2$$
$$+2(2i_2+3)(2i_0+1)\,d_{i_0-3}^{i_2,i_0}c_0c_2=0.$$

If $i_0=2$, then we have a contradiction, because $c_1\neq0$. If $i_0\geq3$, then we have $c_0c_2c_1^{-2}>0$, which also contradicts the previous result. Hence we have contradictions in any case under the assumption $d_{i_0}=2i_0+1$.

II. $d_{i_0}=2i_0$. In this case, $h_{2i_0+1}^{\nu}=c_0x_1x_2^{2i_0}$ and $h_{2i_1+1}^{\nu}=c_1x_1^{2i_1+1-d_{i_1}}x_2^{d_{i_1}}$, where $\nu=(\nu_2,\cdots,\nu_{n+1})$ is chosen such that $c_0\neq0$, $c_1\neq0$. Then it follows that

$$\left(G_t^* F_{i_0-2}(x_1^{2i_1+1-d_{i_1}} x_2^{d_{i_1}},\ x_1 x_2^{2i_0})\right)_{Q_{2i_0+2i_1-2,4}} = 0 .$$

We have

$$\begin{aligned} &F_{i_0-2}(x_1^{2i_1+1-d_{i_1}} x_2^{d_{i_1}},\ x_1 x_2^{2i_0}) \\ &= \frac{(2i_0)!}{3!} x_1^{2i_1+1-d_{i_1}} x_2^{2i_0-3+d_{i_1}} \zeta_1 \zeta_2^3 + (2i_0-3)\frac{(2i_0)!}{4!} x_1^{2i_1+2-d_{i_1}} x_2^{2i_0-4+d_{i_1}} \zeta_2^4 . \end{aligned}$$

By using the formula

$$\begin{aligned} \tilde{X}_{E_0}(x_1^{2i_1+2-d_{i_1}} x_2^{2i_0-3+d_{i_1}} \zeta_2^3) &= (2i_1+2-d_{i_1})\, x_1^{2i_1+1-d_{i_1}} x_2^{2i_0-3+d_{i_1}} \zeta_1 \zeta_2^3 \\ &+ (2i_0-3+d_{i_1})\, x_1^{2i_1+2-d_{i_1}} x_2^{2i_0-4+d_{i_1}} \zeta_2^4 - 3x_1^{2i_1+2-d_{i_1}} x_2^{2i_0-2+d_{i_1}} \zeta_2^2 , \end{aligned}$$

we have

$$\begin{aligned} &\left(G_t^* F_{i_0-2}(x_1^{2i_1+1-d_{i_1}} x_2^{d_{i_1}},\ x_1 x_2^{2i_0})\right)_{Q_{2i_0+2i_1-2,4}} \\ &= \frac{(2i_0)!}{3!}\left(-\frac{2i_0-3+d_{i_1}}{2i_1+2-d_{i_1}} + \frac{2i_0-3}{4}\right)\left(G_t^*(x_1^{2i_1+2-d_{i_1}} x_2^{2i_0-4+d_{i_1}} \zeta_2^4)\right)_{Q_{2i_0+2i_1-2,4}} . \end{aligned}$$

Since $2i_1+2-d_{i_1} \geq 2(i_0+1)+2-(d_{i_0}-1)=5$, we see by lemma 3. 9 that

$$\left(G_t^*(x_1^{2i_1+2-d_{i_1}} x_2^{2i_0-4+d_{i_1}} \zeta_2^4)\right)_{Q_{2i_0+2i_1-2,4}} \neq 0 .$$

Hence it follows that

$$-\frac{2i_0-3+d_{i_1}}{2i_1+2-d_{i_1}} + \frac{2i_0-3}{4} = 0 .$$

Remarking the conditions $2 \leq i_0 < i_1$ and $d_{i_1} \leq d_{i_0}-1 = 2i_0-1$, we then obtain

$$i_0 = 2 , \quad i_1 = 6 , \quad d_{i_0} = 4 , \quad d_{i_1} = 2 .$$

Let i_2 be as in Case I. We shall prove that $i_2=10$ and $d_{i_2}=0$. First assume that either $d_{i_2}=1$ or $d_{i_2}=0$ and $i_2>10$ are satisfied. Let V_4 and V_4' be as in I-2. Then we have

$$\begin{aligned} 0 &= \left(G_t^* F(f,f)\right)_{V_4'} \\ &= 2\left(G_t^* F(h_{2i_2+1},\ h_5)\right)_{V_4'} + 2\left(G_t^* F(f_{2i_2+1}-h_{2i_2+1},\ h_5)\right)_{V_4'} \\ &+ 2\left(G_t^* F(h_{2i_2+1},\ f_5-h_5)\right)_{V_4'} + 2\left(G_t^* F(f_{2i_2+1}-h_{2i_2+1},\ f_5-h_5)\right)_{V_4'} \\ &+ \left(G_t^* F(f_{2i+1},\ f_{2j+1})\right)_{V_4'} , \end{aligned}$$

where the sum in the last term is taken over all 2-tuples of integers (i, j) such that $2 \leq i, j \leq m$, $i+j \geq i_2+2$, $\{i, j\} \neq \{i_2, 2\}$. Such (i, j) with $i \geq j$ satisfies $i \geq j > 2$ or $i > i_2$, $j=2$. Thus we have

$$\deg_2 f_{2i+1} + \deg_2 f_{2j+1} \leq 4 .$$

Moreover we have

$$\deg_2 f_{2i+1} + \deg_2 f_{2j+1} \leq 2$$

in the case where $i_2 > 10$ and $d_{i_2} = 0$, because $\deg_2 f_{2i+1}$ or $\deg_2 f_{2j+1}$ must be zero (or $-\infty$) in this case. Thus we can apply Lemma 4. 11 to each case and obtain

$$\left(G\iota^* F_2(h^{\nu}_{2i_2+1}, h^{\nu}_5)\right)_{Q_{2i_2+2,4}} = 0 .$$

Since $h^{\nu}_{2i_2+1} = c_2 x_1^{2i_2+1-d_{i_2}} x_2^{d_{i_2}}$ and $h^{\nu}_5 = c_0 x_1 x_2^4$, and $\nu = (\nu_2, \cdots, \nu_{n+1})$ can be chosen so that $c_0 \neq 0$, $c_2 \neq 0$, it follows that

$$\left(G\iota^* F_2(x_1^{2i_2+1-d_{i_2}} x_2^{d_{i_2}},\ x_1 x_2^4)\right)_{Q_{2i_2+2,4}} = 0 .$$

But, as we have already seen, this equality holds if and only if $i_2 = 6$ and $d_{i_2} = 2$. This is a contradiction. Thus we have $i_2 \leq 10$ and $d_{i_2} = 0$.

Next assume that $i_2 < 10$. Let V_3 and V_3' be as in I-1. Then we have

$$\begin{aligned} 0 &= \left(G\iota^* F(f, f)\right)_{V_3'} \\ &= \left(G\iota^* F(h_{13}, h_{13})\right)_{V_3'} + 2\left(G\iota^* F(f_{13} - h_{13}, h_{13})\right)_{V_3'} \\ &\quad + \left(G\iota^* F(f_{13} - h_{13}, f_{13} - h_{13})\right)_{V_3'} + \sum \left(G\iota^* F(f_{2i+1}, f_{2i+1})\right)_{V_3'} , \end{aligned}$$

where the sum in the last term is taken over all 2-tuples of integers (i, j) such that $2 \leq i,\ j \leq m,\ i+j \geq 12,\ (i, j) \neq (6, 6)$. Since such (i, j) must satisfy either $2 < i,\ 6 < j$ or $2 < j,\ 6 < i$, it follows that

$$\deg_2 f_{2i+1} + \deg_2 f_{2j+1} \leq 2 .$$

Hence by Lemma 4. 11 we have

$$\left(G\iota^* F_4(h^{\nu}_{13}, h^{\nu}_{13})\right)_{Q_{22,4}} = 0 .$$

But since $h^{\nu}_{13} = c x_1^{11} x_2^2$ $(c \in \boldsymbol{C})$ and ν can be chosen so that $c \neq 0$, this contradicts to Proposition 3. 8 (ii). Consequently we have

$$i_2 = 10 , \qquad d_{i_2} = 0 .$$

Since $d_{i_2} = 0$, i_2 must be equal to m. Hence it follows that

$$m = 10 , \qquad d_i \leq 0 \qquad (7 \leq i \leq 10) .$$

This completes the proof of Proposition 4. 5.

§ 5. The case where deg $f=21$

In this section we shall prove the rest part of Theorem 4.1. Let $f\in\boldsymbol{R}[x]_{od}$ satisfy the conditions (i) (ii) (iii) stated before Lemma 4.4. Suppose that f satisfies the condition $Gt^*F(f,f)\in G(\mathscr{K}^2)$. Then we have seen in Proposition 4.5 that either f is of the form (1) in Theorem 4.1, or

(a) $\deg f=21$, $i_0=2$, $i_1=6$, $d_{i_0}=4$, $d_{i_1}=2$, $d_i\leq 0$ $(7\leq i\leq 10)$. Let $f=\sum_i f_{2i+1}(f_{2i+1}\in\boldsymbol{R}[x]_{2i+1})$ be the decomposition of f into its homogeneous parts, and h_{2i+1} the homogeneous part of degree d_i of f_{2i+1} in the variables $(x_2,\cdots,x_{n+1})$. In case f satisfies the above condition (a), we further consider the following conditions on f:

(b) $f_{21}=x_1^{21}$;

(c) $h_{13}=\left(\sum_{i=2}^{k}\lambda_i x_i^2\right)x_1^{11}$, where $\lambda_i\in\boldsymbol{R}-\{0\}$ $(2\leq i\leq k)$, $2\leq k\leq n$, and $\{\lambda_i\}$ do not satisfy $\lambda_2=\cdots=\lambda_n$ if $k=n$.

LEMMA 5.1. *Let $f'\in\boldsymbol{R}[x]_{od}$ satisfy the conditions* (i) (ii) (iii) *stated before Lemma* 4.4 *and* (a) *above. Then there are $A\in O(n+1,\boldsymbol{R})$, $c\in\boldsymbol{R}-\{0\}$, $u_1\in\boldsymbol{R}[x]_1$, $u_3\in\boldsymbol{R}[x]_3$, and $f\in\boldsymbol{R}[x]_{od}$ such that f satisfies the conditions* (i) (ii) (iii) *before Lemma* 4.4 *and* (a) (b) (c) *above, and*

$$A^*f'\equiv u_1+u_3+cf \qquad \operatorname{mod}\left(1-\sum_{i=1}^{n+1}x_i\right).$$

PROOF. Let f'_{2i+1} $(2\leq i\leq 10)$ be the homogeneous part of degree $2i+1$ of f', and h'_{2i+1} the homogeneous part of degree $\deg_2 f'_{2i+1}$ of f'_{2i+1} in the variables $(x_2,\cdots,x_{n+1})$. Since h'_{13}/x_1^{11} is a real quadratic form, there is an orthogonal transformation A in the variables $(x_2,\cdots,x_{n+1})$ such that $A^*h'_{13}$ is of the form

$$\left(\sum_{i=1}^{k}\lambda_i x_i^2\right)x_1^{11},$$

where $\lambda_2\geq\lambda_3\geq\cdots\geq\lambda_k$, $\lambda_i\neq 0$ $(2\leq i\leq k)$, $2\leq k\leq n+1$. Since h'_{13} and h'_5 do not belong to the ideal $\left(\sum_{i=1}^{n+2}x_i^2\right)$, so do not $A^*h'_{13}$ and $A^*h'_5$. Let $c\in\boldsymbol{R}-\{0\}$ be the constant such that $f'_{21}=cx_1^{21}$. By substituting $1-\sum_{i=1}^{n}x_i^2$ for x_{n+1}^2 in A^*f', we can find $f''=\sum_{i=2}^{10}f''_{2i+1}$ $(f''_{2i+1}\in\boldsymbol{R}[x]_{2i+1})$, $u_1\in\boldsymbol{R}[x]_1$, $u_3\in\boldsymbol{R}[x]_3$ such that f'' satisfies the conditions (i) (ii) (iii) and (a) (b), and

$$A^*f'\equiv u_1+u_3+cf'' \qquad \operatorname{mod}\left(1-\sum_i x_i^2\right).$$

Moreover h''_{13}, the homogeneous part of degree 2 of f''_{13} in the variables $(x_2, \cdots, x_{n+1})$, is equal to $c^{-1}A^*h'_{13}$ if $k\leq n$, and equal to $c^{-1}\sum_{i=2}^{n}(\lambda_i-\lambda_{n+1})x_i^2x_1^{11}$ if $k=n+1$. Hence if $k<n$, or if $k\geq n$ and $\{\lambda_i\}$ do not satisfy $\lambda_2=\cdots=\lambda_n$, then f'' also satisfies the condition (c). If $k\geq n$ and $\lambda_2=\cdots=\lambda_n$, then h''_{13} is of the form

$$c'\left(\sum_{i=2}^{n}x_i^2\right)x_1^{11}, \qquad c'\in\boldsymbol{R}-\{0\}.$$

Let B be the orthogonal transformation such that $B^*x_2=-x_{n+1}$, $B^*x_{n+1}=x_2$, and the other variables are fixed by B. By substituting $1-\sum_{i=1}^{n}x_i^2$ for x_{n+1}^2 in B^*f'', we can find $f=\sum_{i=2}^{10}f_{2i+1}$ $(f_{2i+1}\in\boldsymbol{R}[x]_{2i+1})$, $u'_1\in\boldsymbol{R}[x]_1$, and $u'_3\in\boldsymbol{R}[x]_3$ such that f satisfies the conditions (i) (ii) (iii) and (a) (b), and

$$B^*f''\equiv u'_1+u'_3+f \qquad \mathrm{mod}\,(1-\textstyle\sum x_i^2).$$

Since h_{13}, the homogeneous part of degree 2 of f_{13} in the variables $(x_2, \cdots, x_{n+1})$, is of the form

$$dx_1^{11}x_2^2 \qquad (d\in\boldsymbol{R}-\{0\}),$$

f also satisfies the condition (c). This completes the proof of the lemma.

Now we fix $f\in\boldsymbol{R}[x]_{od}$ which satisfies the conditions (i) (ii) (iii) stated before Lemma 4.4 and (a) (b) (c) above, and satisfies

$$G_t^*F(f,f)\in G(\mathscr{K}^2).$$

Then f is of the form

$$f=(A_5x_1^5+B_5x_1^4+C_5x_1^3+D_5x_1^2+E_5x_1)$$
$$+\sum_{i=3}^{6}(A_{2i+1}x_1^{2i+1}+B_{2i+1}x_1^{2i}+C_{2i+1}x_1^{2i-1})+\sum_{i=7}^{10}A_{2i+1}x_1^{2i+1},$$

where $A_{2i+1}\in\boldsymbol{R}$ $(2\leq i\leq 10)$, $B_{2i+1}\in\boldsymbol{R}[x_2, \cdots, x_{n+1}]_1$, $C_{2i+1}\in\boldsymbol{R}[x_2, \cdots, x_{n+1}]_2$ $(2\leq i\leq 6)$, $D_5\in\boldsymbol{R}[x_2, \cdots, x_{n+1}]_3$, $E_5\in\boldsymbol{R}[x_2, \cdots, x_{n+1}]_4$, $A_{21}=1$, $E_5\neq 0$, $C_{13}=\sum_{i=2}^{k}\lambda_ix_i^2$, $\lambda_i\in\boldsymbol{R}-0$ $(2\leq i\leq k)$, $2\leq k\leq n$, and $\{\lambda_i\}$ do not satisfy $\lambda_2=\cdots=\lambda_n$ if $k=n$.

We shall prove the following

PROPOSITION 5.2. $B_{2i+1}\in\boldsymbol{R}[x_2]_1$, $C_{2i+1}\in\boldsymbol{R}[x_2]_2$ $(2\leq i\leq 6)$, $D_5\in\boldsymbol{R}[x_2]_3$, $E_5\in\boldsymbol{R}[x_2]_4$.

We need some lemmas to prove this proposition.

LEMMA 5.3. *Let J be the ideal* $\left(\sum_{i=1}^{n+1}x_1^2, \sum_{i=1}^{n+1}\zeta_i^2, \sum_{i=1}^{n+1}x_i\zeta_i\right)$ *in* $\boldsymbol{C}[x, \zeta]$, *and put* $J_0=J\cap\boldsymbol{C}[x_1, \cdots, x_n, \zeta_1, \cdots, \zeta_n]$. *Then*

$$J_0=\left(\sum_{i=1}^n x_i^2\sum_{i=1}^n \zeta_i^2-\left(\sum_{i=1}^n x_i\zeta_i\right)^2\right)\boldsymbol{C}[x_1, \cdots, x_n, \zeta_1, \cdots, \zeta_n]\,.$$

PROOF. First remark that the polynomial $\sum_{i=1}^n x_i^2\sum_{i=1}^n \zeta_i^2-\left(\sum_{i=1}^n x_i\zeta_i\right)^2$ is irreducible, provided $n\geq 3$. Let u be a polynomial in the ideal J_0. Fix a point $(p_1, \cdots, p_n, q_1, \cdots, q_n)\in \boldsymbol{C}^{2n}$ such that

$$\sum_{i=1}^n p_i^2\sum_{i=1}^n q_i^2-\left(\sum_{i=1}^n p_i q_i\right)^2=0\,.$$

Let p_{n+1} (resp. q_{n+1}) be one of the square roots of $-\sum_{i=1}^n p_i^2$ $\left((\text{resp. } -\sum_{i=1}^n q_i^2\right)$. Since

$$p_{n+1}^2 q_{n+1}^2=\left(\sum_{i=1}^n p_i q_i\right)^2,$$

we can take p_{n+1} and q_{n+1} such that $\sum_{i=1}^{n+1} p_i q_i=0$. Since $u\in J_0$, there are polynomials v_1, v_2, v_3 such that

$$u=\sum_{i=1}^{n+1} x_i^2 v_1+\sum_{i=1}^{n+1}\zeta_i^2 v_2+\sum_{i=1}^{n+1} x_i\zeta_i v_3\,.$$

By substituting $(p_1, \cdots, p_{n+1}, q_1, \cdots, q_{n+1})$ into both sides, we have

$$u(p_1, \cdots, p_n, q_1, \cdots, q_n)=0\,.$$

This implies that $u\in\left(\sum_{i=1}^n x_i^2\sum_{i=1}^n \zeta_i^2-\left(\sum_{i=1}^n x_i\zeta_i\right)^2\right)$. Hence

$$J_0\subset\left(\sum_{i=1}^n x_i^2\sum_{i=1}^n \zeta_i^2-\left(\sum_{i=1}^n x_i\zeta_i\right)^2\right).$$

On the other hand, since

$$\sum_{i=1}^n x_i^2\sum_{i=1}^n \zeta_i^2-\left(\sum_{i=1}^n x_i\zeta_i\right)^2 \equiv x_{n+1}^2\zeta_{n+1}^2-(x_{n+1}\zeta_{n+1})^2\equiv 0 \qquad \text{mod } J\,,$$

we also have

$$\left(\sum_{i=1}^n x_i^2\sum_{i=1}^n \zeta_i^2-\left(\sum_{i=1}^n x_i\zeta_i\right)^2\right)\subset J_0\,.$$

Therefore the lemma follows.

Let v, $w\in \boldsymbol{R}[x_2, \cdots, x_{n+1}]$ be homogeneous polynomials with $\deg v=a_1$ and $\deg w=a_2$. Put $a=a_1+a_2$ and let b and k be integers such that $a\leq b$, $a+b=2k$, $k\geq 4$. Let p be an integer such that $0\leq p\leq b$ and $p+a_2$ is even, and put

$$u=x_1^{b-p}\zeta_1^p v(x)\, w(\zeta)\,.$$

Then $\deg_1 u=b$, $\deg_2 u=a$, and

$$u\in \boldsymbol{R}[x,\zeta]_{b-p+a_1,p+a_2}\subset \boldsymbol{R}[x,\zeta]_{2k}\,.$$

In this case we have

LEMMA 5.4. *If a is even, then*

$$G_t^* u\equiv cG_t^*\left(x_1^{b-a}v(x)\, w(x)\,\zeta_1^a\right)\qquad \mathrm{mod}\ G(P^{a-2})$$

for some $c\in \boldsymbol{R}$. If a is odd, then

$$G_t^* u\equiv 0\qquad \mathrm{mod}\ G(P^{a-1})\,.$$

PROOF. Put $Y=\sum_{i=1}^{n+1} x_i\frac{\partial}{\partial\zeta_i}$. Then $u=\frac{1}{a_1!}x_1^{b-p}\zeta_1^p(Y^{a_1}v(\zeta))\, w(\zeta)$.

For integers t, r, s satisfying $1\le t\le b+1$, $0\le r\le a_1$, $0\le s\le a_2$, we have

$$\begin{aligned}&\tilde{X}_{E_0}\left(x_1^{b-t+1}\zeta_1^{t-1}Y^r v(\zeta)\, Y^s w(\zeta)\right)\\ &\quad=-(b-t+1)\,x_1^{b-t}\zeta_1^t Y^r v(\zeta)\, Y^s w(\zeta)-(t-1)\,x_1^{b-t+2}\zeta_1^{t-2}Y^r v(\zeta)\, Y^s w(\zeta)\\ &\qquad+r(a_1-r+1)\,x_1^{b-t+1}\zeta_1^{t-1}Y^{r-1}v(\zeta)\, Y^s w(\zeta)-x_1^{b-t+1}\zeta_1^{t-1}Y^{r+1}v(\zeta)\, Y^s w(\zeta)\\ &\qquad+s(a_2-s+1)\,x_1^{b-t+1}\zeta_1^{t-1}Y^r v(\zeta)\, Y^{s-1}w(\zeta)-x_1^{b-t+1}\zeta_1^{t-1}Y^r v(\zeta)\, Y^{s+1}w(\zeta)\,.\end{aligned}$$

Let $W_{2q,t}\left(0\le q\le \frac{p+a_2}{2},\ \max\{0,2q-a\}\le t\le \min\{2q,b\}\right)$ be the subspace of $C^\infty(S^*S^n)$ spanned by the functions

$$G_t^*\left(x_1^{b-t}\zeta_1^t Y^r v(\zeta)\, Y^s w(\zeta)\right)\qquad (0\le r\le a_1,\ 0\le s\le a_2,\ t+a-(r+s)=2q)\,.$$

Put $W_{2q}=\sum_t W_{2q,t}$, where the sum is taken over all possible t. Then

$$W_{2q}\subset G(P^{2q})\,.$$

In the case where $p+a_2<a$, we have $p+a_2\le a-2$ if a is even, and $p+a_2\le a-1$ if a is odd. Hence the lemma follows in this case by putting $c=0$. If $p+a_2\ge a+1$, then the above formula shows that

$$W_{2q,t}\subset W_{2q,t-1}+W_{2q-2}\,,\qquad W_{2q,2q-a}\subset W_{2q-2}$$

for each q and t with $[a/2]+1\le q\le (p+a_2)/2$, $2q-a+1\le t\le \min\{2q,b\}$. Hence in this case we have $W_{2q}\subset W_{2q-2}$, and consequently

$$W_{2q}\subset W_{2[a/2]}$$

for each q with $[a/2]+1\le q\le (p+a_2)/2$.

Assume that a is odd and $p+a_2\geq a$. Then we have $p+a_2\geq a+1$. Hence it follows that

$$W_{p+a_2}\subset W_{2[a/2]}=W_{a-1},$$

which implies

$$G_t^* u\equiv 0 \qquad \mathrm{mod}\; G(P^{a-1}).$$

Next assume that a is even and $p+a_2\geq a$. In this case we see from the above consideration that

$$W_{p+a_2}\subset W_a.$$

Therefore, in order to show the lemma it is enough to prove that

$$W_a\subset W_{a,a}+W_{a-2}.$$

By considering the above formula in the case where $1\leq t\leq a$, $0\leq r\leq a_1$, $0\leq s\leq a_2$, and $t=r+s$, we have

$$W_{a,t}\subset W_{a,t-1}+W_{a-2}$$

and hence

$$W_a\subset W_{a,0}+W_{a-2}.$$

Remark that $W_{a,0}$ is generated by $G_t^*(x_1{}^b v(\zeta)\, w(\zeta))$ and $W_{a,a}$ is generated by $G_t^*(x_1{}^{b-a}\zeta_1{}^a v(x)\, w(x))$. We then consider the following formula;

$$\begin{aligned}&\tilde{X}_{E_0}\left(x_1{}^{b-s+1}\zeta_1{}^{s-1}Y^s\left(v(\zeta)\, w(\zeta)\right)\right)\\ &\quad=(b-s+1)\, x_1{}^{b-s}\zeta_1{}^s Y^s\left(v(\zeta)\, w(\zeta)\right)-(s-1)\, x_1{}^{b-s+2}\zeta_1{}^{s-2}Y^s\left(v(\zeta)\, w(\zeta)\right)\\ &\quad\quad+s(a-s+1)\, x_1{}^{b-s+1}\zeta_1{}^{s-1}Y^{s-1}\left(v(\zeta)\, w(\zeta)\right)-x_1{}^{b-s+1}\zeta_1{}^{s+1}Y^{s+1}\left(v(\zeta)\, w(\zeta)\right),\\ &\quad\quad(1\leq s\leq a).\end{aligned}$$

This formula shows that there is a non-zero constant $c'\in \boldsymbol{R}$ such that

$$x_1{}^b v(\zeta)\, w(\zeta)\equiv c'\, x_1{}^{b-a}\zeta_1{}^a v(x)\, w(x) \qquad \mathrm{mod}\; W_{a-2}.$$

Hence we have

$$W_a\subset W_{a,a}+W_{a-2},$$

which proves the lemma.

COROLLARY 5.5. *Let* $u\in \boldsymbol{R}[x, \zeta]_{2k}$ $(k\geq 4)$.

(i) *If* $\deg_2 u\leq 3$, *then* $G_t^* u\in G(\mathscr{H}^2)$.

(ii) *If* $\deg_2 u=4$, *then there is a homogeneous polynomial* $v\in \boldsymbol{R}[x_2, \cdots, x_{n+1}]_4$ *such that*

$$G\iota^* u \equiv G\iota^*\left(x_1^{2k-8} v(x)\zeta_1^4\right) \qquad \bmod G(\mathscr{A}^2),$$

and such that the degree of v in the variable x_{n+1} is not greater than that of u. Especially $(G\iota^* u)_{Q_{2k-4,4}}=0$ if and only if $G\iota^* u \in G(\mathscr{A}^2)$.

PROOF. (i) and the former part of (ii) are immediate consequences of the previous lemma. For the latter part of (ii) we put

$$w = x_1^{2k-8} v(x)\,\zeta_1^4 + 12a_1^{k-3} x_1^{2k-6} v(x)\,\zeta_1^2 + 24a_2^{k-3} x_1^{2k-4} v(x).$$

Then we have

$$\begin{aligned}(G\iota^* u)_{Q_{2k-4,4}} &= \left(G\iota^*(x_1^{2k-8} v(x)\,\zeta_1^4)\right)_{Q_{2k-4,4}} \\ &= (G\iota^* w)_{G(Q_{2k})}\end{aligned}$$

by Corollary 2.8 (ii). Hence we see by Corollary 3.7 that $(G\iota^* u)_{Q_{2k-4,4}}=0$ if and only if $\tilde{G}w \in (\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i\zeta_i)$. Now assume that $\tilde{G}w$ belongs to the ideal $(\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i\zeta_i)$. Since $\tilde{G}w$ is homogeneous in the variables $(x_2, \cdots, x_{n+1}, \zeta_2, \cdots, \zeta_{n+1})$, it is easily seen by Lemma 4.8 that

$$\tilde{G}w \in (\sum_{i\geq 2} x_i^2, \sum_{i\geq 2} \zeta_i^2, \sum_{i\geq 2} x_i\zeta_i).$$

Hence for any $\nu=(\nu_2, \cdots, \nu_{n+1}) \in \boldsymbol{C}^n$ satisfying $\sum_{i=2}^{n+1} \nu_i^2=0$ we have

$$\tilde{G}w^\nu = 0.$$

Since $w^\nu = v(\nu)\,(x_1^{2k-8} x_2^4 \zeta_1^4 + 12a_2^{k-3} x_1^{2k-6} x_2^4 \zeta_1^2 + 24a_2^{k-3} x_1^{2k-4} x_2^4)$, it thus follows that

$$0 = (G\iota^* w^\nu)_{G(Q_{2k})} = v(\nu)\left(G\iota^*(x_1^{2k-8} x_2^4 \zeta_1^4)\right)_{Q_{2k-4,4}}.$$

Since $(G\iota^*(x_1^{2k-8} x_2^4 \zeta_1^4))_{Q_{2k-4,4}} \neq 0$ by Lemma 3.9, we have $v(\nu)=0$. This implies that $v(x)$ belongs to the ideal $(\sum_{i\geq 2} x_i^2)$. Put

$$v = \sum_{i\geq 2} x_i^2 v', \qquad v' \in \boldsymbol{R}[x_2, \cdots, x_{n+1}]_2.$$

Then $G\iota^*(x_1^{2k-8} v(x)\,\zeta_1^4) = G\iota^*(x_1^{2k-8} v'(x)\,\zeta_1^4) - G\iota^*(x_1^{2k-6} v'(x)\,\zeta_1^4)$. Since $\deg v' = 2$ unless $v'=0$, we see from (i) that the right-hand side of this formula is in $G(\mathscr{A}^2)$. This proves the corollary.

LEMMA 5.6. *Let $w \in \boldsymbol{R}[x, \zeta]_{2k-2p,2p}$ $(0 \leq p \leq [k/2])$. Suppose that w is also homogeneous in the variables $(x_2, \cdots, x_{n+1}, \zeta_2, \cdots, \zeta_{n+1})$ with $\deg_2 w = 2q$ $(p \leq q \leq [k/2])$ and that $(G\iota^* w)_{Q_{2k-2p,2p}}=0$. Then*

$$(G\iota^* w^\nu)_{Q_{2k-2p,2p}} = 0$$

for all $\nu=(\nu_2, \cdots, \nu_{n+1})\in \boldsymbol{C}^n$ *such that* $\sum_{i=2}^{n+1} \nu_i^2=0$.

PROOF. Corollary 2. 8 (ii) shows that

$$(G_t^* w)_{Q_{2k-2p,2p}} = \left(G_t^* \sum_{l=0}^{p} a_l^{k-2p+1}\left(\sum_j x_j \frac{\partial}{\partial \zeta_j}\right)^{2l} w\right)_{G(Q_{2k})}.$$

Then the assumption implies that the polynomial

$$\tilde{G} \sum_{l=0}^{p} a_l^{k-2p+1}\left(\sum_j x_j \frac{\partial}{\partial \zeta_j}\right)^{2l} w$$

belongs to the ideal $(\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i\zeta_i)$. Since w is homogeneous in the variables $(x_2, \cdots, x_{n+1}, \zeta_2, \cdots, \zeta_{n+1})$, we see by Lemma 4. 8 that this polynomial also belongs to the ideal $(\sum_{i\geq 2} x_i^2, \sum_{i\geq 2} \zeta_i^2, \sum_{i\geq 2} x_i\zeta_i)$. Hence it follows that

$$\tilde{G} \sum_{l=0}^{p} a_l^{k-2p+1}\left(\sum_{j=1}^{2} x_j \frac{\partial}{\partial \zeta_j}\right)^{2l} w^\nu = 0$$

for all $\nu=(\nu_2, \cdots, \nu_{n+1})\in \boldsymbol{C}^n$ with $\sum_{i\geq 2} \nu_i^2=0$. This shows that

$$(G_t^* w^\nu)_{Q_{2k-2p,2p}} = 0 ,$$

which proves the lemma.

In general, for homogeneous polynomials $u_1, u_2\in \boldsymbol{R}[x]$ we set

$$F^1(u_1, u_2) = -\sum_{i,j}(x_i x_j+\zeta_i\zeta_j)\frac{\partial u_1}{\partial x_i}\int_0^\pi \frac{\partial u_2}{\partial x_j}(x\cos t+\zeta \sin t)\sin t\, dt ,$$

$$F^2(u_1, u_2) = \sum_i \frac{\partial u_1}{\partial x_i}\int_0^\pi \frac{\partial u_2}{\partial x_i}(x\cos t+\zeta \sin t)\sin t\, dt .$$

Then $F(u_1, u_2)=F^1(u_1, u_2)+F^2(u_1, u_2)$. Furthermore we set

$$F^3(u_1, u_2) = \frac{\partial u_1}{\partial x_1}\int_0^\pi \frac{\partial u_2}{\partial x_1}(x\cos t+\zeta \sin t)\sin t\, dt ,$$

$$F^4(u_1, u_2) = \sum_{i\geq 2} \frac{\partial u_1}{\partial x_i}\int_0^\pi \frac{\partial u_2}{\partial x_i}(x\cos t+\zeta \sin t)\sin t\, dt .$$

Then $F^2(u_1, u_2)=F^3(u_1, u_2)+F^4(u_1, u_2)$. It is easily seen from the proof of Proposition 1. 7 that $\tilde{G}F^i(u_2, u_1)=\tilde{G}F^i(u_1, u_2)$ for each i $(1\leq i\leq 4)$.

PROOF OF PROPOSITION 5. 2. Put

$$F(f,f) = \sum_{i=4}^{21} R_{2i} , \qquad R_{2i}\in \boldsymbol{R}[x, \zeta]_{2i} .$$

Then

$$R_{2i}=\sum_{l+m=i-1}F^1(f_{2l+1},f_{2m+1})+\sum_{l+m=i}F^2(f_{2l+1},f_{2m+1}).$$

We first see that $\deg_2 R_{2i}\leq 2$ when $14\leq i\leq 21$. Thus for such i we have

$$G\iota^* R_{2i}\in G(\mathscr{H}^2)$$

by Corollary 5. 5 (i).

Next we shall prove that $G\iota^* R_{2i}\in G(\mathscr{H}^2)$ $(10\leq i\leq 13)$ and that E_5 is a constant multiple of C_{13}^2. The above fact shows that

$$0=\left(G\iota^* F(f,f)\right)_{Q_{22,4}}=\sum_{i=13}^{21}(G\iota^* R_{2i})_{Q_{22,4}}=(G\iota^* R_{26})_{Q_{22,4}}.$$

In view of Corollary 5. 5 (i),

$$G\iota^* R_{26}\equiv 2G\iota^* F^1(x_1^{21}, E_5x_1)+G\iota^* F^1(C_{13}x_1^{11}, C_{13}x_1^{11}) \qquad \mathrm{mod}\ G(\mathscr{H}^2).$$

Then it follows from Corollary 5. 5 (ii) that $G\iota^* R_{26}\in G(\mathscr{H}^2)$ and

$$46d_0^{10,2}\left(G\iota^* F_0(x_1^{21}, E_5x_1)\right)_{Q_{22,4}}+15d_4^{6,6}\left(G\iota^* F_4(C_{13}x_1^{11}, C_{13}x_{11})\right)_{Q_{22,4}}=0.$$

This implies that

$$\begin{aligned}&46d_0^{10,2}E_5(\nu)\, G\iota^* F_0(x_1^{11}, x_1x_2^4)\big)_{Q_{22,4}}\\&\quad+15d_4^{6,6}C_{13}(\nu)^2\left(G\iota^* F_4(x_1^{11}x_2^2, x_1^{11}x_2^2)\right)_{Q_{22,4}}=0\end{aligned}$$

for all $\nu=(\nu_2,\cdots,\nu_{n+1})\in \boldsymbol{C}^n$ satisfying $\sum_{i=2}^{n+1}\nu_i^2=0$. We shall see later in Lemma 5. 10 that there are non-zero constants c and c' such that

$$\left(G\iota^* F_0(x_1^{21}, x_1x_2^4)\right)_{Q_{22,4}}=c\left(G\iota^*(x_1^{22}\zeta_2^4)\right)_{Q_{22,4}},$$

$$\left(G\iota^* F_4(x_1^{11}x_2^2, x_1^{11}x_2^2)\right)_{Q_{22,4}}=c'\left(G\iota^*(x_1^{22}\zeta_2^4)\right)_{Q_{22,4}}.$$

Since $(G\iota^*(x_1^{22}\zeta_2^4))_{Q_{22,4}}$ does not vanish by Lemma 3. 9, it follows that

$$46cd_0^{10,2}E_5+15c'\,d_4^{6,6}C_{13}^2\equiv 0 \qquad \mathrm{mod}\ (\sum_{i\geq 2}x_i^2).$$

Since C_{13} does not contain the variable x_{n+1}, and the degree of E_5 in x_{n+1} is at most 1, we thus obtain

$$46cd_0^{10,2}E_5+15c'\,d_4^{6,6}C_{13}^2=0.$$

Take an integer i_0 such that $10<i_0+1\leq 13$ and fix it. Assume that $G\iota^* R_{2i}\in G(\mathscr{H}^2)$ for all i satisfying $i_0+1\leq i\leq 13$. Then

$$\begin{aligned}0&=\left(G\iota^* F(f,f)\right)_{Q2i_0-4,4}\\&=\sum_{i=i_0}^{21}(G\iota^* R_{2i})_{Q2i_0-4,4}=(G\iota^* R_{2i_0})_{Q2i_0-4,4}.\end{aligned}$$

Remarking that $2i_0\geq 20$, we see that $\deg_2 R_{2i_0}\leq 4$. Thus it follows that $G\iota^* R_{2i_0}\in G(\mathscr{H}^2)$ from Corollary 5. 5 (ii). Therefore we have

$$G\iota^* R_{2i}\in G(\mathscr{H}^2) \qquad (10\leq i\leq 13)$$

by induction on i.

Next we shall prove the following:

(i) There are $v_i\in \boldsymbol{R}[x_2, \cdots, x_{n+1}]_4$ such that

$$G\iota^* F(f,f)\equiv \sum_{j=4}^{i-1} G\iota^* R_{2j}+G\iota^*\left(x_1^{2i-10} v_i(x)\, \zeta_1^4\right) \qquad \mathrm{mod}\ G(\mathscr{H}^2)\ (6\leq i\leq 9)\,;$$

(ii) D_5 is a constant multiple of $B_{13}C_{13}$;

(iii) C_{2i-5} and B_{2i-5} are constant multiples of C_{13} and B_{13} respectively $(6\leq i\leq 9)$.

In view of Corollary 5. 5 we see that

$$\begin{aligned} G\iota^* R_{18} \equiv\ & 2G\iota^* F^1(C_{13}x_1^{11}, E_5x_1)+2G\iota^* F^1(C_{13}x_1^{11}, D_5x_1^2) \\ & +2G\iota^* F^1(B_{13}x_1^{12}, E_5x_1)+G\iota^*\left((x_1^{10}w_9(x)\, \zeta_1^4\right) \qquad \mathrm{mod}\ G(\mathscr{H}^2) \end{aligned}$$

for some $w_9\in \boldsymbol{R}[x_2, \cdots, x_{n+1}]_4$. Thus it follows that

$$\begin{aligned} G\iota^* R_{18} \equiv\ & -30d_0^{6,2}G\iota^*\Big\{F_0(C_{13}x_1^{11}, E_5x_1)+F_0(C_{13}x_1^{11}, D_5x_1^2) \\ & +F_0(B_{13}x_1^{12}, E_5x_1)\Big\}+G\iota^*\left(x_1^{10}w_9(x)\, \zeta_1^4\right) \qquad \mathrm{mod}\ G(\mathscr{H}^2)\,. \end{aligned}$$

Put

$$X=-30d_0^{6,2}\tilde{G}\left(\sum_{i=0}^{2} a_i^6 F_i(C_{13}x_1^{11}, E_5x_1)\right),$$

$$Y=-30d_0^{6,2}\tilde{G}\left(\sum_{i=0}^{2} a_i^6 F_i(C_{13}x_1^{11}, D_5x_1^2)+\sum_{i=0}^{2} a_i^6 F_i(B_{13}x_1^{12}, E_5x_1)\right)$$

$$Z=\tilde{G}\left(x_1^{10}w_9(x)\, \zeta_1^4+12a_1^6x_1^{12}w_9(x)\, \zeta_1^2+24a_2^6x_1^{14}w_9(x)\right).$$

Then

$$G\iota^* R_{18}\equiv \iota^*(X+Y+Z) \qquad \mathrm{mod}\ G(\mathscr{H}^2)\,,$$

and

$$(G\iota^* R_{18})_{Q_{14,4}}=\left(\iota^*(X+Y+Z)\right)_{G(Q_{18})}.$$

Since $0=(G\iota^* F(f,f))_{Q_{14,4}}=(G\iota^* R_{18})_{Q_{14,4}}$, it follows that

$$X+Y+Z\in (\sum_i x_i^2,\ \sum_i \zeta_i^2,\ \sum_i x_i\zeta_i)\,.$$

Remarking the degrees in the variables $(x_2, \cdots, x_{n+1}, \zeta_2, \cdots, \zeta_{n+1})$, we see that

$X+Z$ and Y belong to the ideal $(\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i\zeta_i)$. Then by Lemma 4.8 there are polynomials H_i, $H_i'\in \boldsymbol{R}[x,\zeta]_{16}$ $(i=1,2,3)$ such that $\deg_2 H_i\leq 4$, $\deg_2 H_i'\leq 3$ and

$$X+Z=\sum_i x_i^2 H_1+\sum_i \zeta_i^2 H_2+\sum_i x_i\zeta_i H_3,$$

$$Y=\sum_i x_i^2 H_i'+\sum_i \zeta_i^2 H_i'+\sum_i x_i\zeta_i H_3'.$$

Then $\iota^*(X+Z)=\iota^*(H_1+H_2)$ and $\iota^* Y=\iota^*(H_1'+H_2')$. Thus we see by Corollary 5.5 that $\iota^* Y\in G(\mathscr{A}^2)$ and

$$\iota^*(X+Z)\equiv G\iota^*\left(x_1^8 v_9(x)\,\zeta_4^1\right) \qquad \mathrm{mod}\ G(\mathscr{A}^2)$$

for some $v_9\in \boldsymbol{R}[x_2,\cdots,x_{n+1}]_4$. Hence it follows that

$$G^* R_{18}\equiv G\iota^*\left(x_1^8 v_9(x)\,\zeta_1^4\right) \qquad \mathrm{mod}\ G(\mathscr{A}^2)$$

and that

$$G\iota^* F(f,f)\equiv \sum_{i=4}^{9} G\iota^* R_{2i}$$

$$\equiv \sum_{i=4}^{8} G\iota^* R_{2i}+G\iota^*\left(x_1^8 v_9(x)\,\zeta_1^4\right) \qquad \mathrm{mod}\ G(\mathscr{A}^2).$$

Now consider the formula

$$Y=\sum_i x_i^2 H_1'+\sum_i \zeta_i^2 H_2'+\sum_i x_i\zeta_i H_3'.$$

By taking the homogeneous part of degree 5 in the variables $(x_2,\cdots,x_{n+1}, \zeta_2,\cdots,\zeta_{n+1})$, we see that Y belong to the ideal $(\sum_{i\geq 2} x_i^2, \sum_{i\geq 2}\zeta_i^2, \sum_{i\geq 2} x_i\zeta_i)$. This implies that

$$C_{13}(\nu)\,D_5(\nu)\left(G\iota^* F_0(x_1^{11}x_2^2,\ x_1^2x_2^3)\right)_{Q_{14,4}}$$

$$+B_{13}(\nu)\,E_5(\nu)\left(G\iota^* F_0(x_1^{12}x_2,\ x_1x_2^4)\right)_{Q_{14,4}}=0$$

for all $\nu=(\nu_2,\cdots,\nu_{n+1})\in \boldsymbol{C}^n$ satisfying $\sum_{i=2}^{n+1}\nu_i^2=0$. Lemma 5.10 stated later shows that

$$\left(G\iota^* F_0(x_1^{11}x_2^2,\ x_1^2x_2^3)\right)_{Q_{14,4}}=c\left(G\iota^*(x_1^{13}x_2\zeta_2^4)\right)_{Q_{14,4}},$$

$$\left(G\iota^* F_0(x_1^{12}x_2,\ x_1x_2^4)\right)_{Q_{14,4}}=c'\left(G\iota^*(x_1^{13}x_2\zeta_2^4)\right)_{Q_{14,4}}$$

for some non-zero constants c and c'. Since $(G\iota^*(x_1^{13}x_2\zeta_2^4))_{Q_{14,4}}$, does not vanish by Lemma 3.9, it follows that

$$cC_{13}D_5+c'B_{13}E_5$$

belongs to the ideal $(\sum_{i\geq 2}x_i^2)$. But the degree of this polynomial in the variable x_{n+1} being at most 1, it must be zero. Since E_5 is a constant multiple of C_{13}^2, it therefore follows that D_5 is a constant multiple of $B_{13}C_{13}$.

Fix an integer m $(6\leq m\leq 8)$ and assume that

$$G\iota^* F(f,f)\equiv\sum_{i=4}^{m}G\iota^* R_{2i}+G\iota^*\left(x_1^{2m-8}v_{m+1}(x)\,\zeta_1^4\right)\qquad \mathrm{mod}\ G(\mathscr{A}^2)$$

for some $v_{m+1}\in\boldsymbol{R}[x_2,\cdots,x_{n+1}]_4$, and that C_{2i-5} and B_{2i-5} are constant multiples of C_{13} and B_{13} respectively for all i satisfying $m+1\leq i\leq 9$. By Corollary 5. 5 we see that

$$\begin{aligned}G\iota^* R_{2m}\equiv\ &2G\iota^* F^1(C_{2m-5}x_1^{2m-7},E_5x_1)+2G\iota^* F^1(C_{2m-5}x_1^{2m-7},D_5x_1^2)\\&+2G\iota^* F^1(B_{2m-5}x_1^{2m-6},E_5x_1)+2G\iota^* F^3(C_{2m-3}x_1^{2m-5},E_5x_1)\\&+2G\iota^* F^3(C_{2m-3}x_1^{2m-5},D_5x_1^2)+2G\iota^* F^3(B_{2m-3}x_1^{2m-4},E_5x_1)\\&+G\iota^*\left(x_1^{2m-8}w_m(x)\,\zeta_1^4\right)\qquad \mathrm{mod}\ G(\mathscr{A}^2)\end{aligned}$$

for some $w_m\in\boldsymbol{R}[x_2,\cdots,x_{n+1}]_4$. We have

$$G\iota^* F^3(u_1,u_2)\equiv\int_0^\pi(\sin t)^5dt\,G\iota^*\left(\frac{\partial u_1}{\partial x_1}\,\frac{\partial u_2(\zeta)}{\partial\zeta_1}\right)\qquad \mathrm{mod}\ G(\mathscr{A}^2)$$

for $u_1\in\boldsymbol{R}[x]_{2m-3}$ and $u_2\in\boldsymbol{R}[x]_5$, and $d_0^{m-3,2}=\int_0^\pi(\sin t)^5dt$. Put

$$\begin{aligned}X'=&-2(2m-3)\,d_0^{m-3,2}\tilde{G}\left(\sum_{i=0}^{2}a_i^{m-3}F_i(C_{2m-5}x_1^{2m-7},E_5x_1)\right)\\&+2(2m-5)\,d_0^{m-3,2}\tilde{G}\left(\sum_{i=0}^{2}a_i^{m-3}\left(\sum_j x_j\frac{\partial}{\partial\zeta_j}\right)^{2i}x_1^{2m-6}C_{2m-3}(x)\,E_5(\zeta)\right),\end{aligned}$$

$$\begin{aligned}Y'=&-2(2m-3)\,d_0^{m-3,2}\tilde{G}\left(\sum_{i=0}^{2}a_i^{m-3}F_i(C_{2m-5}x_1^{2m-7},D_5x_1^2)\right)\\&-2(2m-3)\,d_0^{m-3,2}\tilde{G}\left(\sum_{i=0}^{2}a_i^{m-3}F_i(B_{2m-5}x_1^{2m-6},E_5x_1)\right)\\&+4(2m-5)\,d_0^{m-3,2}\tilde{G}\left(\sum_{i=0}^{2}a_i^{m-3}\left(\sum_j x_j\frac{\partial}{\partial\zeta_j}\right)^{2i}x_1^{2m-6}C_{2m-3}(x)\,\zeta_1D_5(\zeta)\right)\\&+2(2m-4)\,d_0^{m-3,2}\tilde{G}\left(\sum_{i=0}^{2}a_i^{m-3}\left(\sum_j x_j\frac{\partial}{\partial\zeta_j}\right)^{2i}x_1^{2m-5}B_{2m-3}(x)\,E_5(\zeta)\right),\end{aligned}$$

$$Z=\tilde{G}\left(\sum_{i=0}^{2}a_i^{m-3}\left(\sum_j x_j\frac{\partial}{\partial\zeta_j}\right)^{2i}x_1^{2m-8}w_m(x)\,\zeta_1^4\right),$$

$$W'=\tilde{G}\left(\sum_{i=0}^{2}a_i^{m-3}\left(\sum_j x_j\frac{\partial}{\partial\zeta_j}\right)^{2i}x_1^{2m-8}v_{m+1}(x)\,\zeta_1^4\right).$$

Then $G\iota^* R_{2m} \equiv \iota^*(X'+Y'+Z') \mod G(\mathscr{A}^2)$. Moreover,

$$0=\left(G\iota^* F(f,f)\right)_{Q_{2m-4,4}}=\left(G\iota^* R_{2m}+G\iota^*\left(x_1^{2m-8} v_{m+1}(x)\,\zeta_1^4\right)\right)_{Q_{2m-4,4}}$$
$$=\left(\iota^*(X'+Y'+Z'+W')\right)_{G(Q_{2m})}.$$

Remarking the degrees in the variables $(x_2, \cdots, x_{n+1}, \zeta_2, \cdots, \zeta_{n+1})$, we see that $X'+Z'+W'$ and Y' belong to the ideal $(\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i \zeta_i)$. Hence as before, we have $\iota^* Y' \in G(\mathscr{A}^2)$ and

$$\iota^*(X'+Z'+W') \equiv G\iota^*\left(x_1^{2m-6} v_m(x)\,\zeta_1^4\right) \qquad \mod G(\mathscr{A}^2)$$

for some $v_m \in \boldsymbol{R}[x_2, \cdots, x_{n+1}]_4$. From these it follows that

$$G\iota^* F(f,f) \equiv \sum_{i=4}^{m-1} G\iota^* R_{2i} + G\iota^*\left(x_1^{2m-10} v_m(x)\,\zeta_1^4\right) \qquad \mod G(\mathscr{A}^2).$$

Moreover we see that X' and Y' belong to the ideal $(\sum_{i\geq 2} x_i^2, \sum_{i\geq 2} \zeta_i^2, \sum_{i\geq 2} x_i \zeta_i)$. This implies that

$$-(2m-3)\,C_{2m-5}(\nu)\,E_5(\nu)\left(G\iota^* F_0(x_1^{2m-7}x_2^2, x_1 x_2^4)\right)_{Q_{2m-4,4}}$$
$$+(2m-5)\,C_{2m-3}(\nu)\,E_5(\nu)\left(G\iota^*(x_1^{2m-6}x_2^2\zeta_2^4)\right)_{Q_{2m-4,4}}=0,$$
$$-(2m-3)\,C_{2m-5}(\nu)\,D_5(\nu)\left(G\iota^* F_0(x_1^{2m-7}x_2^2, x_1^2 x_2^3)\right)_{Q_{2m-4,4}}$$
$$-(2m-3)\,B_{2m-5}(\nu)\,E_5(\nu)\left(G\iota^* F_0(x_1^{2m-6}x_2, x_1 x_2^4)\right)_{Q_{2m-4,4}}$$
$$+2(2m-5)\,C_{2m-3}(\nu)\,D_5(\nu)\left(G\iota^*(x_1^{2m-6}x_2^2\zeta_1\zeta_2^3)\right)_{Q_{2m-4,4}}$$
$$+(2m-4)\,B_{2m-3}(\nu)\,E_5(\nu)\left(G\iota^*(x_1^{2m-5}x_2\zeta_2^4)\right)_{Q_{2m-4,4}}=0$$

for all $\nu=(\nu_2, \cdots, \nu_{n+1}) \in \boldsymbol{C}^n$ with $\sum_{i=1}^{n+1} \nu_i^2=0$. By Lemma 5.10 stated later, $(G\iota^* F_0(x_1^{2m-7}x_2^2, x_1^2 x_2^3))_{Q_{2m-4,4}}$ is a non-zero constant multiple of

$$\left(G\iota^*(x_1^{2m-6}x_2^2\zeta_2^4)\right)_{Q_{2m-4,4}}, \quad \text{and} \quad \left(G\iota^* F_0(x_1^{2m-7}x_2^2, x_1^2 x_2^3)\right)_{Q_{2m-4,4}},$$
$$\left(G\iota^* F_0(x_1^{2m-6}x_2, x_1 x_2^4)\right)_{Q_{2m-4,4}}, \quad \text{and} \quad \left(G\iota^*(x_1^{2m-6}x_2^2\zeta_1\zeta_2^3)\right)_{Q_{2m-4,4}}$$

are non-zero constant multiples of $(G\iota^*(x_1^{2m-5}x_2\zeta_2^4))_{Q_{2m-4,4}}$. Hence there are constants c, c' such that

$$C_{2m-5}E_5 \equiv cC_{13}E_5, \qquad B_{2m-5}C_{13}^2 \equiv c' B_{13}C_{13}^2 \qquad \mod (\sum_{i\geq 2} x_i^2).$$

Since the degrees of both sides of the above congruences in the variable x_{n+1} are at most 1, it follows that

$$C_{2m-5}=cC_{13}, \qquad B_{2m-5}=c'B_{13}.$$

Thus the assertions (i), (ii), and (iii) have been shown by induction.

Next we shall show that B_{13} and B_5 (resp. C_{13} and C_5) are constant multiples of x_2 (resp. x_2^2), from which Proposition 5. 2 will follows. We shall consider two cases according as B_{13} is zero or not.

I. $B_{13}\neq 0$. We have already seen that the polynomial

$$\tilde{G}\Big(\sum_{i=0}^{2} a_i^6 F_i(C_{13}x_1^{11}, D_5x_1^2)\Big)+\tilde{G}\Big(\sum_{i=0}^{2} a_i^6 F_i B_{13}x_1^{12}, E_5x_1)\Big)$$

belongs to the ideal $(\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i\zeta_i)$. Let $(p_2, \cdots, p_{n+1})$ and $(q_2, \cdots, q_{n+1})$ be non-zero vectors in $\boldsymbol{C}^n$ such that

$$\sum_{i=2}^{n+1} p_i^2=\sum_{i=2}^{n+1} p_iq_i=0, \qquad 1+\sum_{i=1}^{n+1} q_i^2=0.$$

Then the subspace of $\boldsymbol{C}^{n+1}$ spanned by $(0, p_2, \cdots, p_{n+1})$ and $(1, q_2, \cdots, q_{n+1})$ is contained in $\tilde{S}$. Let

$$\kappa:\ \boldsymbol{C}^4=\Big\{(x_1, x_2, \zeta_1, \zeta_2)\Big\}\longrightarrow \boldsymbol{C}^{2n+2}=\Big\{(x, \zeta)\Big\}$$

be the linear map defined by

$$\kappa^*x_1=x_1,\ \kappa^*\zeta_1=\zeta_1,\ \kappa^*x_j=q_jx_1+p_jx_2,\ \kappa^*\zeta_j=q_j\zeta_1+p_j\zeta_2 \qquad (j\geq 2).$$

Then as in § 3 we have

$$\tilde{G}\Big(\sum_{j=0}^{2} a_j^6 F_j(\kappa^*C_{13}x_1^{11}, \kappa^*D_5x_1^2)\Big)+\tilde{G}\Big(\sum_{j=0}^{2} a_j^6 F_j(\kappa^*B_{13}x_1^{12}, \kappa^*E_5x_1)\Big)=0,$$

which implies that

$$\Big(G\iota^*F_0(\kappa^*C_{13}x_1^{11}, \kappa^*D_5x_1^2)\Big)_{Q_{14,4}}+\Big(G\iota^*F_0(\kappa^*B_{13}x_1^{12}, \kappa^*E_5x_1)\Big)_{Q_{14,4}}=0.$$

Put

$$B_{13}=\sum_{i=2}^{n+1} b_ix_i, \qquad E_5=dC_{13}^2, \qquad D_5=d'B_{13}C_{13} \qquad (b_i, d, d'\in\boldsymbol{R}).$$

Then

$$\kappa^*C_{13}=\Big(\sum_{j=2}^{k}\lambda_jq_j^2\Big)x_1^2+2\Big(\sum_{j=2}^{k}\lambda_jp_jq_j\Big)x_1x_2+\Big(\sum_{j=2}^{k}\lambda_jp_j^2\Big)x_2^2,$$

$$\kappa^*B_{13}=\Big(\sum_{j=2}^{n+1} b_jq_j\Big)x_1+\Big(\sum_{j=2}^{n+1} b_jp_j\Big)x_2,$$

and we have

$$\Big(G\iota^*F_0(\kappa^*C_{13}x_1^{11}, \kappa^*D_5x_1^2)\Big)_{Q_{14,4}}$$

$$= d'\Big\{(\sum \lambda_j p_j^2)^2 (\sum b_j p_j) \Big(G\iota^* F_0(x_1^{11} x_2^2, x_1^2 x_2^3)\Big)_{Q_{14,4}}$$
$$+2(\sum \lambda_j p_j q_j)(\sum \lambda_j p_j^2)(\sum b_j p_j)\Big(G\iota^* F_0(x_1^{12} x_2, x_1^2 x_2^3)\Big)_{Q_{14,4}}$$
$$+2(\sum \lambda_j p_j q_j)(\sum \lambda_j p_j^2)(\sum b_j p_j)\Big(G\iota^* F_0(x_1^{11} x_2^2, x_1^3 x_2^2)\Big)_{Q_{14,4}}$$
$$+(\sum \lambda_j p_j^2)^2 (\sum b_j q_j)\Big(G\iota^* F_0(x_1^{11} x_2^2, x_1^3 x_2^2)\Big)_{Q_{14,4}}\Big\},$$

$$\Big(G\iota^* F_0(\kappa^* B_{13} x_1^{12}, \kappa^* E_5 x_1)\Big)_{Q_{14,4}}$$
$$= d\Big\{(\sum \lambda_j p_j^2)^2 (\sum b_j p_j)\Big(G\iota^* F_0(x_1^{12} x_2, x_1 x_2^4)\Big)_{Q_{14,4}}$$
$$+(\sum \lambda_j p_j^2)^2 (\sum b_j q_j)\Big(G\iota^* F_0(x_1^{13}, x_1 x_2^4)\Big)_{Q_{14,4}}$$
$$+4(\sum \lambda_j p_j^2)(\sum \lambda_j p_j q_j)(\sum b_j p_j)\Big(G\iota^* F_0(x_1^{12} x_2, x_1^2 x_2^3)\Big)_{Q_{14,4}}\Big\}.$$

By Lemma 5.10 stated later,

$$\Big(G\iota^* F_0(x_1^{11} x_2^2, x_1^2 x_2^3)\Big)_{Q_{14,4}} = -\frac{5}{26}\Big(G\iota^*(x_1^{13} x_2 \zeta_2^4)\Big)_{Q_{14,4}},$$
$$\Big(G\iota^* F_0(x_1^{12} x_2, x_1 x_2^4)\Big)_{Q_{14,4}} = \frac{5}{13}\Big(G\iota^*(x_1^{13} x_2 \zeta_2^4)\Big)_{Q_{14,4}},$$
$$\Big(G\iota^* F_0(x_1^{12} x_2, x_1^2 x_2^3)\Big)_{Q_{14,4}} = -\frac{10}{13\cdot 7}\Big(G\iota^*(x_1^{14} \zeta_2^4)\Big)_{Q_{14,4}},$$
$$\Big(G\iota^* F_0(x_1^{11} x_2^2, x_1^3 x_2^2)\Big)_{Q_{14,4}} = \frac{5}{13\cdot 14}\Big(G\iota^*(x_1^{14} \zeta_2^4)\Big)_{Q_{14,4}},$$
$$\Big(G\iota^* F_0(x_1^{13}, x_1 x_2^4)\Big)_{Q_{14,4}} = \frac{5}{7}\Big(G\iota^*(x_1^{14} \zeta_2^4)\Big)_{Q_{14,4}}.$$

Since $(G\iota^*(x_1^{13} x_2 \zeta_2^4))_{Q_{14,4}}$ and $(G\iota^*(x_1^{14} \zeta_2^4))_{Q_{14,4}}$ do not vanish, we have

$$(-d' + 2d)\Big(\sum_{j=2}^{k} \lambda_j p_j^2\Big)(\sum_j b_j p_j) = 0,$$
$$(d' + 26d)\Big(\sum_{j=2}^{k} \lambda_j p_j^2\Big)^2 (\sum_j b_j q_j)$$
$$-(6d' + 16d)\Big(\sum_{j=2}^{k} \lambda_j p_j^2\Big)\Big(\sum_{j=2}^{k} \lambda_j p_j q_j\Big)(\sum_i b_j p_j) = 0.$$

Since $B_{13} \neq 0$, $B_{13} C_{13}$ does not belong to the ideal $(\sum_{i \geq 2} x_i^2)$. Then there is a vector $(p_2, \cdots, p_{n+1}) \in \boldsymbol{C}^n$ such that

$$\sum_{i=2}^{n+1} p_i^2 = 0, \qquad \Big(\sum_{j=2}^{k} \lambda_j p_j^2\Big)(\sum_j b_j p_j) \neq 0.$$

It is easy to see that for such $(p_2, \cdots, p_{n+1})$ there is a vector $(q_2, \cdots, q_{n+1}) \in \boldsymbol{C}^n$

such that $1+\sum_{i=2}^{n+1} q_i^2=0$ and $\sum_{i=2}^{n+1} p_i q_i=0$. Therefore it follows that

$$d = 2d .$$

If some b_i $(3\leq i\leq n)$ does not vanish, then put $p_2=1$, $p_{n+1}=\sqrt{-1}$, $p_j=0$ $(3\leq j\leq n)$, and $q_i=\sqrt{-1}$, $q_j=0$ $(j\neq i)$ in the above formula. Then we have $d'+26d=0$, which is a contradiction because $d\neq 0$. Hence

$$b_i=0 \qquad (3\leq i\leq n) .$$

Next assume that b_{n+1} does not vanish. Take (p_i) and (q_i) as follows:

$$q_{n+1}=\sqrt{-1},\ q_i=0\ (2\leq i\leq n);$$
$$p_2=1,\ p_n=\sqrt{-1},\ p_i=0\ (i\neq 2, n)\ \text{if}\ k<n ,$$
$$p_2=1,\ p_{i_0}=\sqrt{-1},\ p_i=0\ (i\neq 2, i_0)\ \text{if}\ k=n ,$$

where i_0 is chosen such that $\lambda_2\neq\lambda_{i_0}$.
By substituting these (p_i) and (q_i) into the above formula, we have $d'+26d=0$, which is again a contradiction. Hence

$$b_{n+1}=0 ,$$

and B_{13} is a constant multiple of x_2. Moreover, if $k\geq 3$, then put $p_3=1$, $p_{n+1}=\sqrt{-1}$, $p_i=0$ $(i\neq 3, n+1)$, and $q_2=\sqrt{-1}$, $q_i=0$ $(3\leq i\leq n+1)$ in the above formula. Then we have $d'+26d=0$, which is a contradiction. Hence we also see that C_{13} is a constant multiple of x_2^2 under the assumption $B_{13}\neq 0$.

Next we shall show that C_5 is a constant multiple of x_2^2. We have already seen that $X+Z$ belongs to the ideal $(\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i\zeta_i)$, where

$$X=-30d_0^{6,2}\tilde{G}\Big(\sum_{i=0}^{2} a_i^6 F_i(C_{13}x_1^{11}, E_5x_1)\Big),$$

$$Z=\tilde{G}\Big(\sum_{i=0}^{2} a_i^6\Big(\sum_j x_j\frac{\partial}{\partial\zeta_j}\Big)^{2i} x_1^{10}w_9(x)\,\zeta_1^4\Big),$$

and $w_9\in \boldsymbol{R}[x_2,\cdots,x_{n+1}]$ is defined by the condition

$$\begin{aligned}&2G\iota^* F^1(A_{13}x_1^{13}, E_5x_1)+2G\iota^* F^1(B_{13}x_1^{12}, D_5x_1^2)\\&\quad+\sum_{i+j=8} G\iota^* F^1(C_{2i+1}x_1^{2i-1}, C_{2j+1}x_1^{2j-1})+2G\iota^* F^3(A_{15}x_1^{15}, E_5x_1)\\&\quad+\sum_{i+j=9} G\iota^* F^3(C_{2i+1}x_1^{2i-1}, C_{2j+1}x_1^{2j-1})\equiv G\iota^*\Big(x_1^{10}w_9(x)\,\zeta_1^4\Big)\\&\quad \bmod G(\mathscr{A}^2) .\end{aligned}$$

In view of Corollary 5. 5 (ii) we may assume that the degree of w_9 in the variable x_{n+1} is at mosy 1. Remarking that

$$\left(\tilde{G}F^3(C_{2i+1}x_1^{2i-1}, C_{2j+1}x_1^{2j-1})\right)^\nu = C_{2i+1}(\nu)\, C_{2j+1}(\nu)\, \tilde{G}F^3(x_1^{2i-1}x_2^2, x_1^{2j-1}x_2^2),$$

we have by Lemma 5. 6.

$$\begin{aligned} w_9(\nu)\left(G\iota^*(x_1^{10}x_2^4\zeta_1^4)\right)_{Q_{14,4}} &= -30d_0^{6,2}C_5(\nu)\, C_{13}(\nu)\left(G\iota^* F_0(x_1^{11}x_2^2, x_1^3x_2^2)\right)_{Q_{14,4}} \\ &\quad -30d_0^{6,2}B_{13}(\nu)\, D_5(\nu)\left(G\iota^* F_0(x_1^{12}x_2, x_1^2x_2^3)\right)_{Q_{14,4}} \\ &\quad +cC_{13}(\nu)^2\left(G\iota^*(x_1^{14}\zeta_2^4)\right)_{Q_{14,4}} \qquad (c\in \boldsymbol{R}) \end{aligned}$$

for all $\nu=(\nu_2, \cdots, \nu_{n+1})\in \boldsymbol{C}^n$ with $\sum_i \nu_i^2=0$. We see from the last part of the proof of Lemma 5. 4 that

$$\left(G\iota^*(x_1^{10}x_2^4\zeta_1^4)\right)_{Q_{14,4}} = \frac{1}{7\cdot 11\cdot 13}\left(G\iota^*(x_1^{14}\zeta_2^4)\right)_{Q_{14,4}},$$

and from Lemma 5. 10 stated leater that

$$-30d_0^{6,2}\left(G\iota^* F_0(x_1^{11}x_2^2, x_1^3x_2^2)\right)_{Q_{14,4}} = -\frac{3\cdot 25}{7\cdot 13}d_0^{6,2}\left(G\iota^*(x_1^{14}\zeta_2^4)\right)_{Q_{14,4}}$$

and $(G\iota^* F_0(x_1^{12}x_2, x_1^2x_2^3))_{Q_{14,4}}$ is a constant multiple of $(G\iota^*(x_1^{14}\zeta_2^4))_{Q_{14,4}}$. Hence there are constants c', $c''\in \boldsymbol{R}$ such that

$$w_9 \equiv -3\cdot 25\cdot 11 d_0^{6,2}C_5C_{13}+c' B_{13}D_5+c'' C_{13}^2 \qquad \mathrm{mod}\,(\sum_{i\geq 2} x_i^2).$$

On the other hand, we have already seen in the part II of the proof of Proposition 4. 5 that $(G\iota^* F_0(x_1^{11}x_2^2, x_1x_2^4))_{Q_{14,4}}=0$. Since C_{13} and E_5 are constant multiples of x_2^2 and x_2^4 respectively in the present case, it follows that $X=0$. Hence $z\in(\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i\zeta_i)$, and $(G\iota^*(x_1^{10}w_9(x)\,\zeta_1^4))_{Q_{14,4}}=0$. Then by Lemma 5. 6 we have

$$w_9(\nu)\left(G\iota^*(x_1^{10}x_2^4\zeta_1^4)\right)_{Q_{14,4}}=0$$

for all $\nu=\nu_2, \cdots, \nu_{n+1})\in \boldsymbol{C}^n$ with $\sum_i \nu_i^2=0$. Since $(G\iota^*(x_1^{10}x_2^4\zeta_1^4))_{Q_{14,4}}$ does not vanish and the degree of w_9 in the variable x_{n+1} is at most 1, it follows that $w_9=0$. This shows that C_5 is a constant multiple of x_2^2.

Next we shall show that B_5 is a constant multiple of x_2. We have already shown that there is $v_6\in \boldsymbol{R}\,[x_2, \cdots, x_{n+1}]_4$ such that

$$G\iota^* F(f,f) \equiv \sum_{j=4}^{5} G\iota^* R_{2j}+G\iota^*\left(x_1^2v_6(x)\,\zeta_1^4\right) \qquad \mathrm{mod}\, G(\mathscr{K}^2).$$

This implies that

$$0=(G\iota^* R_{10})_{Q_{6,4}}+\left(G\iota^*(x_1^2 v_6(x)\,\zeta_1^4)\right)_{Q_{6,4}}.$$

Remark that in the present case $G\iota^*(E_5x_1,\ D_5x_1^2)\in G(\mathscr{H}^2)$ by Corollary 5. 5 (i). By taking the parts of odd degree in the variables $(x_2, \cdots, x_{n+1}, \zeta_2, \cdots, \zeta_{n+1})$, we then have

$$-7d_0^{2,2}\left\{G\iota^* F_0(E_5x_1,\ B_5x_1^4)\right)_{Q_{6,4}}+\left(G\iota^* F_0(D_5x_1^2,\ C_5x_1^3)\right)_{Q_{6,4}}\Big\}$$
$$+\left(G\iota^* F^3(B_7x_1^6,\ E_5x_1)\right)_{Q_{6,4}}+\left(G\iota^* F^3(C_7x_1^5,\ D_5x_1^2)\right)_{Q_{6,4}}=0\,.$$

By Lemma 5. 10 stated later there are non-zero constants b_i $(1\leq i\leq 4)$ such that

$$\left\{b_1E_5(v)\,B_5(v)+b_2D_5(\nu)\,C_5(\nu)+b_3B_7(\nu)\,E_5(\nu)+b_4C_7(\nu)\,D_5(\nu)\right\}$$
$$\times\left(G\iota^*(x_1^5x_2\zeta_2^4)\right)_{Q_{6,4}}=0\,.$$

Since $(G\iota^*(x_1^5x_2\zeta_2^4))_{Q_{6,4}}\neq 0$ and the degree of B_5 in the variable x_{n+1} is at most 1, it follows that B_5 is a constant multiple of x_2.

II. $B_{13}=0$. In this case we have $X+Z\in(\sum_i x_i^2,\ \sum_i \zeta_i^2,\ \sum_i x_i\zeta_i)$, X and Z being as in I, and

$$w_9=-3\cdot 25\cdot 11 d_0^{6,2}C_5C_{13}+c''\,C_{13}^2\,,\qquad c''\in\boldsymbol{R}\,.$$

By considering the homogeneity in the variables $(x_2, \cdots, x_{n+1}, \zeta_2, \cdots, \zeta_{n+1})$, we also have $X\in(\sum_{i\geq 2} x_i^2,\ \sum_{i\geq 2}\zeta_i^2,\ \sum_{i\geq 2} x_i\zeta_i)$. First assume that $k\leq n-1$ $\left(C_{13}=\sum_{i=2}^{k}\lambda_i x_i^2\right)$. Then X does not contain the variables $(x_n, x_{n+1}, \zeta_n, \zeta_{n+1})$. Since for each $(x_2, \cdots, x_{n-1}, \zeta_2, \cdots, \zeta_{n-1})\in\boldsymbol{C}^{2n-4}$ we can choose $(x_n, x_{n+1}, \zeta_n, \zeta_{n+1})\in\boldsymbol{C}^4$ such that $\sum_{i\geq 2} x_i^2=\sum_{i\geq 2}\zeta_i^2=\sum_{i\geq 2} x_i\zeta_i=0$, it follows that $X=0$. Next assume that $k=n$ and $n\geq 4$. Then, since X does not contain the variables (x_{n+1}, ζ_{n+1}), we have

$$X\in\left(\sum_{i=2}^{n} x_i^2\sum_{i=2}^{n}\zeta_{i2}-\left(\sum_{i=2}^{n} x_i\zeta_i\right)^2\right)$$

by Lemma 5. 3. Put $\mu=\sum_{i=2}^{n} x_i^2$, and consider the following polynomial;

$$X_0=\tilde{G}\left(\sum_{i=0}^{2} a_i^6 F_i(\mu x_1^{11},\ \mu^2 x_1)\right).$$

We have seen that $(G\iota^* F_0(x_1^{11}x_{n+1}^2,\ x_1x_{n+1}^4))_{Q_{14,4}}=0$. By substituting $1-x_1^2-\mu$ for x_{n+1}^2 we have

$$\left(G\iota^* F_0(x_1^{11}\mu,\ x_1\mu^2)\right)_{Q_{14,4}}+\left(G\iota^* F_0(x_1^{13},\ x_1\mu^2)\right)_{Q_{14,4}}$$
$$+2\left(G\iota^* F_0(x_1^{11}\mu,\ x_1^3\mu)\right)_{Q_{14,4}}=0\,.$$

This implies that $X_0 \in (\sum_{i\geq 2} x_i^2, \sum_{i\geq 2} \zeta_i^2, \sum_{i\geq 2} x_i\zeta_i)$, and hence

$$X_0 \in \left(\sum_{i=2}^n x_i^2 \sum_{i=2}^n \zeta_i^2 - \left(\sum_{i=2}^n x_i\zeta_i\right)^2\right).$$

Note that C_{13} is the image of μ under the transformation $x_i \to \sqrt{\lambda_i}\, x_i$ $(2 \leq i \leq n)$, $x_1 \to x_1$, $x_{n+1} \to x_{n+1}$. Since this transformation commutes with the operators $\tilde{G}$ and $\sum_i x_i \frac{\partial}{\partial \zeta_i}$, it follows that

$$X \in \left(\sum_{i=2}^n \lambda_i x_i^2 \sum_{i=2}^n \lambda_i \zeta_i^2 - \left(\sum_{i=2}^n \lambda_i x_i\zeta_i\right)^2\right).$$

Since $n \geq 4$, the polynomials $\sum_{i=2}^n x_i^2 \sum_{i=2}^n \zeta_i^2 - \left(\sum_{i=2}^n x_i\zeta_i\right)^2$ and $\sum_{i=2}^n \lambda_i x_i^2 \sum_{i=2}^n \lambda_i \zeta_i^2 - \left(\sum_{i=2}^n \lambda_i x_i\zeta_i\right)^2$ are irreducible and mutually prime. Hence X must be divided by their product. But since $\deg_2 X = 6$, it follows that $X = 0$.

Now we assume that $X = 0$ and $k \geq 3$. Then an explicit computation shows that the coefficient of $x_1^{12} x_2^2 \zeta_3^4$ in the polynomial X is, putting $I'_{a,b} = \frac{1}{2\pi}\int_0^{2\pi} (\cos t)^{2a} (\sin t)^{2b}\, dt$ and $E_5 = d_1 C_{13}^2$,

$$\begin{aligned} &-30 d_1 d_0^{6,2} \lambda_2 \lambda_3^2 \Big\{ (I'_{9,0} - 4I'_{8,1} + 6I'_{7,2} - 4I'_{6,3}) + a_1^6 (48 I'_{8,1} - 96 I'_{7,2} + 36 I'_{6,3}) \\ &\qquad + a_2^6 \cdot 360 I'_{7,2} \Big\} \\ &= -30 d_1 d_0^{6,2} \lambda_2 \lambda_3^2 \times \left(-\frac{5979}{5 \cdot 13^2 \cdot 17} \right) I'_{9,0}, \end{aligned}$$

which is not zero. This is a contradiction. Hence C_{13} is a constant multiple of x_2^2 if $X = 0$. Moreover that $X = 0$ implies that $Z \in (\sum_{i\geq 2} x_i^2, \sum_{i\geq 2} \zeta_i^2, \sum_{i\geq 2} x_i\zeta_i)$, and we see that C_5 is a constant multiple of x_2^2 as in the case I. For B_5 we can also use the argument in I, from which that $B_5 = 0$ is easily deduced.

It remains to consider the case where $n = 3$ and $k = 3$. We shall compute C_5 in two ways, and show a contradiction. Let X, Z, and X_0 be as above. A direct conputation shows that

$$X_0 = 44\tilde{G}\left(x_1^{10}(x_2^2 + x_3^2)\, \zeta_1^2 + 2a_1^6 x_1^{12} (x_2^2 + x_3^2)\right) (x_2\zeta_3 - x_3\zeta_2)^2.$$

Put

$$W = -30 d_0^{6,2} d_1 \times 44 \lambda_2 \lambda_3 \tilde{G}\left(x_1^{10} (\lambda_2 x_2^3 + \lambda_3 x_3^2)\, \zeta_1^2 + 2a_1^6 x_1^{12} (\lambda_2 x_2^2 + \lambda_3 x_3^2)\right),$$

where $d_1 = E_5/C_{13}^2$ and $C_{13} = \lambda_2 x_2^2 + \lambda_3 x_3^2$. Then

$$X = W(x_2\zeta_3 - x_2\zeta_2)^2.$$

Since $X+Z\in(\sum_i x_i^2,\ \sum_i \zeta_i^2,\ \sum_i x_i\zeta_i)$, we can write

$$X+Z=\sum_i x_i^2\sum_{j=0}^{2} H_1^{2j}+\sum_i \zeta_i^2\sum_{j=0}^{2} H_2^{2j}+\sum_i x_i\zeta_i\sum_{j=0}^{2} H_3^{2j}\,,$$

where H_i^{2j} are homogeneous in both variables (x_1, ζ_1) and $(x_2, \cdots, x_4, \zeta_2, \cdots, \zeta_4)$ deg $H_i^{2j}=16$, $\deg_2 H_i^{2j}=2j$. By comparing both sides we have

$$X=\sum_{i\geq 2} x_i^2 H_1^4+\sum_{i\geq 2}\zeta_i^2 H_2^4+\sum_{i\geq 2} x_i\zeta_i H_3^4\,,$$

$$Z=x_1^2 H_1^4+\zeta_1^2 H_2^4+x_1\zeta_1 H_3^4+\sum_{i\geq 2} x_i^2 H_1^2+\sum_{i\geq 2}\zeta_i^2 H_2^2+\sum_{i\geq 2} x_i\zeta_i H_3^2\,.$$

Since

$$\begin{aligned}(x_2\zeta_2-x_3\zeta_2)^2&=\frac{1}{2}(\zeta_2^2+\zeta_3^2-\zeta_4^2)(x_2^2+x_3^2+x_4^2)\\&\quad+\frac{1}{2}(x_2^2+x_3^2-x_4^2)(\zeta_2^2+\zeta_3^2+\zeta_4^2)\\&\quad-(x_2\zeta_2+x_3\zeta_3-x_4\zeta_4)(x_2\zeta_2+x_3\zeta_3+x_4\zeta_4)\,,\end{aligned}$$

it follows that

$$\begin{aligned}&\left(H_1^4-\frac{1}{2}\left((\zeta_2^2+\zeta_3^2-\zeta_4^2)\,W\right)\sum_{i=2}^{4} x_i^2+\left(H_2^4-\frac{1}{2}(x_2^2+x_3^2-x_4^2)\,W\right)\sum_{i=2}^{4}\zeta_i^2\right.\\&\quad+\left(H_3^4+(x_2\zeta_2+x_3\zeta_3-x_4\zeta_4)\,W\right)\sum_{i=2}^{4} x_i\zeta_i=0\,.\end{aligned}$$

This implies that the polynomials $H_1^4-(\zeta_2^2+\zeta_3^2)\,W$, $H_2^4-(x_2^2+x_3^2)\,W$, and $H_3^4+2(x_2\zeta_2+x_3\zeta_3)\,W$ belong to the ideal $(\sum_{i=2}^{4} x_i^2,\ \sum_{i=2}^{4}\zeta_i^2,\ \sum_{i=2}^{4} x_i\zeta_i)$. Hence we have

$$\begin{aligned}Z\equiv&\left\{x_1^2(\zeta_2^2+\zeta_3^2)+\zeta_1^2(x_2^2+x_3^2)-2x_1\zeta_1(x_2\zeta_2+x_3\zeta_3)\right\}W\\&\bmod\left(\sum_{i=2}^{4} x_i^2,\ \sum_{i=2}^{4}\zeta_i^2,\ \sum_{i=2}^{4} x_i\zeta_i\right),\end{aligned}$$

which implies that

$$Z^\nu=(\nu_2^2+\nu_3^2)(x_1\zeta_2-x_2\zeta_1)^2 W^\nu$$

for all $\nu=(\nu_2, \nu_3, \nu_4)\in \boldsymbol{C}^3$ with $\sum_{i=2}^{4}\nu_i^2=0$. We see that

$$W^\nu=-30d_0^{6,2}d_1\times 44\lambda_2\lambda_3(\lambda_2\nu_2^2+\lambda_3\nu_3^2)\,\tilde{G}(x_1^{10}x_2^2\zeta_1^2+2a_1^6x_1^{12}x_2^2)\,,$$

$$Z^\nu=w_9(\nu)\,\tilde{G}(x_1^{10}x_2^4\zeta_1^4+12a_1^6x_1^{12}x_2^4\zeta_1^2+24a_2^6x_1^{14}x_2^4)\,.$$

Moreover an easy computation shows that

$$\tilde{G}(x_1^{10}x_2^2\zeta_1^2+2a_1^6x_1^{12}x_2^2)(x_1\zeta_2-x_2\zeta_1)^2$$
$$=\frac{35}{2}\tilde{G}(x_1^{10}x_2^4\zeta_1^4+12a_1^6x_1^{12}x_2^4\zeta_1^2+24a_2^6x_1^{14}x_2^4).$$

Hence we have

$$w_9(\nu)=-30\cdot 44\cdot\frac{35}{2}d_0^{6,2}d_1\lambda_2\lambda_3(\lambda_2\nu_2^2+\lambda_3\nu_3^2)(\nu_2^2+\nu_3^2).$$

Since the degree of w_9 in the variable x_4 is at most one, it follows that

$$w_9(x)=-30\cdot 44\cdot\frac{35}{2}d_0^{6,2}d_1\lambda_2\lambda_3(\lambda_2x_2^2+\lambda_3x_3^2)(x_2^2+x_3^2).$$

On the other hand we have already seen that

$$w_9=-3\cdot 25\cdot 11d_0^{6,2}C_5C_{13}+c''C_{13}^2,\qquad c''\in\boldsymbol{R}.$$

Hence it follows that

$$C_5=28d_1\lambda_2\lambda_3(x_2^2+x_3^2)+eC_{13},\qquad e\in\boldsymbol{R}.$$

Next we shall consider $Q_{6,4}$-component. We have already seen that there is a polynomial $v_6\in\boldsymbol{R}[x_2,x_3,x_4]_4$ such that

$$G\iota^*F(f,f)\equiv G\iota^*R_{10}+G\iota^*R_8+G\iota^*\left(x_1^2v_6(x)\zeta_1^4\right)\qquad\text{mod } G(\mathscr{A}^2).$$

Then we have

$$(G\iota^*R_{10})_{Q_{6,4}}+\left(G\iota^*\left(x_1^2v_6(x)\zeta_1^4\right)\right)_{Q_{6,4}}=0.$$

This implies that

$$\left(G\iota^*F^1(E_5x_1,E_5x_1)\right)_{Q_{6,4}}+2\left(G\iota^*F^1(C_5x_1^3,E_5x_1)\right)_{Q_{6,4}}$$
$$+2\left(G\iota^*F^3(C_7x_1^5,E_5x_1)\right)_{Q_{6,4}}+\left(G\iota^*\left(x_1^2w_5(x)\zeta_1^4\right)\right)_{Q_{6,4}}=0$$

for some $w_5\in\boldsymbol{R}[x_2,x_3,x_4]_4$, and

$$\left(G\iota^*F^1(B_5x_1^4,E_5x_1)\right)_{Q_{6,4}}=0.$$

Put

$$S=-7d_0^{2,2}\tilde{G}\left(\sum_{i=0}^{2}a_i^2F_i(E_5x_1,E_5x_1)\right),$$

$$T=-14d_0^{2,2}\tilde{G}\left(\sum_{i=0}^{2}a_i^2F_i(C_5x_1^3,E_5x_1)\right)$$
$$+10d_0^{2,2}\tilde{G}\left(\sum_{i=0}^{2}a_i^2\left(\sum_j x_j\frac{\partial}{\partial\zeta_j^2}\right)^{2i}x_1^4C_7(x)E_5(\zeta),\right.$$

$$U=G\left(x_1^2 w_5(x)\,\zeta_1^4+12a_1^2 x_1^4 w_5(x)\,\zeta_1^2+24a_2^2 x_1^6 w_5(x)\right).$$

Then there are homogeneous polynomials $H_i \in \boldsymbol{R}[x, \zeta]_8$ $(i=1, 2, 3)$ with $\deg_2 H_i \leq 6$ such that

$$S+T+U=\sum_i x_i^2 H_1+\sum_i \zeta_i^2 H_2+\sum_i x_i \zeta_i H_3 .$$

Let H_i^j be the homogeneous part of H_i of degree j in the variables $(x_2, \cdots, x_4, \zeta_2, \cdots, \zeta_4)$. Then we have

$$S=\sum_{i=2}^{4} x_i^2 H_1^6+\sum_{i=2}^{4} \zeta_i^2 H_2^6+\sum_{i=2}^{4} x_i \zeta_i H_3^6 ,$$

$$T=x_1^2 H_1^6+\zeta_1^2 H_2^6+x_1\zeta_1 H_3^6+\sum_{i=2}^{4} x_i^2 H_1^4+\sum_{i=2}^{4} \zeta_i^2 H_2^4+\sum_{i=2}^{4} x_i \zeta_i H_3^4 .$$

Now consider the following polynomial;

$$S_0=\tilde{G}\left(\sum_{i=0}^{2} a_i^2 F_i(x_1\mu^2, x_1\mu^2)\right),$$

where $\mu=x_2^2+x_3^2$. A direct computation shows that

$$S_0=-\frac{8}{35}\tilde{G}\left(x_1^2(x_2^2+x_3^2)(\zeta_2^2+\zeta_3^2)+9x_1^2(x_2\zeta_2+x_3\zeta_3)^2-2x_1^2(x_2^2+x_3^2)^2\right)$$
$$\times(x_2\zeta_3-x_3\zeta_2)^2 .$$

Hence, by putting

$$V=\frac{8}{5}d_0^{2,2}d_1^2\lambda_2\lambda_3\tilde{G}\left(x_1^2(\lambda_2 x_2^2+\lambda_3 x_3^2)(\lambda_2\zeta_2^2+\lambda_3\zeta_3^2)\right)$$
$$+9x_1^2(\lambda_2 x_2\zeta_2+\lambda_3 x_3\zeta_3)^2-2x_1^2(\lambda_2 x_2^2+\lambda_3 x_3^2)^2\Big),$$

we have

$$S=(x_2\zeta_3-x_3\zeta_2)^2 V ,$$

where $d_1=E_5/(\lambda_2 x_2^2+\lambda_3 x_3^2)^2$. Then a similar argument as before shows that

$$T\equiv\left\{x_1^2(\zeta_2^2+\zeta_3^2)+\zeta_1^2(x_2^2+x_3^2)-2x_1\zeta_1(x_2\zeta_2+x_3\zeta_3)\right\} V$$
$$\bmod\left(\sum_{i=2}^{4} x_i^2,\ \sum_{i=2}^{4}\zeta_i^2,\ \sum_{i=2}^{4} x_i\zeta_i\right),$$

and $T^\nu=(\nu_2^2+\nu_3^2)(x_1\zeta_2-x_2\zeta_1)^2 V^\nu$ for all $\nu=(\nu_2, \nu_3, \nu_4)\in \boldsymbol{C}^3$ with $\sum_{i=2}^{4}\nu_i^2=0$. We have

$$V^\nu=\frac{8}{5}d_0^{2,2}d_1^2\lambda_2\lambda_3(\lambda_2\nu_2^2+\lambda_3\nu_3^2)^2\tilde{G}(10x_1^2x_2^2\zeta_2^2-2x_1^2x_2^4),$$

$$T^{\nu} = -14d_0^{2,2}C_5(\nu)\,E_5(\nu)\,\tilde{G}\Big(\sum_{i=0}^{2} a_i^2 F_i(x_1^3x_2^2,\ x_1x_2^4)\Big)$$
$$+10d_0^{2,2}C_7(\nu)\,E_5(\nu)\,\tilde{G}(x_1^4x_2^2\zeta_2^4+12a_1^2x_1^4x_2^4\zeta_2^2+24a_2^2x_1^4x_2^6)\,.$$

By Lemma 5. 10 stated later we see that

$$\Big(G\iota^* F_0(x_1^3x_2^2,\ x_1x_2^4)\Big)_{Q_{6,4}} = -2\Big(G\iota^*(x_1^4x_2^2\zeta_2^4)\Big)_{Q_{6,4}}\,.$$

From this it follows that

$$\tilde{G}\Big(\sum_{i=0}^{2} a_i^2 F_i(x_1^3x_2^2,\ x_1x_2^4)\Big)$$
$$= -2\tilde{G}(x_1^4x_2^2\zeta_2^4+12a_1^2x_1^4x_2^4\zeta_2^2+24a_2^2x_1^4x_2^6)\,.$$

Hence

$$T^{\nu} = \Big(28C_5(\nu)+10C_7(\nu)\Big)\,d_0^{2,2}E_5(\nu)\,\tilde{G}(x_1^4x_2^2\zeta_2^4+12a_2^2x_1^4x_2^4\zeta_2^2+24a_2^2x_1^4x_2^6)\,.$$

Moreover a direct a direct computation shows that

$$\tilde{G}(10x_1^2x_2^2\zeta_2^2-2x_1^2x_2^4)\,(x_1\zeta_2-x_2\zeta_1)^2$$
$$= 35\tilde{G}(x_1^4x_2^2\zeta_2^4+12a_1^2x_1^4x_2^4\zeta_2^2+24a_2^2x_1^4x_2^6)\,.$$

Since

$$\Big(\iota^*\tilde{G}(x_1^4x_2^2\zeta_2^4+12a_1^2x_1^4x_2^4\zeta_2^2+24a_2^2x_1^4x_2^6)\Big)_{G(Q_{10})} = \Big(G\iota^*(x_1^4x_2^2\zeta_2^4)\Big)_{Q_{6,4}}\,,$$

it does not vanish by Lemma 3. 9. Thus we have

$$\Big(28C_5(\nu)+10C_7(\nu)\Big)\,(\lambda_2\nu_2^2+\lambda_3\nu_3^2)^2 = 56d_1\lambda_2\lambda_3(\nu_2^2+\nu_3^2)\,(\lambda_2\nu_2^2+\lambda_3\nu_3^2)^2$$

for all $\nu=(\nu_2, \nu_3, \nu_4)\in \boldsymbol{C}^3$ with $\sum_{i=2}^{4}\nu_i^2=0$. Since the ideal $\Big(\sum_{i=2}^{4}x_i^2\Big)$ is prime, it follows that

$$C_5 = 2d_1\lambda_2\lambda_3(x_2^2+x_3^2)-\frac{5}{14}C_7\,.$$

But we have already seen that $C_5=28d_1\lambda_2\lambda_3(x_2^2+x_3^2)+eC_{13}$, $e\in\boldsymbol{R}$. Since C_7 is a constant multiple of C_{13}, and since $x_2^2+x_3^2$ and C_{13} are mutually prime, it is a contradiction.

This concludes the proof of Proposition 5. 2.

Now we shall prove the following proposition, which will complete the proof of Theorem 4. 1.

PROPOSITION 5. 7. *Let f be a polynomial of the form*

$$f=\sum_{i=2}^{10}\alpha_{2i+1}x_1^{2i+1}+\sum_{i=2}^{6}\beta_{2i+1}x_1^{2i}x_2+\sum_{i=2}^{6}\gamma_{2i+1}x_1^{2i-1}x_2^2+\delta_5x_1^2x_2^3+\varepsilon_5x_1x_2^4\,,$$

where the coefficients are real numbers, $\alpha_{21}=1$, $\gamma_{13}\neq 0$, $\varepsilon_5\neq 0$. *Then* $G_t^*F(f,f)\in G(\mathscr{A}^2)$ *if and only if the coefficients satisfy the relations described in Theorem* 4.1 (ii).

To prove this proposition we need some lemmas. Let R_{2i} $(4\leq i\leq 21)$ be the homogeneous part of degree $2i$ of $F(f,f)$.

LEMMA 5.8. $G_t^*F(f,f)\in G(\mathscr{A}^2)$ *if and only if* $(G_t^*R_{2i})_{Q_{2i-4,4}}=0$ $(4\leq i\leq 13)$.

PROOF. Since $\deg_2 R_{2i}\leq 2$ $(14\leq i\leq 21)$, it follows that $G_t^*R_{2i}\in G(\mathscr{A}^2)$ $(14\leq i\leq 21)$ in view of Corollary 5.5 (i). Hence the condition $G_t^*F(f,f)\in G(\mathscr{A}^2)$ is equivalent to the condition

$$\sum_{i=4}^{13} G_t^*R_{2i}\in G(\mathscr{A}^2),$$

Let f_{2i+1} $(2\leq i\leq 10)$ be the homogeneous part of degree $2i+1$ of f. If i, $j\geq 3$, then $\deg_2 F^1(f_{2i+1}, f_{2j+1})$ and $\deg_2 F^2(f_{2i+1}, f_{2j+1})$ are at most 4. Thus for such i, j we have

$$\left(G_t^*F^1(f_{2i+1}, f_{2j+1})\right)_{Q_{2i+2j+2-2k,2k}}=0 \qquad (3\leq k\leq[(i+j+1)/2])$$

and

$$\left(G_t^*F^2(f_{2i+1}, f_{2j+1})\right)_{Q_{2i+2j-2k,2k}}=0 \qquad (3\leq k\leq[(i+j)/2])$$

by Corollary 5.5 (ii). Moreover, since the degrees of $F^1(f_{2i+1}, f_5)$ and $F^2(f_{2i+1}, f_5)$ $(2\leq i\leq 10)$ in the variables ζ are at most 4, we also have

$$\left(G_t^*F^1(f_{2i+1}, f_5)\right)_{Q_{2i+6-2k,2k}}=0 \qquad (3\leq k\leq[(i+3)/2])$$

and

$$\left(G_t^*F^2(f_{2i+1}, f_5)\right)_{Q_{2i+4-2k,2k}}=0 \qquad (3\leq k\leq[(i+2)/2])$$

Therefore we see that

$$(G_t^*R_{2i})_{Q_{2i-2k,2k}}=0 \qquad (3\leq k\leq[i/2],\ 6\leq i\leq 13).$$

Now fix an index i $(4\leq i\leq 13)$ and assume that $(G_t^*R_{2i})_{Q_{2i-4,4}}=0$. Then by the above fact we have

$$(G_t^*R_{2i})_{G(Q_{2i})}=(G_t^*R_{2i})_{Q_{2i-2,2}}+(G_t^*R_{2i})_{Q_{2i,0}}.$$

In view of Corollary 2.8 (ii) there is a polynomial $h\in \boldsymbol{R}[x_1, x_2, \zeta_1, \zeta_2]_{2i-2,2}+\boldsymbol{R}[x_1, x_2, \zeta_1, \zeta_2]_{2i,0}$ such that

$$(G_t^*R_{2i})_{Q_{2i-2,2}}+(G_t^*R_{2i})_{Q_{2i,0}}=(G_t^*h)_{G(Q_{2i})}.$$

Then $(Gt^*(R_{2i}-h))_{G(Q_{2i})}=0$, and it follows that the polynomial $\tilde{G}(R_{2i}-h)$ belongs to the ideal $(\sum_i x_i^2, \sum_i \zeta_i^2, \sum_i x_i\zeta_i)$. But, since $\tilde{G}(R_{2i}-h)$ is a polynomial only in four variables $(x_1, x_2, \zeta_1, \zeta_2)$, and since $n\geq 3$, it follows that $\tilde{G}(R_{2i}-h)=0$. Hence $Gt^*R_{2i}=Gt^*h\in G(\mathscr{H}^2)$.

On the other hand, if $\sum_{j=4}^{i} Gt^*R_{2j}\in G(\mathscr{H}^2)$ for some i, then

$$0=\left(\sum_{j=4}^{i} Gt^*R_{2j}\right)_{Q_{2i-4,4}}=(Gt^*R_{2i})_{Q_{2i-4,4}}.$$

Therefore we see that $\sum_{j=4}^{i} Gt^*R_{2j}\in G(\mathscr{H}^2)$ if and only if $\sum_{j=4}^{i-1} Gt^*R_{2j}\in G(\mathscr{H}^2)$ and $(Gt^*R_{2i})_{Q_{2i-4,4}}=0$. The lemma now follows by induction.

We have defined positive constants $d_l^{i,j}$ for integers i, j, l satisfying $0\leq l\leq j\leq i$ in § 3. Here we further define $d_l^{i,i+1}$ for integers i, l with $0\leq l\leq i+1$ by the same formula, i. e., $d_0^{i,i+1}=I_0^{i+1}$ and

$$d_l^{i,i+1}=\frac{I_l^{i+1}}{(2l)\,!}-\sum_{p=0}^{l-1} d_p^{i,i+1}a_{l-p}^{2p+1} \qquad (l\geq 1)\,.$$

The proof of Lemma 3. 4 (ii) is also valid in this case, and we have

$$d_l^{i,i+1}>0 \qquad (0\leq l\leq i+1)\,.$$

LEMMA 5. 9. *For* $u\in \boldsymbol{R}[x]_{2i+1}$ *and* $v\in \boldsymbol{R}[x]_{2j+1}$ $(2\leq j\leq i)$,

$$\left(Gt^*F^3(u, v)\right)_{Q_{2i+2j-4,4}}=d_{j-2}^{i-1,j}\left(Gt^*\left(\sum_k x_k\frac{\partial}{\partial\zeta_k}\right)^{2j-4}\frac{\partial u(x)}{\partial x_1}\frac{\partial v(\zeta)}{\partial\zeta_1}\right)_{Q_{2i+2j-4,4}},$$

$$\left(Gt^*F^4(u, v)\right)_{Q_{2i+2j-4,4}}$$
$$=d_{j-2}^{i-1,j}\left(Gt^*\left(\sum_k x_k\frac{\partial}{\partial\zeta_k}\right)^{2j-4}\sum_{l=2}^{n+1}\frac{\partial u(x)}{\partial x_l}\frac{\partial v(\zeta)}{\partial\zeta_l}\right)_{Q_{2i+2j-4,4}}.$$

PROOF. Since

$$\int_0^\pi\frac{\partial v}{\partial x_l}(x\cos t+\zeta\sin t)\sin t\,dt=\sum_{p=1}^{j}\frac{I_p^j}{(2p)\,!}\left(\sum_k x_k\frac{\partial}{\partial\zeta_k}\right)^{2p}\frac{\partial v(\zeta)}{\partial\zeta_l},$$

it follows that

$$F^3(u, v)=\sum_{p=0}^{j}\frac{I_p^j}{(2p)\,!}\left(\sum_k x_k\frac{\partial}{\partial\zeta_k}\right)^{2p}\frac{\partial u(x)}{\partial x_1}\frac{\partial v(\zeta)}{\partial\zeta_1},$$

$$F^4(u, v)=\sum_{p=0}^{j}\frac{I_p^j}{(2p)\,!}\left(\sum_k x_k\frac{\partial}{\partial\zeta_k}\right)^{2p}\sum_{l=2}^{n+1}\frac{\partial u(x)}{\partial x_l}\frac{\partial v(\zeta)}{\partial\zeta_l}.$$

Then from the definition of $d_p^{i,j}$ we have

$$F^3(u, v)=\sum_{p=0}^{j} d_p^{i-1,j}\sum_{q=p}^{j} a_{q-p}^{i-j+1+2p}\left(\sum_k x_k\frac{\partial}{\partial\zeta_k}\right)^{2q}\frac{\partial u(x)}{\partial x_1}\frac{\partial v(\zeta)}{\partial\zeta_1}.$$

Hence

$$\begin{aligned}&\Big(G\iota^* F^3(u, v)\Big)_{Q_{2i+2j-4,4}}\\&=d_{j-2}^{i-1,j}\Big(G\iota^* \sum_{q=j-2}^{j} a_{q-j+2}^{i+j-3}\Big(\sum_k x_k \frac{\partial}{\partial\zeta_k}\Big)^{2q} \frac{\partial u(x)}{\partial x_1}\frac{\partial v(\zeta)}{\partial\zeta_1}\Big)\Big)_{G(Q_{2i+2j})}\\&=d_{j-2}^{i-1,j}\Big(G\iota^*\Big(\sum_k x_k \frac{\partial}{\partial\zeta_k}\Big)^{2j-4}\frac{\partial u(x)}{\partial x_1}\frac{\partial v(\zeta)}{\partial\zeta_1}\Big)_{Q_{2i+2j-4,4}}.\end{aligned}$$

The formula for F^4 is obtained in the same way.

LEMMA 5.10.

(I) $(2\leq i\leq 10)$

$$\Big(G\iota^* F^1(x_1^{2i+1}, x_1x_2^4)\Big)_{Q_{2i+2,4}}=-\frac{(2i+3)(i-1)}{i+1}d_0^{i,2}\Big(G\iota^*(x_1^{2i+2}\zeta_2^4)\Big)_{Q_{2i+2,4}},$$

$$\Big(G\iota^* F^2(x_1^{2i+1}, x_1x_2^4)\Big)_{Q_{2i,4}}=(2i+1)\,d_0^{i-1,2}\Big(G\iota^*(x_1^{2i}\zeta_2^4)\Big)_{Q_{2i,4}}$$

(II) $(2\leq j\leq i\leq 6)$

$$\begin{aligned}&\Big(G\iota^* F^1(x^{2i-1}x_2^2, x_1^{2j-1}x_2^2)\Big)_{Q_{2i+2j-2,4}}\\&=-\frac{(2j-1)!\,(2i+3)(2i-1)(2i-2)}{(2i+2j-2)\cdots\cdots(2i+2j-5)}d_{j-2}^{i,j}\Big(G\iota^*(x_1^{2i+2j-2}\zeta_2^4)\Big)_{Q_{2i+2j-2,4}},\end{aligned}$$

$$\begin{aligned}&\Big(G\iota^* F^2(x_1^{2i-1}x_2^2, x_1^{2j-1}x_2^2)\Big)_{Q_{2i+2j-2,4}}\\&=\frac{(2j-1)!\,(2i-1)(2i-2)(2i-3)}{(2i+2j-4)\cdots\cdots(2i+2j-7)}d_{j-2}^{i-1,j}\Big(G\iota^*(x_1^{2i+2j-4}\zeta_2^4)\Big)_{Q_{2i+2j-4,4}}.\end{aligned}$$

(III) $(2\leq i\leq 6)$

$$\Big(G\iota^* F^1(x_1^{2i}x_2, x_1^2x_2^3)\Big)_{Q_{2i+2,4}}=\frac{2(2i+3)(2i-2)}{(2i+1)(2i+2)}d_0^{i,2}\Big(G\iota^*(x_1^{2i+2}\zeta_2^4)\Big)_{Q_{2i+2,4}},$$

$$\Big(G\iota^* F^2(x_1^{2i}x_2, x_1^2x_2^3)\Big)_{Q_{2i,4}}=-2d_0^{i-1,2}\Big(G\iota^*(x_1^{2i}\zeta_2^4)\Big)_{Q_{2i,4}}.$$

(IV) $(2\leq i\leq 6)$

$$\Big(G\iota^* F^1(x_1^{2i-1}x_2^2, x_1x_2^4)\Big)_{Q_{2i+2,4}}=-\frac{(2i+3)(i-6)}{i}d_0^{i,2}\Big(G\iota^*(x_1^{2i}x_2^2\zeta_2^4)\Big)_{Q_{2i+2,4}},$$

$$\Big(G\iota^* F^3(x_1^{2i-1}x_2^2, x_1x_2^4)\Big)_{Q_{2i,4}}=(2i-1)\,d_0^{i-1,2}\Big(G\iota^*(x_1^{2i-2}x_2^2\zeta_2^4)\Big)_{Q_{2i,4}},$$

$$\Big(G\iota^* F^4(x_1^{2i-1}x_2^2, x_1x_2^4)\Big)_{Q_{2i,4}}=-\frac{4}{i}d_0^{i-1,2}\Big(G\iota^*(x_1^{2i}\zeta_2^4)\Big)_{Q_{2i,4}}.$$

(V) $(2\leq i\leq 6)$

$$\left(G_t^* F^1(x_1^{2i} x_2, x_1 x_2^4)\right)_{Q_{2i+2,4}} = -\frac{(2i+3)(2i-7)}{2i+1} d_0^{i,2}\left(G_t^*(x_1^{2i+1} x_2 \zeta_2^4)\right)_{Q_{2i+2,4}},$$

$$\left(G_t^* F^2(x_1^{2i} x_2, x_1 x_2^4)\right)_{Q_{2i,4}} = 2i d_0^{i-1,2}\left(G_t^*(x_1^{2i-1} x_2 \zeta_2^4)\right)_{Q_{2i,4}}.$$

(VI) $(2 \leq i \leq 6)$

$$\left(G_t^* F^1(x_1^{2i-1} x_2^2, x_1^2 x_2^3)\right)_{Q_{2i+2,4}} = \frac{(2i+3)(4i-9)}{(2i+1)\, i} d_0^{i,2}\left(G_t^*(x_1^{2i+1} x_2 \zeta_2^4)\right)_{Q_{2i+2,4}},$$

$$\left(G_t^* F^2(x_1^{2i-1} x_2^2, x_1^2 x_2^3)\right)_{Q_{2i,4}} = -4 d_0^{i-1,2}\left(G_t^*(x_1^{2i-1} x_2 \zeta_2^4)\right)_{Q_{2i,4}}.$$

(VII)

$$\left(G_t^* F^1(x_1^2 x_2^3, x_1^2 x_2^3)\right)_{Q_{6,4}} = -\frac{21}{2} d_0^{2,2}\left(G_t^*(x_1^4 x_2^2 \zeta_2^4)\right)_{Q_{6,4}},$$

$$\left(G_t^* F^2(x_1^2 x_2^3, x_1^2 x_2^3)\right)_{Q_{4,4}} = \frac{3}{2} d_0^{1,2}\left(G_t^*(x_1^4 \zeta_2^4)\right)_{Q_{4,4}}.$$

PROOF. We shall only prove (I). The other cases will be verified in the same way. By the proof of Proposition 3. 5 we have

$$\begin{aligned}
&\left(G_t^* F^1(x_1^{2i+1}, x_1 x_2^4)\right)_{Q_{2i+2,4}} \\
&= -(2i+3)\, d_0^{i,2}\left(G_t^* F_0(x_1^{2i+1}, x_1 x_2^4)\right)_{Q_{2i+2,4}} \\
&= -(2i+3)\, d_0^{i,2}\left(G_t^*(x_1^{2i+2}\zeta_2^4 + 4 x_1^{2i+1} x_2 \zeta_1 \zeta_2^3)\right)_{Q_{2i+2,4}}.
\end{aligned}$$

Since

$$\tilde{X}_{E_0}(x_1^{2i+2} x_2 \zeta_2^3) = (2i+2)\, x_1^{2i+1} x_2 \zeta_1 \zeta_2^3 + x_1^{2i+2} \zeta_2^4 - 3 x_1^{2i+2} x_2^2 \zeta_2^2,$$

it follows that

$$\left(G_t^*(x_1^{2i+1} x_2 \zeta_1 \zeta_2^3)\right)_{Q_{2i+2,4}} = -\frac{1}{2i+2}\left(G_t^*(x_1^{2i+2} \zeta_2^4)\right)_{Q_{2i+2,4}}.$$

Hence the first formula is obtained. The second formula immediately follows from Lemma 5. 9.

LEMMA 5. 11.

$$d_{j-2}^{i,j} = \frac{(2i+2j-1)(2i+2j-3)(2i+2j-5)}{(2i+3)(2i+1)(2i-1)} \frac{16}{(2j+1)!!\,(2j-4)!!}.$$

PROOF. We have

$$d_{j-2}^{i,j} = \sum_{q=0}^{j-2} \frac{I_q^j}{(2q)!} J_{j-2-q}^{i-2+q}$$

$$= \sum_{q=0}^{j-2} \frac{2(2q-1)!!(2j-2q)!!}{(2q)!(2j+1)!!} \frac{(2i+2q-3)!!}{(2j-2q-4)!!(2i+2j-7)!!}.$$

Thus we must show the following formula:

$$\sum_{q=0}^{j-2}(j-q)(j-q-1)\frac{(2i+2q-3)!!}{(2q)!!(2i-3)!!} = \frac{(2i+2j-1)!!}{(2i+3)!!(2j-4)!!}\times 2.$$

Put

$$h(z) = \sum_{q=0}^{j-2}(j-q)(j-q-1)\frac{z(z+1)\cdots(z+q-1)}{q!}.$$

$h(z)$ is a polynomial of degree $j-2$ in the variable z. Let k be an integer such that $0 \leq k \leq j-2$. Then we have

$$h(-k) = \sum_{q=0}^{k}(j-q)(j-q-1)(-1)^q\binom{k}{q}$$
$$= j^2\sum_{q=0}^{k}(-1)^q\binom{k}{q} - j\sum_{q=0}^{k}(-1)^q(2q+1)\binom{k}{q} + \sum_{q=0}^{k}(-1)^q(q+1)q\binom{k}{q},$$

and hence $h(0)=j(j-1)$, $h(-1)=2(j-1)$, $h(-k)=0$ $(3 \leq k \leq j-2)$. This implies that

$$h(z) = \frac{2}{(j-2)!}(z+3)(z+4)\cdots(z+j).$$

Then by considerding $h\left(i-\frac{1}{2}\right)$ we have the lemma.

PROOF OF PROPOSITION 5.7. In view of Lemmas 5.8, 5.10, and 5.11, the condition $G_t^* F(f,f) \in G(\mathscr{K}^2)$ turns out to be a system of algebraic equations in the indeterminates α_{2i+1} $(2 \leq i \leq 9)$, β_{2i+1}, γ_{2i+1} $(2 \leq i \leq 6)$, δ_5, ε_5, which is as follows:

$$\varepsilon_5 + \frac{25}{13^2\cdot 12}\gamma_{13}{}^2 = 0,$$

$$\alpha_{19}\varepsilon_5 + \frac{25}{13\cdot 11\cdot 6}\gamma_{11}\gamma_{13} - \frac{5}{4}\left(\varepsilon_5 + \frac{25}{13^2\cdot 12}\gamma_{13}{}^2\right) = 0,$$

$$\alpha_{17}\varepsilon_5 + \frac{25}{13\cdot 56}\gamma_9\gamma_{13} + \frac{15}{11^2\cdot 7}\gamma_{11}{}^2 - \frac{9}{7}\left(\alpha_{19}\varepsilon_5 + \frac{25}{13\cdot 11\cdot 6}\gamma_{11}\gamma_{13}\right) = 0,$$

$$\alpha_{15}\varepsilon_5 + \frac{25}{13\cdot 49}\gamma_7\gamma_{13} + \frac{10}{11\cdot 21}\gamma_9\gamma_{11} - \frac{4}{3}\left(\alpha_{17}\varepsilon_5 + \frac{25}{13\cdot 56}\gamma_9\gamma_{13} + \frac{15}{11^2\cdot 7}\gamma_{11}{}^2\right) = 0,$$

$$\alpha_{13}\varepsilon_5 - \frac{2}{13}\beta_{13}\delta_5 + \frac{1}{26}\gamma_5\gamma_{13} + \frac{4}{11\cdot 7}\gamma_7\gamma_{11} + \frac{1}{36}\gamma_9{}^2$$

$$-\frac{7}{5}\left(\alpha_{15}\varepsilon_5+\frac{25}{13\cdot 49}\gamma_7\gamma_{13}+\frac{10}{11\cdot 21}\gamma_9\gamma_{11}\right)=0,$$

$$\alpha_{11}\varepsilon_5-\frac{2}{11}\beta_{11}\delta_5+\frac{3}{11\cdot 5}\gamma_5\gamma_{11}+\frac{1}{14}\gamma_7\gamma_9+\frac{1}{13}\gamma_{13}\varepsilon_5$$
$$-\frac{3}{2}\left(\alpha_{13}\varepsilon_5-\frac{2}{13}\beta_{13}\varepsilon_5+\frac{1}{26}\gamma_5\gamma_{13}+\frac{4}{11\cdot 7}\gamma_7\gamma_{11}+\frac{1}{36}\gamma_9^2\right)=0,$$

$$\alpha_9\varepsilon_5-\frac{2}{9}\beta_9\delta_5+\frac{1}{12}\gamma_5\gamma_9+\frac{5}{49\cdot 2}\gamma_7^2+\frac{4}{11\cdot 3}\gamma_{11}\varepsilon_5$$
$$-\frac{5}{3}\left(\alpha_{11}\varepsilon_5-\frac{2}{11}\beta_{11}\delta_5+\frac{3}{11\cdot 5}\gamma_5\delta_5+\frac{1}{14}\gamma_7\gamma_9\right)=0,$$

$$\alpha_7\varepsilon_5-\frac{2}{7}\beta_7\delta_5+\frac{1}{7}\gamma_5\gamma_7+\frac{2}{9}\gamma_9\varepsilon_5$$
$$-2\left(\alpha_9\varepsilon_5-\frac{2}{9}\beta_9\delta_5+\frac{1}{12}\gamma_5\gamma_9+\frac{5}{49\cdot 2}\gamma_7^2\right)=0,$$

$$\alpha_5\varepsilon_5-\frac{2}{5}\beta_5\delta_5+\frac{3}{20}\gamma_5^2+\frac{4}{7}\gamma_7\varepsilon_5-3\left(\alpha_7\varepsilon_5-\frac{2}{7}\beta_7\delta_5+\frac{1}{7}\gamma_5\gamma_7\right)=0,$$

$$\alpha_5\varepsilon_5-\frac{2}{5}\beta_5\delta_5+\frac{3}{20}\gamma_5^2-\frac{2}{5}\gamma_5\varepsilon_5+\frac{3}{20}\delta_5^2=0,$$

$$-\beta_{13}\varepsilon_5+\frac{1}{2}\gamma_{13}\delta_5=0,$$

$$-\frac{3}{11}\beta_{11}\varepsilon_5+\frac{1}{5}\gamma_{11}\delta_5+\frac{12}{13}\beta_{13}\varepsilon_5-\frac{4}{13}\gamma_{13}\delta_5=0,$$

$$-\frac{1}{9}\beta_9\varepsilon_5+\frac{7}{36}\gamma_9\delta_5+\frac{10}{11}\beta_{11}\varepsilon_5-\frac{4}{11}\gamma_{11}\delta_5=0,$$

$$\frac{1}{7}\beta_7\varepsilon_5+\frac{1}{7}\gamma_7\delta_5+\frac{8}{9}\beta_9\varepsilon_5-\frac{4}{9}\gamma_9\delta_5=0,$$

$$\frac{3}{5}\beta_5\varepsilon_5-\frac{1}{10}\gamma_5\delta_5+\frac{6}{7}\beta_7\varepsilon_5-\frac{4}{7}\gamma_7\delta_5=0,$$

$$\gamma_{11}=-\frac{55}{13}\gamma_{13},\ \gamma_9=-\frac{18}{11}\gamma_{11},\ \gamma_7=-\frac{7}{9}\gamma_9,$$

$$\gamma_5\varepsilon_5-\frac{3}{8}\delta_5^2+\frac{5}{14}\gamma_7\varepsilon_5=0.$$

Then it is easy to see that the indeterminates satisfy these equations if and only if they satisfy the relations described in Theorem 4.1 (ii) under the condition $\gamma_{13}\neq 0$. This finishes the proof of the proposition.

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