Hokkaido Mathematical Journal, Vol. 13 (1984) p. 145-150

A remark on socles and normal subgroups

By Tadashi IKEDA

(Received September 19, 1983)

1. Introduction

In modular representation theory of finite groups, it is often usefull to study socles of indecomposable modules. For example, the famous Brauer's result on socles and heads of projective indecomposable modules, or Green's recent result [5] on socles and heads of indecomposable direct summands of some permutation modules. In this paper, we are concerned with socles of direct summand of modules indeuced from normal subgroups.

Let G be a finite group, p a rational prime, and k a field of characteristic p. Let K be a normal subgroup of G and W an indecomposable kK-module. A number of authoers, including A. H. Clifford, have investigated the induced module W^{G} and the endomorphism ring $\mathrm{End}_{kG}(W^{G})$. In his paper [1], S. B. Conlon proved $\mathrm{End}_{kG}(W^{G})$ is almost isomorphic to a twisted group algebra over G/K. After, P. A. Tucker [7], [8] and H. N. Ward [9] studied the relationship between submodules of W^{G} and left ideals of $\mathrm{End}_{kG}(W^{G})$ in case W is a simple kK-module. Their results are found in the book [2]. As corollary of them, we see easily, if W is a simple kK-module and the innertia group $T_{G}(W)$ equals G, then any indecomposable direct summand of W^{G} has a simple socle and a simple head. We shall extend it and prove the following Theorem:

THEOREM 3.2 *Let k be an algebraically closed field. Suppose K is a normal subgroup of G and W is an indecomposable kK-module satisfying $T_{G}(W)=G$. If the socle* $\mathrm{soc}(W)$ *is a simple kK-module, then for an indecomposable direct summand V of W^{G} the socle* $\mathrm{soc}(V)$ *is a simple kG-module.*

Notation. Maps are usually on the left with the corresponding conversion for writing compositions. Let A and B sets and f a map of A to B. For a subset C of A we denote by $f|_{C}$ the restriction of f to C. For a subgroup H of G we denote by (G/H) a set of representatives of the left coset gH in G, containing the identity element. All kG-modules are finite generated left kG-modules. For a ring A we denote by $J(A)$ the Jacobson radical of A. Let M and M' be kG-modules. The socle of M is the maxmal semi-simple submodule of M, denoted by $\mathrm{soc}(M)$ and the head of M is $M/J(kG)M$, denoted by $hd(M)$. The notation $M|M'$ means that M is isomorphic to a direct summand of M'. If K is a normal subgroup of G and W is a kK-module, then the innertia group $T_G(W)$ is the subgroup consist of the element x satisfying W and W^x are isomorphic as kK-modules. For other definitions we refer to the books [2] and [3].

2. Preliminary

Throughout this paper, K is a normal subgroup of G, Y is a quotiant group G/K, W is an indecomposable kK-module and E is the endomorphism ring $\mathrm{End}_{kG}(W^G)$. The kG-module W^G equals $\bigoplus_{y\in(G/K)} y\otimes W$, where the symbol $\otimes$ will be $\otimes_{kK}$. We denote by E_y the k-subspace $\{\phi\in E|\phi(1\otimes W)\subset y\otimes W\}$ in E. It is easy to see E_1 is isomorphic to $\mathrm{End}_{kK}(W)$ as k-algebra. In this sence, we shall identify $\mathrm{End}_{kK}(W)$ with E_1. The following Theorem is the well-known Cliflord-Conlon's Theorem:

THEOREM 2.1 *Let k be an algebraicaly closed field. Then the k-algebra $E/J(E_1)E$ is isomorphic to a twisted group algebra over Y. In particular, if W is a simple kK-module, then E is isomorphic to a twisted group algebra over Y.*

In case W is a simple kK-module, H. Ward prove the following Theorem. See [9].

THEOREM 2.2 (H. Ward) *Let W be a simple kK-module satisfying $T_G(W)=G$. Then there is a lattice isomorphism between the set of left ideals of E and kG-submodules of W^G, given by;*

$$I \longmapsto \{\phi(w)|\phi\in I,\ w\in W^G\}$$

and

$$\{\phi\in E|\phi(W^G)\subset M\} \longleftarrow\!\shortmid\ M$$

where I is a left ideal of E and M is a kG-submodule of W^G.

As a corollary of Theorem 2.2, we can prove Theorem 3.5 in case W is a simple kK-module.

COROLLARY 2.3 *Let k be an algebraicaly closed field. If W is a simple kK-module satisfying $T_G(W)=G$, then for an indecomposable direct summand V of W^G the socle* $\mathrm{soc}(V)$ *and the head $hd(V)$ are both simple kG-modules.*

PROOF. By Theorem 2. 1, the endomorphism ring E is isomorphic to a twisted group algebra. In particular E is a quassi-Frobenius algebra. Therefore a projective indecomposable E-module has a simple socle and a simple head. Now the lattice isomorphism in Theorem 2. 2 induces the bijective correspondence between indecomposable direct summands of W^G and projective indecomposable E-modules. Thus Theorem 2. 2 implies Corollary.

We shall extend the above fact and prove Theorem 3. 2 in the following section.

3. Simplicity

Now we assume that the indecomposable kK-module W has a simple socle L. Then L^G is naturally embedded in W^G. Let ψ be an element of E. Clealy ψ is the element of $\mathrm{End}_{kK}(W^G{}_K)$. Then the normality of K and Mackey decomposition implies $\mathrm{soc}\,(W^G{}_K)=L^G{}_K$. Therefore $\psi(L^G{}_K)$ is included in $L^G{}_K$. Hence the following maps are well-defined:

$$\begin{aligned}\alpha:\ E_1 &\longrightarrow \mathrm{End}_{kK}(L)\\ \phi &\longmapsto \phi|_L\end{aligned}$$

and

$$\begin{aligned}\beta:\ E &\longrightarrow \mathrm{End}_{kG}(L^G)\\ \psi &\longmapsto \psi|_{L^G}\,.\end{aligned}$$

LEMMA 3. 1 *Let k be an algebraically closed field. If $T_G(W)=G$, then* $\ker(\beta)=J(E_1)\,E$.

PROOF. Since $T_G(W)=G$ for each element y of (G/K) there exists an element ψ_y which is a unit of E satisfying $E_y=E_1\cdot\psi_y=\psi_y\cdot E_1$. Therefore we have $E=\bigoplus_{y\in(G/K)} E_1\cdot\psi_y$. On the othere hand, let E' be the endomorphism ring $\mathrm{End}_{kG}(L^G)$ and E'_y the k-subspace $\{\phi\in E'\,|\,\phi(1\otimes L)\subset y\otimes L\}$ in E'. Then $E'=\bigoplus_{y\in(G/K)} E'_y$ and the element $\psi_y|_{L^G}$ of E'_y is a unit of E' satisfying $E'_y=E'_1\cdot(\psi_y|_{L^G})=(\psi_y|_{L^G})\,E'_1$ for $y\in(G/K)$.

Suppose that $\psi=\sum_{y\in(G/K)}\phi_y\psi_y$ is a element of $\ker\beta$, where $\phi_y\in E_1$. By definition, we have

$$\begin{aligned}\beta(\psi) &= \psi|_{L^G}\\ &= \textstyle\sum_{y\in(G/K)}(\phi_y\psi_y)|_{L^G}\\ &= \textstyle\sum_{y\in(G/K)}(\phi_y|_{L^G})(\psi_y|_{L^G})\\ &= 0\,.\end{aligned}$$

Since $(\phi_y|_{L^G})(\psi_y|_{L^G})$ is in E'_y this implies $(\phi_y|_{L^G})(\psi_y|_{L^G})$ is 0 for all $y\in(G/K)$.

Therefore $\alpha(\phi_y)=\phi_y|_{L^G}=0$ for all $y\in(G/K)$, and so ϕ_y is a element of $\ker\alpha$ for all $y\in(G/K)$.

By the way, $\ker\alpha$ equals $J(E_1)$ because $E_1/J(E_1)$ is isomorphic to k and the image of α is isomorphic to k. Thus ϕ_y is in $J(E_1)$ for all $y\in(G/K)$. So ϕ is contained in $J(E_1)E$. Hence $\ker\beta\subset J(E_1)E$.

Conversely, $\ker\beta\supset J(E_1)E$ is easily checked by reversing the above process.

THEOREM 3.2 *Let k be an algebraically closed field. Suppose that W is an indecomposable kK-module satisfying $T_G(W)=G$. If* $\mathrm{soc}(W)$ *is a simple kK-module, then for an indecomposable direct summand V of W^G,* $\mathrm{soc}(V)$ *is a simple kG-module.*

PROOF. Suppose that $\mathrm{soc}(W)=L$ is a simple kK-module. By Lemma 3.1, β induces the following inclusion:

$$\bar{\beta}:\ E/\ker\beta=E/J(E_1)E\hookrightarrow E'.$$

By Theorem 2.1 $E/J(E_1)E$ and E' are both twisted group algebras over Y. In particular, the dimension of $E/J(E_1)E$ over k equals that of E'. Therefore the above inclusion is an isomorphism of $E/J(E_1)E$ onto E'.

Let V be an indecomposable direct summand of W^G and f a primitive idempotent of E satisfying $V=fW^G$. Since $\bar{\beta}$ is an isomorphism $\beta(f)=f|_{L^G}$ is a primitive idempotent of E'. Thus fL^G is an indecomposable direct summand of L^G, and so $V\cap L^G$ is that of L^G because $fL^G=fW^G\cap L^G=V\cap L^G$. On the other hand the normality of K implies $\mathrm{soc}(W^G{}_K)=L^G{}_K$. Therefore $\mathrm{soc}(V)$ is contained in L^G. Thus we obtain $\mathrm{soc}(V)=\mathrm{soc}(V\cap L^G)$. Hence we can reduce this Theorem in case W is simple. By Corollary 2.3 we have completed the proof.

COROLLARY 3.3 *Let k, K, W are defined in Theorem* 3.2. *If* $\mathrm{soc}(W)$ *is a simple kK-modules, then the number of indecomposable direct summand of W^G is equal to that of* $\mathrm{soc}(W)^G$.

PROOF. The statement of Corollary is immediately from the proof of Theorem 2.5.

COROLLARY 3.4 *Let k be an algebraically closed field of characteristic p and W an indecomposable kK-module satisfying $T_G(W)=G$. Suppose that* $\mathrm{soc}(W)=L$ *is a simple kK-module and $p\nmid|G:K|$. Then W is extedible to G if, and only if, L is extendible to G.*

PROOF. Suppose that W is extendible to G. Then there exists an indecomposable kG-module V such that $V_K\simeq W$. Since K is a normal subgroup of G we have $\mathrm{soc}(V)_K \subset \mathrm{soc}(V_K) \simeq \mathrm{soc}(W) = L$. Therefore $\mathrm{soc}(V)$ is a simple kG-module and $\mathrm{soc}(V)_K \simeq L$. Hence L is extendible.

Conversely, suppose that L is extendible to G. There exists a simple kG-module M such that $M_K \simeq L$. Since $\mathrm{soc}(W) = L$, Frobenius reciprocity implies

$$\begin{aligned} k &\simeq \mathrm{Hom}_{kK}(L, W) \\ &\simeq \mathrm{Hom}_{kK}(M_K, W) \\ &\simeq \mathrm{Hom}_{kG}(M, W^G) \\ &\simeq \mathrm{Hom}_{kG}\left(M, \mathrm{soc}(W^G)\right). \end{aligned}$$

Therefore $\mathrm{soc}(W^G) \simeq M +$(other simple kG-modules). By Theorem 3. 2, there exists an indecomposable direct summand V of W^G such that $\mathrm{soc}(V) \simeq M$. Since $T_G(W) = G$ Mackey decomposition Theorem implies $V_K \simeq mW$, where m is a positive integer. So Frobenius reciprocity implies

$$\begin{aligned} mk &\simeq \mathrm{Hom}_{kK}(L, mW) \\ &\simeq \mathrm{Hom}_{kK}(L, V_K) \\ &\simeq \mathrm{Hom}_{kG}(L^G, V). \end{aligned}$$

On the other hand, $\mathrm{End}_{kG}(L^G)$ is a semi-simple algebra because $p \nmid |G:K|$ and Theorem 2. 1. Therefore by Theorem 3. 2, L^G is a semi-simple kG-module. Furthermore L^G has M as a direct summand with multiplicity one. So $\mathrm{Hom}_{kG}(L^G, M) \simeq k$. Hence we obtain $m=1$, and $V_K \simeq W$. Thus W is extendible.

The results of Theorem 3. 2 and two corollaries are concerned with socles, but by using of contragradient modules, we can prove similar facts of them in case of heads.

Acknowledgement

The author wish to thank Dr. T. Okuyama for his helpfull advice.

References

[1] S. B. CONLON: Twisted group algebras and their representations, *J. Austral. Math. Soc.* *4* (1964), 152–173.

[2] C. W. CURTIS and I. REINER: Methods of Representation Theory: With Application to Finite Groups and Orders, Vol. 1, Pure and Applied Mathematics, Wieley-Interscience, New York/Chichester/Brisbane/Toronto, 1981.

[3] W. FEIT: The Representation Theory of Finite Groups, Mathematical Library, Vol. 25, North-Holland, Amsterdam/New York/Oxford, 1982.

[4] J. A. GREEN: A transfer theorem for modular representations, *J. Alg. 1* (1964), 73–84.

[5] J. A. GREEN: On a theorem of H. Sawada, *Proc. London Math. Soc.* (2) *18* (1978), 247–252.

[6] T. OKUYAMA: Module correspondence in finite groups, *Hokkaido Math. J. 10* (1981), 299–318.

[7] P. TUCKER: Endomorphism ring of an induced module, *Michigan Math. J. 12* (1965), 197–203.

[8] P. TUCKER: On the reduction of induced indecomposable representation, *Amer. J. Math. 87* (1965), 798–806.

[9] H. N. WARD: The analysis of representations induced from a normal subgroup, *Mich. Math. J. 15* (1968), 417–428.

Department of Mathematics
Hokkaido University
Sapporo 060, Japan